# PUTNAM 2009 WEEK 8: ALGEBRA AND GROUPS.

## Easier Problems

**1.** *(The interpolation formula)* Suppose $a_1$, ..., $a_n$ are distinct numbers, and $b_1$, ..., $b_n$ are given numbers, and $P(x)$ is a degree at most $n-1$ polynomial such that $P(a_i) = b_i$ for all $i$. Show that

$$P(x) = b_1 \frac{(x-a_2)(x-a_3)\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + b_2 \frac{(x-a_1)(x-a_3)\cdots(x-a_n)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} + \cdots + b_n \frac{(x-a_1)(x-a_2)\cdots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}.$$

**2.** If $P(x)$ is a polynomial of degree $n-1$ such that $P(k) = k/(k+1)$ for $k = 0, \ldots, n-1$, determine $P(n)$.

**3.** In the additive group of ordered pairs of integers $(m, n)$ [with addition defined componentwise: $(m, n) + (m', n') = (m + m', n + n')$] consider the subgroup $H$ generated by three elements:

$$(3, 8),\ (4, -1),\ (5, 4).$$

Then $H$ has another set of generators of the form

$$(1, b),\ (0, a)$$

for some integers $a, b$ with $a > 0$. Find $a$.

**4.** $[\sqrt{44}] = 6$, $[\sqrt{4444}] = 66$. Generalize and prove.

**5.** $A$ is a subset of a finite group $G$ (with group operation called multiplication) and $A$ contains more than one half of the elements of $G$. Prove that each element of $G$ is the product of two elements of $A$.

## Harder Problems

**6.** Let $A$ and $B$ be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$, and $B^{2009} = 1$. Prove $B = 1$.

**7.** Show that four points on the parabola $y = x^2$ with coordinates

$$(a, a^2),\ (b, b^2),\ (c, c^2),\ (d, d^2)$$

($a, b, c, d$ distinct) lie on a circle if and only if $a + b + c + d = 0$.

**8.** Let $f(x)$ be a polynomial of degree $n$ such that some power of $f(x)$ is divisible by some power of its derivative $f'(x)$. Prove that $f(x)$ has a unique root of multiplicity $n$.

**9.** The product of two of the four zeros of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is $-32$. Find $k$.

**10.** Suppose that a finite group has exactly $n$ elements of order $p$, where $p$ is a prime. Prove that either $n = 0$ or $p$ divides $n + 1$.