PUTNAM 2009 WEEK 6:
GEOMETRY, VECTORS, COMPLEX NUMBERS.

Easier Problems

1. Draw two lines through each vertex of a triangle ABC and assume that they divide an opposite side into three equal segments. These six lines form a hexagon. Prove that diagonals of this hexagon intersect in one point. (Hint: this is a *simple* problem).

2. Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

3. Points A and B move clock-wise around circumferences C_1 and C_2 with equal angular speeds. Prove that the vertex C of the equilateral triangle ABC also moves around some circumference.

4. Let $ABCDEF$ be a hexagon inscribed in a circle of radius r. Show that if $|AB| = |CD| = |EF| = r$, then the midpoints of BC, DE, FA are the vertices of an equilateral triangle.

5. Let a, b, c, d be four different complex numbers. Prove that the corresponding points of the plane lie on a circle (or on a line) if and only if $\frac{a-c}{a-d}/\frac{b-c}{b-d}$ is a real number.

Harder Problems

6. Take points A, C, E, F on a circle. Let $t(A)$ (resp. $t(C)$) be the line tangent to the circle at A (resp. at C). Let CE, EF, FA be the lines that pass through the corresponding points. Now define $Q = t(A) \cap CE$, $R = AC \cap EF$, $S = t(C) \cap FA$. Prove that points Q, R, S are collinear.

7. Let $u_1, \ldots, u_n$ be vectors on the plane such that $|u_i| \leq 1$ for any i. Show that one can choose signs in the expression $\pm u_1 \pm u_2 \pm \ldots \pm u_n$ such that the resulting vector has length at most $\sqrt{2}$.

8. For any line L and a point P of the plane such that $P \notin L$, let $\pi_{P,L}$ be the projection onto L away from P. It is defined as follows: take any point $Q \neq P$. Then $\pi_{P,L}(Q)$ is the point where L intersects the line PQ.

Now consider four points A, B, C, D such that no three of them lie on a line. Let M_1 be any point of the line AB. Now we define inductively

$$M_2 = \pi_{D,BC}(M_1),\ M_3 = \pi_{A,CD},\ M_4 = \pi_{B,DA},\ M_5 = \pi_{C,AB}, \ldots.$$

Prove that $M_{13} = M_1$.

9. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that $P^2 + Q^2 = X^{2n} + 1$ and $\deg P > \deg Q$.