PUTNAM 2010 WEEK 4. GAMES.

Easier Problems

1. Alice and Bob play a game in which they have two piles of stones and they alternatively pick any number of stones, but from just one pile. The person who takes the last stone (a) loses; (b) wins. Alice goes first. Who has the winning strategy?

2. If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

3. Can you make change for 5 dollars using 100 coins if you are not allowed to use nickels?

4. Let $P_1, \dots, P_{1001}$ be distinct points in the plane. Connect the points with the line segments P_1P_2, P_2P_3, $\dots$, $P_{1000}P_{1001}$, $P_{1001}P_1$. Can one draw a line that passes through the interior of every one of these segments?

5. Any infinite sequence of real numbers contains either an infinite increasing subsequence or an infinite decreasing subsequence.

6. Let S be a set of points in the plane and let R be a fixed positive real number. We want to place discs of radius R on the plane such that,

(i) Every point in S is covered by exactly one disc, and
(ii) Ever disc is centered at a point in S.

This may or may not be possible, but if it is possible, show that all solutions use the exact same number of discs.

Harder Problems

7. Suppose $n \geq 2$ light bulbs are arranged in a row, numbered 1 through n. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the neighbors' states. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which n is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off?

8. Start with the set $\{3, 4, 12\}$. You are then allowed to replace any two numbers a and b with the new pair $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you transform the set into $4, 6, 12$?

9. Initially, we are given a sequence $1, 2, 3, \ldots, 100$. Every minute, we erase any two numbers u and v and replace them with the value $u + v + uv$. Clearly, we will be left with just one number after 99 minutes. Does this number depend on the choices that we made?

10. Alice and Bob play a game in which they take turns filling entries of an initially empty 100×100 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

11. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k - 1 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$