PUTNAM 2011 WEEK 5: NUMBER THEORY.

Basic number theory concepts to remember and use: modular arithmetic, unique factorization, greatest common divisor (a,b) can be written as a linear combination ax+by, Chinese Remainder Theorem, positional notation, Fermat's little theorem $a^p \equiv a \pmod{p}$, quadratic residues.

Easier Problems

0. Find the smallest positive integer n such that one half of n is a square, one third of n is a cube, and one fifth of n is a fifth power.

1. Let a and b be positive integers such that $56a = 65b$. Show that $a + b$ cannot be a prime number.

2. Show, without using a calculator, that $2^9 + 2^{99}$ is divisible by 100.

3. Find ten different positive integers such that their sum is divisible by each of them.

4. Show that, for every positive integer n, the number $1998^n - 1$ is not divisible by $1000^n - 1$.

5. Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

Harder Problems

6. A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

7. Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

8. Let p be an odd prime and let

$$F(n) = 1 + 2n + 3n^2 + \ldots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p.

9. Prove that there are exactly three right-angled triangles T whose sides are integers and such that the area of T is equal to the perimeter of T (the triangle with sides 6, 8, 10 is one example).

10. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.