PUTNAM 2012 WEEK 4. INEQUALITIES.

Easier Problems

1. In each case, which number is larger (or largest): (a) $\sqrt{19}+\sqrt{99}$ or $\sqrt{20}+\sqrt{98}$? (b) $2000!$ or 1000^{2000}? (c) 1999^{1999}, 2000^{1998}, or 1998^{2000}?

2. Prove that $\sqrt{ab} \leq (a+b)/2$ with equality if and only if $a = b$.

Digression. The previous problem is the $n = 2$ case of the powerful AG (arithmetic mean – geometric mean) inequality

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

if the a_i are non-negative real numbers. To see this, let's take logarithms of both sides. Then we have to prove that

$$\frac{1}{n}\sum_{i=1}^{n} \ln(a_i) \leq \ln\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right).$$

In fact, the analogous inequality holds for *any concave function* $f(x)$:

$$\frac{1}{n}\sum_{i=1}^{n} f(a_i) \leq f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right).$$

This is called *Jensen's inequality*. You can read its proof in any real analysis textbook.

3. What changes in Jensen's inequality if $f(x)$ is a convex function? What does this inequality say if $f(x) = x^2$? If $f(x) = 1/x$?

4. Let $a_1, \ldots, a_n$ be positive, with a sum of 1. Show that

$$\sum a_i^2 \geq 1/n.$$

5. Given a, b, c positive real numbers such that $(a+1)(b+1)(c+1) \leq 8$, prove that $abc \leq 1$ (Hint: Lagrange multipliers).

6. Determine the maximum value of $(\sin A_1)(\sin A_2)\cdots(\sin A_n)$ if

$$(\tan A_1)(\tan A_2)\cdots(\tan A_n) = 1.$$

Harder Problems

8. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.
9. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx$, where $a_1, \ldots, a_n$ are real numbers. Given that $|f(x)| \leq |\sin x|$ for all x, prove that

$$|a_1 + 2a_2 + \ldots + na_n| \leq 1.$$

Digression. Another very useful inequality is the *Cauchy–Schwarz inequality*

$$|a_1 b_1 + \cdots + a_n b_n| \leq \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}.$$

An incarnation of the same inequality is

$$\left| \int_0^1 f(x) g(x)\, dx \right| \leq \sqrt{\int_0^1 f^2(x)\, dx} \sqrt{\int_0^1 g^2(x)\, dx}.$$

The proof can be found in any linear algebra textbook.
10. Show that if the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is convergent, where $a_1, a_2, \ldots$ are positive real numbers, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(a_1 + \ldots + a_n)^2} a_n$$

is also convergent.
11. Let f be a continuous and monotonically increasing function such that $f(0) = 0$ and $f(1) = 1$. Prove that

$$f(0.1) + f(0.2) + \cdots + f(0.9) + f^{-1}(0.1) + f^{-1}(0.2) + \cdots + f^{-1}(0.9) \leq 9.9$$

(here $f^{-1}(x)$ is the inverse function of $f(x)$).
12. Find all polynomials whose coefficients are equal to 1 or -1 and which have only real roots.