Problem Solving Seminar. Worksheet 6. Real Analysis.

Basic principles from real analysis:

Calculus. Riemann Sums: if a function is integrable, e.g. if it is continuous on a closed finite interval, then the integral is the limit of the Riemann sums.

Continuity. Remember these via pictures! The Intermediate Value Theorem (if f is continuous on $[a, b]$, then every value between $f(a)$ and $f(b)$ is of the form $f(c)$ for $a \le c \le b$). The Extreme Value Theorem (a continuous function on $[a, b]$ attains its sup and inf). Rolle's Theorem (if f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a point $u \in (a, b)$ at which $f'(u) = 0$). The Mean Value Theorem (if f is continuous on $[a, b]$ and differentiable on (a, b), then there is a point $u \in (a, b)$ at which $f'(u) = \frac{f(b)-f(a)}{b-a}$.

Convergence. Sequences that converge: a bounded monotone sequence converges. A sum (series) in which the entries have alternating sign and which decrease in absolute value converge. A monotone sum whose corresponding integral is bounded converges (the integral comparison test). A sequence bounded above and below by two other convergent sequences must converge (the squeeze principle).

Taylor's Formula with Remainder: if h has continuous nth derivatives, then for any $x > 0$ and integer $n > 0$, there exists $\theta_n \in [0, x]$ such that

$$h(x) = h(0) + h'(0)x + h''(0)\frac{x^2}{2!} + \cdots + h^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + h^{(n)}(\theta_n)\frac{x^n}{n!}.$$

1. Suppose $f(x)$ is a polynomial of odd degree. Then $f(x) = 0$ has a real root. (Follow-ups: Any square matrix with an odd number of rows has a real eigenvalue. Euler's Theorem: Any orthogonal linear transformation of $\mathbb{R}^3$ that preserves orientation is a rotation around some axis.)

2. Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

3. Prove that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

4. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x)\,dx \right| \le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)|.$$

5. Recall integration by parts:

$$\int f\ dg = fg - \int g\ df.$$

Substitute $f(x) = 1/x$, $g(x) = x$, and manipulate, to get

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

Hence $0 = 1$. What has gone wrong?

6. Prove Bernoulli's Inequality: $(1 + x)^a \ge 1 + ax$ for $x > -1$ and $a \ge 1$, with equality when $x = 0$.

7. What happens if you put a random positive number in your (infinite precision) calculator, and repeatedly hit the sequence of buttons "$1/x$", "+" and "1"? (In other words, what happens if you iterate $x \mapsto 1/x + 1$?)

8. Prove that

$$(a) \qquad 2=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}}$$

$$(b) \qquad 3=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\dots}}}}}$$

9. Is $\sqrt{2}$ the limit of a sequence of numbers of the form

$$\sqrt[3]{n}-\sqrt[3]{m}, \quad n,m=0,1,2,\dots?$$

10. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right)=\frac{n^2}{n^2+1}, \qquad n=1,2,3,\dots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k=1,2,3,\dots$.