Semi-orthogonal decompositions of moduli spaces

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- C smooth proj curve $g \geq 2$
- N moduli of stable v.b. E on C $\quad \mathrm{rk}\, E = 2$
  fixed $\det E \simeq \Lambda$ of odd degree d
  stable $\Leftrightarrow$ every line subbundle $L \subset E$ has $\deg L < d/2$

- Th (T.-Torres) $D^b(N)$ has an SOD with blocks $D^b(\mathrm{Sym}^i C)$ (2 blocks for $i = 0, \dots, g-2$; 1 block for $i = g-1$)
  embedded by explicit Fourier–Mukai functors
  and, possibly, an orthogonal "phantom" block
  (conjecturally zero)

- $g = 2 \Rightarrow N = Q_1 \cap Q_2 \subset \mathbb{P}^5$ (Newstead)
  SOD proved by Bondal-Orlov

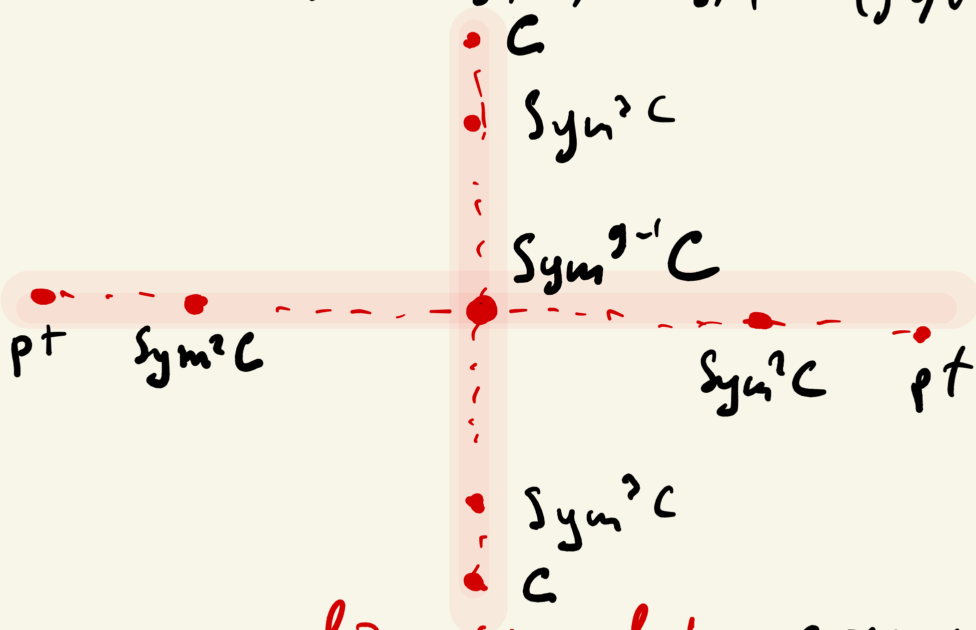
- C hyperelliptic $\Rightarrow N = \mathrm{Fano}(\mathbb{P}^{g-2} \subset Q_1 \cap Q_2 \subset \mathbb{P}^{2g+1})$
  (Desale-Ramanan)
  Fonarev-Kuznetsov: $D^b(N) = \langle D^b(C)^{\perp}, D^b(C) \rangle$ in this case
- Narasimhan: true for all curves
  (paper 1: $g \geq 4$; paper 2: $g = 3$) and introduced a general conjecture (independently by Belmans, Galkin, Mukhopadyay)

- Lee-Narasimhan: $D^b(N)$ contains $D^b(\mathrm{Sym}^2 C)$ as a semi-orthogonal block if $g \geq 16$ & C not hyperelliptic
- After our work has appeared, a different approach was proposed by Hu-Yan

• The shape of the S.O.D. is compatible with Dubrovin conjectures due to

Muñoz $c_1(N) \curvearrowright QH^*(N)$ eigenvalues 8λ

with $\lambda = (1-g), (2-g)i, (3-g), \dots (g-2)i, (g-1)$



dimension of the eigenspace

$\|$

dimension of cohomology of $\mathrm{Sym}^{g-1-|\lambda|}C$

Blocks of the SOD are organized into four "megablocks", one for each side of this Cruz del Sur with blocks within each megablock mutually orthogonal

