PUTNAM 2014 WEEK 1. PROOF BY INDUCTION.

General but surprisingly effective advice. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Easier Problems

1. Find and prove a formula for the sum of the first n consecutive odd positive integers. For example, if $n = 4$ then $1 + 3 + 5 + 7 = 16$.
2. Show that a $2^n \times 2^n$ square with a corner tile removed can be covered without overlaps by L-shaped figures (each figure contains 3 tiles). (If you feel adventurous, how about an $n \times n$ square?)
3. Write numbers $1, 1$ on the blackboard. Write the sum of two numbers between them. We get $1, 2, 1$. Repeat: $1, 3, 2, 3, 1$. Repeat: $1, 4, 3, 5, 2, 5, 3, 4, 1$. Find the sum of all numbers on the board after repeating this 100 times.
4. Show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$$

for any $n > 0$.
5. Suppose we have n closed intervals $[a_i, b_i]$ on the x-axis and suppose that any two of them have a common point. Show that all n of them have a common point.
6. Show that 1 can be written as a sum $\frac{1}{a_1} + \ldots + \frac{1}{a_{100}}$, where $a_1, \ldots, a_{100}$ are different integers.

Harder Problems

7. Start with a rational number q and allow two operations: either erase q and write $q+1$ or erase q and write $1/q$. Show that we can get any positive rational number after a sequence of these steps if we start with 1.
8. The first $2n$ natural numbers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order, i.e., $a_1 < ... < a_n$, and the numbers in the second group are sorted in descending order: $b_1 > ... > b_n$. Find, with proof, the sum

$$|a_1 - b_1| + ... + |a_n - b_n|.$$

9. The sequence $\{p_n(x)\}$ of polynomials is defined as follows:

$$p_1(x) = 1 + x, \quad p_2(x) = 1 + 2x,$$

and

$$p_{2m+1}(x) = p_{2m}(x) + (m+1)xp_{2m-1}(x),$$
$$p_{2m+2}(x) = p_{2m+1}(x) + (m+1)xp_{2m}(x).$$

Let x_n be the largest real solution of $p_n(x) = 0$. Prove that x_n is an increasing sequence and that $\lim_{n\to\infty} x_n = 0$.
10. Suppose we have a map of n countries, where at any point at most three countries come together. Let's call this map 3-*colorable* if one can color it using 3 colors in such a way that any two countries sharing a common border are colored differently. Show that the map is 3-colorable if and only if any country is separated from other countries by an even number of contiguous borders (Hint: look for a country with a small number of borders).
11. Let $g(x, y)$ be a continuous function defined for $0 \leq x \leq 1$, $0 \leq y \leq 1$. Suppose that

$$g(x, y) = \int_0^x \int_0^y g(u, v)\, du\, dv$$

for any $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that $g(x, y)$ is identically equal to 0.
12. Given $T_1 = 2$, $T_{n+1} = T_n^2 - T_n + 1$ for $n > 0$, show that T_n and T_m are coprime for any $n \neq m$.