PUTNAM 2009 WEEK 2: GAMES AND INVARIANTS

Easier Problems.

1. Alice and Bob play a game in which they have two piles of stones and they alternatively pick any number of stones, but from just one pile. The person who takes the last stone (a) loses; (b) wins. Alice goes first. Who has the winning strategy?

2. In a round-robin tournament with n players $P_1, \dots, P_n$, each player plays one game with each of the other players and no ties can occur. Let w_r and l_r be the number of games won and lost, respectively, by P_r. Prove that

$$\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} l_i^2.$$

3. Let $P_1, \dots, P_{2009}$ be distinct points in the plane. Connect the points with the line segments P_1P_2, P_2P_3, $\dots$, $P_{2008}P_{2009}$, $P_{2009}P_1$. Can one draw a line that passes through the interior of every one of these segments?

4. If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

5. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{2006} minutes, could the room contain $3^{1000} + 2$ people?

6. Alice and Bob play a game in which they take turns filling entries of an initially empty 100×100 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Harder Problems.

7. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must a prime number. The winner is the player who takes the last stone. If there is just one stone left then they declare a tie (because 1 is not a prime number). Alice plays first. (a) Can Bob ever win if Alice does not make any mistakes? (b) Prove that there are infinitely many n such that Bob has a non-loosing strategy.

8. Suppose $n \geq 2$ light bulbs are arranged in a row, numbered 1 through n. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the neighbors' states. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which n is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off?

9. A collection of n beetles, each black or white in color, is arranged in a line. On each move, a black beetle turns pink, emitting a chemical which causes its immediate neighbors to switch from black to white, or white to black (as appropriate). Already pink bugs are not affected. Under what starting conditions is it possible for all n bugs to turn pink?

10. Suppose T is a triangle on the plane and its vertices are lattice points (i.e. they have integral coordinates). Suppose also that no other lattice points lie inside the triangle or on its sides. Prove that T has area $1/2$.

11. A 100×100 checkerboard is arbitrarily tiled by 2×1 dominoes. Alice and Bob play a game in which they take turns "gluing" two adjacent squares in the checkerboard that belong to different dominoes. Alice goes first. It is not allowed to glue along the same segment twice. As a result, after each step, the checkerboard will be tiled by some connected figures. The player loses if the whole checkerboard becomes connected after his or her step. Prove that Bob has a winning strategy.