Preliminaries on $D^b(X)$

- X smooth proj. variety $/\mathbb{C}$
- $D^b(X) = \{$bounded complexes of coherent sheaves$\}$

quasi-isomorphic complexes are isomorphic in $D^b(X)$

- Every object is isomorphic to a bounded complex of vector bundles

- **Example** $\mathcal{F}/X$ vector bundle of rank r

$Y \subset X$ zero locus of $s \in \Gamma(X, \mathcal{F})$

$$\operatorname{codim}_X Y = r \Rightarrow \mathcal{O}_Y \simeq \underbrace{[\Lambda^r \mathcal{F}^* \xrightarrow{s} \dots \xrightarrow{s} \mathcal{F}^* \xrightarrow{s} \mathcal{O}_X]}_{\text{Koszul complex}} \text{ in } D^b(X)$$

- $D^b(X)$ is not an abelian but a triangulated category:

$X \xrightarrow{f} Y$ uniquely $\rightsquigarrow$ exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$

via the mapping cone construction — shift functor

- Classical functors of abelian categories are (often) not exact but their derived functors are triangulated

- $\Gamma : \mathrm{Coh}\, X \to \mathrm{Vect}_{\mathbb{C}}$ $\quad R\Gamma : D^b(X) \to D^b(\mathrm{Vect}_{\mathbb{C}})$

$\mathcal{F}$ v.b. $\quad R\Gamma(\mathcal{F}) = 0 \iff H^i(X, \mathcal{F}) = 0 \ \forall i$

$\iff \mathcal{F}$ is acyclic

• **Preliminaries on SODs**

• **Def** A semi-orthogonal decomposition

SOD $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ (1) $R\mathrm{Hom}(B,A) = 0$ $\quad A \in \mathcal{A}, B \in \mathcal{B}$

$\Leftrightarrow \mathrm{Ext}^i(B,A) = 0 \;\; \forall i$

$= \mathrm{Hom}(B, A[i])$

(2) $\forall T \in \mathcal{T}$ $\exists$ exact triangle $B \to T \to A \to B[1]$

• Triangulated subcategories appearing as semi-orthogonal blocks are called

admissible $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{T} \Leftrightarrow$ (1) i is fully-faithful

(2) admits adjoint projection functors $\mathcal{T} \to \mathcal{A}$

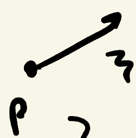
(2) is often automatic in geometric & rep.-theoretic contexts

• "Most" functors are not f.f.: $\{p\} \overset{i}{\hookrightarrow} X$ skyscraper sheaf

$Ri_* : D^b(p) = D^b(\mathrm{Vect}_{\mathbb{C}}) \to D^b(X)$, $\quad \mathbb{C} \mapsto \mathcal{O}_p$

$R\mathrm{Hom}(\mathbb{C},\mathbb{C}) = \mathbb{C}$ but $R\mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_p) = \Lambda^\bullet T_p$!

Tangent vector



$\mathcal{O}_X \longrightarrow \mathcal{O}_z$

$f \mapsto (f(p), f'_z(p))$

extension $0 \to \mathcal{O}_{pt} \to \mathcal{O}_z \to \mathcal{O}_{pt} \to 0$

So the point is rigid on its own but moves in X

- F.f. $D^b(pt) \to D^b(X)$ should send $\mathbb{C}$ to a "rigid" v.b. (or complex) $\mathcal{E}$ called exceptional

$R\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}$

$\Rightarrow \langle \mathcal{E} \rangle \subset D^b(X)$ admissible and $D^b(X) = \langle \underbrace{\mathcal{E}^\perp}_{\{F : R\mathrm{Hom}(\mathcal{E},F)=0\}}, \underbrace{\mathcal{E}}_{\langle \mathcal{E} \rangle} \rangle$

- Ex $R\Gamma(\mathcal{O}_X) = \mathbb{C} \Leftrightarrow \mathcal{O}_X$ is an exceptional line bundle

$\Leftrightarrow D^b(X) = \langle \mathcal{O}_X^\perp, \mathcal{O}_X \rangle$

- This holds for rational & Fano varieties, also for example, Enriques, Dolgachev, Barlow surfaces
- Relative Case: $f: X \to Y$, $Rf_*\mathcal{O}_X = \mathcal{O}_Y$
(e.g. f is birational or a projective bundle)

