Problem Solving Seminar. Worksheet 6. Invariants and Games.

There is a little change in rules this time. When we discuss problems in class, we tend to concentrate on the most important ideas behind the proof and almost never give a complete argument with all the details. But when you write a written test (like the Putnam exam), it is of course very important to persuade the grader that nothing is missing from your proof. So I want to take a look at your proof-writing skills. Pick one of the problems that you solved and write down its complete solution. Don't take the easiest problem: take the hardest and the messiest one you can solve. There will be no grades but I promise a lot of red ink!

In some of these problems in this worksheet you will deal with a process (for example a game) that involves many steps. One common idea is to try to find a quantity (invariant) that does not change (or changes in some simple predictable way, for example increases) as the process develops. Common invariants include parity, total number, total length, etc.

1. Suppose you have an $m \times n$ chocolate bar. What is the minimum number of times you must break it in order to get 1×1 squares?

2. Suppose we have two piles of stones. Two people alternatively pick any number of stones, but just from one pile. The person who takes the last stone wins. What is the winning strategy?

3. (a) If you remove opposite corner squares from the chessboard, can you tile the remainder with 31 (2×1) dominoes? (b) Can a 10×10 square be tiled with 4×1 "quadrominoes"?

4. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{2008} minutes, could the room contain $3^{1000} + 2$ people?

5. Suppose 2007 red points and 2007 blue points are given in the plane, no three collinear (and pairwise distinct). Show that one can match up the red points and the blue points so that no two of the corresponding line segments intersect.

6. A collection of n beetles, each black or white in color, is arranged in a line. On each move, a black beetle turns pink, emitting a chemical which causes its immediate neighbors to switch from black to white, or white to black (as appropriate). Already pink bugs are not affected. Under what starting conditions is it possible for all n bugs to turn pink?

7 (This beautiful problem is attributed to the Fields medalist Maxim Kontsevich). Imagine a checkerboard that is infinite to the right and upward. You play the following "game of life". At the beginning, you have just 3 pieces (see the picture). At each step, you are allowed to remove one piece but then you have to put two new pieces: one to the right and one above of the original piece (see the picture). You are not

allowed to put two pieces in the same square. Is it possible to come up with a sequence of steps that would result in freeing all three originally occupied squares?

8. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)