PUTNAM 2014 WEEK 10
POLYNOMIALS

Basic definitions and facts

A polynomial $a_0x^n + \ldots + a_n$, $a_0 \neq 0$, has *degree* n, *leading term* a_0, and *constant term* a_n. A polynomial with $a_0 = 1$ is called *monic*.

Division with remainder (long division): for any polynomials $f(x)$ and $g(x)$ we can write $f(x) = g(x)q(x)+r(x)$ with $\deg r(x) < \deg g(x)$ (by convention, $\deg 0 = -\infty$). The *greatest common divisor* of polynomials $f(x)$ and $g(x)$ can always be written as $a(x)f(x) + b(x)g(x)$.

A polynomial $f(x)$ has root α if and only if $x - \alpha$ divides $f(x)$. Any complex polynomial admits a *unique factorization* into linear complex factors $f(x) = c(x - \alpha_1)\ldots(x - \alpha_n)$. A *multiple root* is a root repeated several times in this factorization.

Coefficients can be expressed in terms of roots (*Vieta formulas*), for example $-a_1/a_0 = \alpha_1 + \ldots + \alpha_n$ and $(-1)^n a_n/a_0 = \alpha_1 \ldots \alpha_n$.

Rational roots theorem: if a polynomial with integer coefficients has a rational root p/q then p divides a_n and q divides a_0.

Interpolation: a degree n polynomial is uniqiely determined by its values at $n + 1$ points.

Easier Problems

1. Consider two polynomials, $P(x)$ and $Q(x)$. Suppose that each of them has the property that the sum of its coefficients at odd powers of x is equal to the sum of its coefficients at even powers of x. For example, we can take $P(x) = 1+3x+2x^2$ and $Q(x) = -1+4x^2+3x^3$. Is it true that $P(x)Q(x)$ has the same property?

2. Suppose a polynomial $x^3 + px + q$ has roots α, β, γ. Express $\alpha^3 + \beta^3 + \gamma^3$ in terms of p and q.

3. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

4. It is known that a quadratic equation has either 0, 1, or 2 unique real solutions. But consider the equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$$

where a, b, and c are distinct. Notice that $x = a$, $x = b$, and $x = c$ are all solutions – how can this equation have three solutions?

5. Show that a polynomial $f(x)$ has a multiple root if and only if the greatest common divisor of $f(x)$ and $f'(x)$ is not a constant.

6. Show that the polynomial

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

has no multiple roots.

Harder Problems

7. Solve

$$(x^2 - 3x - 4)(x^2 - 5x + 6)(x^2 + 2x) + 30 = 0.$$

8. *(The interpolation formula)* Suppose a_1, ..., a_n are distinct numbers, and b_1, ..., b_n are given numbers, and $P(x)$ is a degree at most $n - 1$ polynomial such that $P(a_i) = b_i$ for all i. Show that

$$P(x) = b_1 \frac{(x - a_2)(x - a_3) \cdots (x - a_n)}{(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)} + b_2 \frac{(x - a_1)(x - a_3) \cdots (x - a_n)}{(a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n)} + \cdots + b_n \frac{(x - a_1)(x - a_2) \cdots (x - a_{n-1})}{(a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})}.$$

9. Let $p(x)$ be a polynomial of even degree with a positive leading coefficient. Suppose $p(x) \geq p''(x)$ for every x. Show that $p(x) \geq 0$ for every x.

10. If $P(x)$ is a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, \ldots, n$, determine $P(n + 1)$.

11. A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

12. Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

($\lfloor a \rfloor$ means the largest integer $\leq a$.)