$\Rightarrow D^b(X) = \langle \underbrace{D^b(Y)^\perp}_{\{F \in D^b(X) \,:\, Rf_* F = 0\}}, \underbrace{D^b(Y)}_{Lf^* D^b(Y)} \rangle$

$\{F \in D^b(X) : Rf_* F = 0\}$ hard to describe unless $X = Bl_Z Y$ or f is a projective bundle (Orlov Theorems)

- Iterating SOD gives $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$
- Ex $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$

Definition full exceptional collection

- Ideally, each block $\mathcal{A}_i$ is indecomposable?
(Warning: no Jordan-Hölder property in general)

- Conjecturally, K_X nef & effective $\Rightarrow D^b(X)$ indecomposable

- The strongest known result is a recent theorem (Lin): $\bigcap_{L \in \mathrm{Pic}^0 X} \mathrm{Bs}|\omega_X \otimes L|$ finite $\Rightarrow D^b(X)$ indecomposable

- Ex: $D^b(\mathrm{Sym}^i C)$ indecomposable $\forall i = 0, \dots, g-1$
$\Rightarrow$ our decomposition of $D^b(N)$ is maximal

- Abel-Jacobi $\mathrm{Sym}^g C \to \mathrm{Pic}^g C$ birational $\Rightarrow$
$D^b(\mathrm{Sym}^g C) = \langle D^b(\mathrm{Pic}^g C)^\perp, D^b(\mathrm{Pic}^g C) \rangle$

CY $\Rightarrow$ indecomposable

AJ is not an iso over $\{ L \in \mathrm{Pic}^g C : h^0(L) \geq 2$
$\Leftrightarrow h^0(\omega_X^* \otimes L) \geq 1 \}$

This locus is resolved by $\mathrm{Sym}^{g-2} C$

Remarkably, $D^b(\mathrm{Pic}^g C)^\perp \simeq D^b(\mathrm{Sym}^{g-2} C)$ (Toda)

- Fourier-Mukai transforms:
  Let $\mathcal{E}$ be a v.b. on $Y \times X$
  $FM_{\mathcal{E}} : D^bY \to D^bX \qquad \mathcal{F} \mapsto R\pi_{2*}(L\pi_1^*\mathcal{F} \otimes_L \mathcal{E})$
- $FM_{\mathcal{E}}$ f.f. $\Rightarrow D^b(X) = \langle D^b(Y)^\perp, D^b(Y) \rangle$ via $FM_{\mathcal{E}}$
- V.b. $\mathcal{E}/Y\times X$ with a f.f. FM functor is a generalization of an exceptional v.b. (case $Y = pt$)
- Bondal-Orlov criterion

  $\forall y \in Y \qquad \mathcal{E}_y = \mathcal{E}|_{\{y\}\times X} = FM_{\mathcal{E}}(\mathcal{O}_y)$ evaluation v.b. on X

  $FM_{\mathcal{E}}$ fully faithful $\Leftrightarrow$

  (1) $\mathrm{Ext}^i(\mathcal{E}_y, \mathcal{E}_{y'}) = 0$, $\forall y \neq y'$, $\forall i$

  (2) $\mathrm{Hom}(\mathcal{E}_y, \mathcal{E}_y) = \mathbb{C}$

  (3) $\mathrm{Ext}^i(\mathcal{E}_y, \mathcal{E}_y) = 0$ for $i > \dim Y$
- Ex. In our case we have

  $\mathcal{E}$ a Poincare vector bundle on $C \times N$:

  $\mathcal{E}|_{C\times\{x\}} = E$ v.b. on C given by $x \in N$

  $\forall p \in C \qquad \mathcal{E}_p = \mathcal{E}|_{p\times N}$ evaluation v.b. on N

  $\det \mathcal{E}_p = \Theta$ ample generator of $\mathrm{Pic}\,N = \mathbb{Z}$ (N is index 2 Fano)
- Th (Narasimhan) $FM_{\mathcal{E}}$ is fully-faithful

To illustrate our approach let's reprove this using windows into derived categories & stable pairs

- Moduli of stable pairs (following Thaddeus)

$$M = \{(F,\varphi) : \varphi \in H^0(C,F),\ \mathrm{rk}\,F = 2,\ \det F = \Lambda\} \qquad d = \deg \Lambda > 0$$
$\varphi \neq 0$

