PUTNAM 2010 WEEK 8: POLYNOMIALS.

Things to try first. Move everything to one side. Factor. Expand. Complete the square. $x^n - y^n =?$, $x^{2m+1} + y^{2m+1} =?$. Interpolation: a degree n polynomial is determined by its values at $n+1$ points. Vieta formulas (coefficients of a polynomial in terms of its roots). How to get sums of powers of roots (and other symmetric polynomials)? Rational roots theorem (if a polynomial with integer coefficients has a rational root then ???). Long division of polynomials.

Easier Problems

1. $1/(x+1)(x+2)(x+3) =?/(x+1)+?/(x+2)+?/(x+3)$.

2. Consider two polynomials, $P(x)$ and $Q(x)$. Suppose that each of them has the property that the sum of its coefficients at odd powers of x is equal to the sum of its coefficients at even powers of x. For example, we can take $P(x) = 1 + 3x + 2x^2$ and $Q(x) = -1 + 4x^2 + 3x^3$. Is it true that $P(x)Q(x)$ has the same property?

3. Factor $x^4 + 4$ into two polynomials with real coefficients.

4. Let $p(x) = x^6 + x^5 + x^4 + \ldots + 1$. Find the remainder when $p(x^7)$ is divided by $p(x)$.

5. Prove that $(2+\sqrt{5})^{1/3}$ is irrational but $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}$ is rational.

6. Solve

$$(x^2 - 3x - 4)(x^2 - 5x + 6)(x^2 + 2x) + 30 = 0.$$

Harder Problems

7. If $P(x)$ is a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, \dots, n$, determine $P(n+1)$.

8. Show that four points on the parabola $y = x^2$ with coordinates

$$(a, a^2),\ (b, b^2),\ (c, c^2),\ (d, d^2)$$

(a, b, c, d distinct) lie on a circle if and only if $a + b + c + d = 0$.

9. A *repunit* is a positive integer which looks like $111\dots1$. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

10. The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

10. Let k be a positive integer. Prove that there exist polynomials

$$P_0(n), P_1(n), \dots, P_{k-1}(n)$$

(which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

($\lfloor a \rfloor$ means the largest integer $\leq a$.)

11. If a and b are two roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.