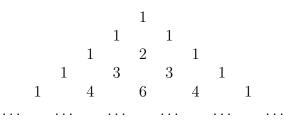
PUTNAM 2010 WEEK 2 FIBONACCI! (AND OTHER SEQUENCES)

Easier Problems

1. Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... are defined by

$$F_1 = 1, \quad F_2 = 1, \quad \dots, \quad F_{n+2} = F_{n+1} + F_n.$$

Prove that any two consecutive Fibonacci numbers are coprime. **2.** Find lines in the Pascal triangle that add up to Fibonacci numbers.



3. In how many ways can a $2 \times n$ rectangle be tiled with 2×1 dominoes?

4. Prove that

(a)
$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$
 (Cassini identity);
(b) $F_{n-r}F_{n+r} - F_n^2 = (-1)^{n-r+1}F_r^2$.

5. What happens if you put a random positive number in your (infinite precision) calculator, and repeatedly hit the sequence of buttons "1/x", "+" and "1"? (In other words, what happens if you iterate $x \mapsto (1/x) + 1$?). How is this problem related to Fibonacci numbers? **6.** $[\sqrt{44}] = 6$, $[\sqrt{444}] = 66$. Generalize and prove.

7. Using the formula

 $(1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \ldots + (n^3 - (n-1)^3) = n^3,$

find a closed formula for $1^2 + 2^2 + \ldots + n^2$. Generalize to the sum of cubes.

8. Find the sum

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!$$

Harder Problems

9. Two players take turns removing stones from the pile. The rules are simple: if the pile contains 2n stones then the player must leave n stones in the pile. If the pile contains 2n+1 stones then the player has a choice of leaving n or n+1 stones. The game ends when one person removes the last stone, thus winning. How many stones should be in the pile for the first player to have a winning strategy? **10.** Returning to Fibonacci numbers, prove that

$$\arctan \frac{1}{F_{2n}} = \arctan \frac{1}{F_{2n+1}} + \arctan \frac{1}{F_{2n+2}}$$

Use this identity to compute the infinite series

$$\arctan \frac{1}{1} + \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{13} + \arctan \frac{1}{34} + \dots$$

11. The sequence a_0, a_1, a_2, \ldots satisfies

$$a_{m+n} + a_{m-n} = \frac{a_{2m} + a_{2n}}{2}$$

for all nonnegative integers with $m \ge n$. If $a_1 = 1$, compute a_{2010} . **12.** You have n coins C_1, C_2, \ldots, C_n . For each k, C_k is biased so that, when tossed, it has probability 1/(2k + 1) of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

13. Consider two lists. List *A* consists of the positive powers of 10 (10, 100, 1000, ...) written in base 2. List *B* consists of the positive powers of 10 written in base 5.

Powers of 10	List A	List B
10	1010 (4 digits)	20 (2 digits)
100	1100100 (7 digits)	400 (3 digits)
1000	1111101000 (10 digits)	13000 (5 digits)
10000	10011100010000 (14 digits)	310000 (6 digits)

Show that, for any integer n > 1, there is exactly one number in exactly one of the lists that is exactly n digits long.