$\forall$ line subbundle $L \subset F$, $\deg L \leq \begin{cases} \frac{d}{2} - \sigma & \varphi \in H^0(L) \\ \frac{d}{2} + \sigma & \varphi \notin H^0(L) \end{cases}$

$\sigma > 0$ is a stability parameter

- No strictly s.s. pairs $\Rightarrow$ M is smooth, $\dim M = d+g-2$
  a universal v.b. $\mathcal{F}$ on $C \times M$
  with a universal section Φ.
- Varying σ gives a diagram of flips

$$\begin{tikzcd} E \subset \mathrm{Bl}_C \mathbb{P}^{d+g-2} \arrow[d, "p"] \arrow[r, phantom, "="] & M_1 \arrow[r, dashed] \arrow[d] & M_2 \arrow[r, dashed] & \cdots \arrow[r, dashed] & M_{\lfloor \frac{d-1}{2} \rfloor} \arrow[d, "\xi"] & (F,\varphi) \arrow[d, mapsto] \\ C \arrow[r, hook, "|\omega_C \otimes \Lambda|"] & \mathbb{P}^{d+g-2} & & & N & F \end{tikzcd}$$

- For pairs $(F,\varphi) \in M_i$, $\varphi \in H^0(F)$ has at most i zeros.
- We need all degrees, including even, for inductive purposes but here I will assume that $d = 2g-1 \Rightarrow \xi$ is birational
- $D^b(M_1) = D^b(\mathrm{Bl}_C \mathbb{P}^{3g-3}) = \langle p^* D^b(C)(-i),\ D^b(\mathbb{P}^{3g-3}) \rangle$
  $i = 1, \ldots, 3g-5$ (Orlov theorem)

torsion subcategories supported on $E \hookrightarrow M_1$.
Need to relate to the universal vector bundle $\mathcal{F}$.

- $E = \{0 \to \mathcal{O}(p) \to F \to \Lambda(-p) \to 0\} = \{(F,\varphi) : \varphi \text{ has a zero}\}$ (unique!)

$\varphi \quad p \in C$

- The universal section vanishes on

$$\begin{array}{ccc} Z(\phi) = \{(p, F, \varphi) : \varphi(p) = 0\} & \xhookrightarrow[\text{codim } 2]{j} & C \times M_1 \\ \wr\| & & \downarrow pr_2 \\ E & \xhookrightarrow{\text{codim } 1} & M_1 \end{array}$$

(Koszul cpx)

$\Rightarrow j_* \mathcal{O}_E \simeq [\Lambda^* \to \mathcal{F}^* \to \mathcal{O}]$ in $D^b(C \times M_1)$

$\Lambda = \Lambda \otimes \Lambda_M$ (Λ_M is defined on any M_i)

$\forall p \in C$, $\mathcal{O}_{E_p} \simeq [\Lambda_M^* \to \mathcal{F}_p^* \to \mathcal{O}]$ in $D^b(M_1)$

$\mathbb{P}^{3g-5} \hookrightarrow M_1$

- $R\mathrm{Hom}(\mathcal{O}_{E_p}, \mathcal{O}_{E_{p'}}) = \begin{cases} 0 & p \neq p' \\ \mathbb{C} \oplus \mathbb{C}[-1] & p = p' \end{cases}$ ($\star$)

(easy calculation $\Leftarrow$ Orlov blow-up theorem)

- $\Rightarrow$ (modulo simple cohomological calc)

$R\mathrm{Hom}(\mathcal{F}_p, \mathcal{F}_{p'}) = $ ($\star$)

$\Leftrightarrow FM_{\mathcal{F}} : D^b(C) \hookrightarrow D^b(M_1)$ is f.f. !

