# PUTNAM 2010 WEEK 2
# FIBONACCI! (AND OTHER SEQUENCES)

## Easier Problems

**1.** Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ are defined by

$$F_1 = 1, \quad F_2 = 1, \quad \ldots, \quad F_{n+2} = F_{n+1} + F_n.$$

Prove that any two consecutive Fibonacci numbers are coprime.

**2.** Find lines in the Pascal triangle that add up to Fibonacci numbers.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots \end{array}$$

**3.** In how many ways can a $2 \times n$ rectangle be tiled with $2 \times 1$ dominoes?

**4.** Prove that

$$(a) \quad F_{n-1}F_{n+1} - F_n^2 = (-1)^n \quad \text{(Cassini identity);}$$

$$(b) \quad F_{n-r}F_{n+r} - F_n^2 = (-1)^{n-r+1}F_r^2.$$

**5.** What happens if you put a random positive number in your (infinite precision) calculator, and repeatedly hit the sequence of buttons "$1/x$", "+" and "1"? (In other words, what happens if you iterate $x \mapsto (1/x) + 1$?). How is this problem related to Fibonacci numbers?

**6.** $[\sqrt{44}] = 6$, $[\sqrt{4444}] = 66$. Generalize and prove.

**7.** Using the formula

$$(1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \ldots + (n^3 - (n-1)^3) = n^3,$$

find a closed formula for $1^2 + 2^2 + \ldots + n^2$. Generalize to the sum of cubes.

**8.** Find the sum

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!$$

**Harder Problems**

**9.** Two players take turns removing stones from the pile. The rules are simple: if the pile contains $2n$ stones then the player must leave $n$ stones in the pile. If the pile contains $2n+1$ stones then the player has a choice of leaving $n$ or $n+1$ stones. The game ends when one person removes the last stone, thus winning. How many stones should be in the pile for the first player to have a winning strategy?

**10.** Returning to Fibonacci numbers, prove that

$$\arctan\frac{1}{F_{2n}} = \arctan\frac{1}{F_{2n+1}} + \arctan\frac{1}{F_{2n+2}}.$$

Use this identity to compute the infinite series

$$\arctan\frac{1}{1} + \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{13} + \arctan\frac{1}{34} + \dots$$

**11.** The sequence $a_0, a_1, a_2, \dots$ satisfies

$$a_{m+n} + a_{m-n} = \frac{a_{2m} + a_{2n}}{2}$$

for all nonnegative integers with $m \geq n$. If $a_1 = 1$, compute $a_{2010}$.

**12.** You have $n$ coins $C_1, C_2, \dots, C_n$. For each $k$, $C_k$ is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the $n$ coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of $n$.

**13.** Consider two lists. List $A$ consists of the positive powers of $10$ ($10$, $100$, $1000$, $\dots$) written in base $2$. List $B$ consists of the positive powers of $10$ written in base $5$.

| Powers of 10 | List $A$ | List $B$ |
|---:|---:|---:|
| 10 | 1010 (4 digits) | 20 (2 digits) |
| 100 | 1100100 (7 digits) | 400 (3 digits) |
| 1000 | 1111101000 (10 digits) | 13000 (5 digits) |
| 10000 | 10011100010000 (14 digits) | 310000 (6 digits) |

Show that, for any integer $n > 1$, there is exactly one number in exactly one of the lists that is exactly $n$ digits long.