

Problem Solving Seminar. Worksheet 2. Pigeonhole (or Dirichlet) principle.

1. Show that if there are 30 people at a party, then two of them know the same number of people (among those present).
2. A *repunit* is a number that contains only “ones” in its decimal expression, for example: 11111111. Prove that one can find a repunit divisible by 1973.
3. Five points lie in an equilateral triangle of size 1. Show that two of the points lie no farther than $1/2$ apart. Can the “ $1/2$ ” be replaced by anything smaller?
4. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist 2 whose midpoint is also a lattice point.
5. Somebody draws 9 straight lines on the plane and finds out that each of them cuts a fixed square into two quadrilateral pieces with areas in ratio 2 : 1. Prove that at least 3 of these lines have a common point.
6. Prove that in any group of six people there are either three mutual friends or three mutual strangers.
7. Astronomers have discovered that there are exactly 57 sunspots on the Sun and each of them is a circular region that covers less than half of the Sun’s surface. Suppose that sunspots don’t overlap (and don’t touch). Prove that there exists a pair of opposite points on the Sun that are not covered by sunspots.
8. In a hotly contested election year, each Senator has slapped the face of one other Senator. Can you form a committee of 34 Senators none of whom has slapped another committee member? Reminder: there are 100 senators in the US Senate.
9. Consider integers $1, 2, 3, \dots, 2n$ and pick more than n of them. Show that regardless of the choice made, one can find two integers picked such that one divides another.
10. Problem A3 from Putnam 2006.
11. Problem B2 from Putnam 2006.