• Next: $FM_{\mathcal{F}} : D^b(C) \hookrightarrow D^b(M_i)\ \forall i = 1, \dots, g-1$

$$\begin{tikzcd}[column sep=small]
 & & & PGL \curvearrowright X & & & \\
 & & X/\!\!/_+ PGL & \text{VGIT} & X/\!\!/_- PGL & & \\
\mathbb{P}W_i^- \arrow[rrrd, "\mathbb{P}^{i-1}\text{-bundle}"'] & \subset & M_{i-1} \arrow[u, phantom, "\shortparallel"] \arrow[rd] & & M_i \arrow[u, phantom, "\shortparallel"] \arrow[ld] & \supset & \mathbb{P}W_i^+ \arrow[llld, "\mathbb{P}^{3g-3-2i}\text{-bundle}"] \\
 & & & \text{singular } \mathcal{M}_{i-1} \supset \mathrm{Sym}^i C & & &
\end{tikzcd}$$

• Theory of windows of Teleman and Halpern-Leistner gives the windows inequality: If $B \in D^b_{PGL}(X)$ has

$$-(3g-3-2i) \leq \text{weights}(B) \leq i-1$$

$$R\Gamma(M_i, B) \overset{\Downarrow}{=} R\Gamma^{PGL}(X, B) \overset{\Downarrow}{=} R\Gamma(M_{i-1}, B)$$

• $\text{Weight}(\mathcal{F}_p) = \{0, -1\} \Rightarrow FM_{\mathcal{F}} : D^b(C) \hookrightarrow D^b(M_i)$ is f.f. $\forall i = 1, \dots, g-1$

Bondal-Orlov

• Finally, we can reprove Narasimhan's Th:

$L\zeta^* : D^b(N) \hookrightarrow D^b(M_{g-1})$ is f.f.

$\zeta^* \mathcal{E} = \mathcal{F}(Z) \Rightarrow FM_{\mathcal{E}} : D^b(C) \hookrightarrow D^b(N)$ f.f.

divisor on M_{g-1}

• Tensor v.b.

$C^\alpha \xrightarrow{\tau} \mathrm{Sym}^\alpha C \qquad S_\alpha$-quotient

CM equidim smooth $\Rightarrow$ τ is flat

any scheme

• $\Rightarrow$ every base change $C^\alpha \times M \xrightarrow{\tau} \mathrm{Sym}^\alpha C \times M$ is flat

• Given a v.b. $\mathcal{E}$ on $C \times M$

$\pi_1^*\mathcal{E} \otimes \cdots \otimes \pi_\alpha^*\mathcal{E}$ S_α-equiv v.b on $C^\alpha \times M$

• τ is flat $\Rightarrow$ $\mathcal{E}^{\otimes\alpha} := \tau_*^{S_\alpha}(\pi_1^*\mathcal{E} \otimes \cdots \otimes \pi_\alpha^*\mathcal{E})$

a v.b on $\mathrm{Sym}^\alpha C \times M$ and construction is functional in M

• The same construction with $\oplus$ gives tautological v.b.

• Th (T- Torres) Let $\mathcal{E}$ be the Poincare v.b. on $C \times N$.

$\Rightarrow FM_{\mathcal{E}^{\otimes\alpha}} : D^b(\mathrm{Sym}^\alpha C) \hookrightarrow D^b(N)$ fully faithful

$\alpha = 0, 1, \dots, g-1$

$$
\begin{aligned}
D^b(N) = \langle\, & \Theta^\wedge \otimes G_0,\ (\Theta^\wedge)^2 \otimes G_2,\ (\Theta^\wedge)^3 \otimes G_4, \dots,\\
& \Theta^\wedge \otimes G_1,\ (\Theta^\wedge)^2 \otimes G_3,\ (\Theta^\wedge)^3 \otimes G_5, \dots,\\
& G_0,\ \Theta^\wedge \otimes G_2,\ (\Theta^\wedge)^2 \otimes G_4, \dots,\\
& G,\ \Theta^\wedge \otimes G_3,\ (\Theta^\wedge)^2 \otimes G_5, \dots \,\rangle
\end{aligned}
$$

