PUTNAM 2009 WEEK 7: GENERATING FUNCTIONS

In this worksheet we will consider infinite power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$$

We will not worry about their convergence at all: we will just add and multiply them formally. This is interesting because this gives a compact way to "store" information about the sequence $a_0, a_1, a_2, \ldots$ and this is very useful for combinatorics. The power series $f(x)$ is called a generating function of the sequence $a_0, a_1, a_2, \ldots$

Easier Problems

1. Take a sequence $1, 2, 4, 8, \ldots$. Prove that its generating function is $\frac{1}{1-2x}$.

2. (a) Prove that $(1+x)^n$ is the generating function of the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots$$

(b) Using the identity $(1+x)^n(1+x)^m = (1+x)^{n+m}$, prove the Vandermonde identity

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}.$$

(Hint: Interpret the x^k term on both sides of the identity).

3. Give a formula for

$$\sum_{i=1}^{n} i^2 \binom{n}{i}$$

where n is a non-negative integer.

4. (a) Prove that

$$f(x) = \frac{1}{1-x-x^2}$$

is the generating function for the Fibonacci sequence

$$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ \ldots$$

(b) Rewrite $f(x)$ using partial fractions. (c) Write a closed-form formula for the Fibonacci sequence.

5. Find a closed-form formula for A_n, where

$$A_0 = 2,\ A_1 = 5,\ A_n = 5A_{n-1} - 6A_{n-2}.$$

6. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

Harder Problems

7. (a) Suppose that in base p, $n = n_0 + n_1 p + \cdots + n_k p^k$ and $a = a_0 + a_1 p + \cdots + a_k p^k$. Show that

$$\binom{n}{a} \equiv \prod_{i=1}^{k} \binom{n_i}{a_i} \pmod{p}$$

(b) how many odd binomial coefficients are there in the 2009-th row of Pascal's triangle?

8. Let $x_0 = 1$ and for $n \geq 0$, let

$$x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor.$$

In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.)

9. Show that for each positive integer n, the number of partitions of n into unequal parts is equal to the number of partitions of n into odd parts. For example, if $n = 6$, there are 4 partitions into unequal parts, namely

$$1+5, \quad 1+2+3, \quad 2+4, \quad 6.$$

And there are also 4 partitions into odd parts,

$$1+1+1+1+1+1, \quad 1+1+1+3, \quad 1+5, \quad 3+3.$$

10. A finite sequence a_1, a_2, …, a_n is called *p-balanced* if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, \ldots, p$. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal to zero.