PUTNAM 2010 WEEK 1
INDUCTION AND PIGEONHOLE PRINCIPLE

Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask. Don't solve problems that you already know how to solve.

Easier Problems

1. Find and prove a formula for the sum of the first n consecutive odd positive integers. For example, if $n = 4$ then $1 + 3 + 5 + 7 = 16$.
2. Prove the formula for the sum of a geometric series:

$$a^n + a^{n-1} + a^{n-2} + \ldots + 1 = \frac{a^{n+1} - 1}{a - 1}.$$

3. Let S be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.
4. Prove that in a room with n people, at least two people know exactly the same number of people. Assume knowing is a symmetric relation: If Paul knows Pete, then Pete knows Paul.
5. On a certain street there are twenty houses, ten along each side of the road. Abe, Bill, Cathy, Dierdre, and Ed each live in one of the houses. Prove that among these five people, there must be two of them who live on the same side of the street separated by no more than three houses between them.
6. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

Harder Problems

7. Prove that given any five points on a sphere, there exists a closed hemisphere which contains four of the points.
8. Choose 51 positive integers from 1 to 100. Prove that one of them is a multiple of another.)
9. For each non-empty subset of $\{1, 2, ..., n\}$ take the sum of the elements divided by the product. Let S be the sum of the resulting quantities. For example, if $n = 2$ then non-empty subsets are $\{1\}$, $\{2\}$, and $\{1, 2\}$, and

$$S = \frac{1}{1} + \frac{2}{2} + \frac{3}{2} = \frac{7}{2}.$$

Prove that

$$S = n^2 + 2n - (n + 1)s_n,$$

where $s_n = 1 + 1/2 + 1/3 + ... + 1/n$.
10. The first $2n$ natural numbers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order, i.e., $a_1 < ... < a_n$, and the numbers in the second group are sorted in descending order: $b_1 > ... > b_n$. Find, with proof, the sum $|a_1 - b_1| + ... + |a_n - b_n|$.
11. Prove that

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \leq \frac{1}{\sqrt{3n}}.$$