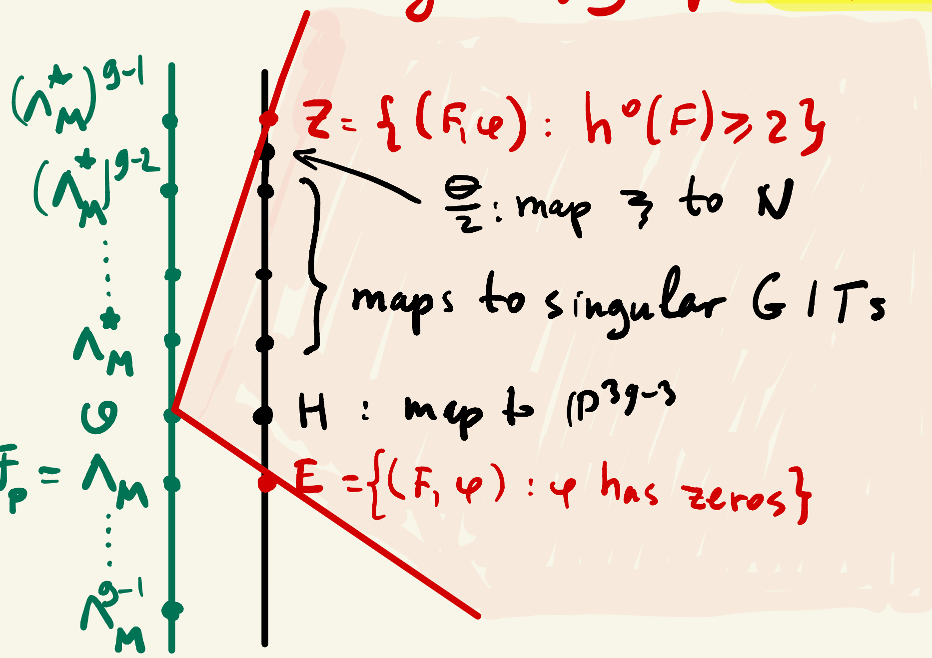
G_{g-1} appears once, G_i for $i<g-1$ twice,

blocks within each line are orthogonal

• To apply the Bondal-Orlov criterion, we need to describe evaluation v.b. $\mathcal{E}_D^{\otimes\alpha}$ on M

for $D = p_1 + \dots + p_\alpha \in \mathrm{Sym}^\alpha C$

- $p_i \neq p_j \Rightarrow \mathcal{E}_D^{\otimes \alpha} = \mathcal{E}_{p_1} \otimes \dots \otimes \mathcal{E}_{p_\alpha}$ $\quad \mathcal{E}_{p_i} = \mathcal{E}|_{\{p_i\} \times M}$
- **Lemma** $\forall D$, $\mathcal{E}_D^{\otimes \alpha}$ is a canonical (functorial in M) deformation of $\mathcal{E}_{p_1} \otimes \dots \otimes \mathcal{E}_{p_\alpha}$
- This allows us to use semi-continuity arguments to prove cohomology vanishing involving $\mathcal{E}_D^{\otimes \alpha}$
- Working on the moduli space of stable pairs and combining windows with syzygies techniques allows to organize sophisticated induction which depends on a few key cases, including very non-trivial acyclicity of line bundles



- Global sections of l.b. in $\mathrm{Eff}(M_i)$ were computed to prove Verlinde formula for $h^0(N, \theta^k)$ (Thaddeus)

Tools: Kodaira vanishing & Grothendieck-Riemann-Roch

- To deal with $\mathrm{Sym}^k C$, we needed

$$R\Gamma(M_{g-1}, \Lambda^k) = 0 \text{ for } |k| \leq g-1,\ k \neq 0$$

Easy for $k>0$ (prove on $M_1 = \mathrm{Bl}_C \mathbb{P}^{3g-3}$ + windows)

Hard for $k<0$ (needs to be checked on M_{-k})

- Use "Thaddeus package" + Hecke correspondences + geometry of divisor E after flips & more general

$$\{(F,\varphi) : \text{zero locus of } \varphi \text{ has degree} \geq r\} \subset M_i,$$

which is a better-behaved locus than the classical locus

$$\{\mathcal{E} : E \text{ contains a l.b. } L \text{ of degree} \geq r\} \subset N$$

- How do "torsion" SOD of M_i and "Poincaré bundle" SOD of N compare?

- Applying the windows theorem of Halpern-Leistner & Ballard-Favero-Katzarkov $\Rightarrow$

SOD $D^b(M_i) = \langle D^b(M_{i-1}), \underbrace{D^b(\mathrm{Sym}^i C)}_{3g-2-3i \text{ copies}} \rangle$

supported on $\mathbb{P}W_i^+$

- Combining these SODs gives

torsion subcategories

$$D^b(M_{g-1}) = \langle D^b(pt), D^b(C), D^b(\mathrm{Sym}^2 C), \dots, D^b(\mathrm{Sym}^{g-1} C) \rangle$$

$$\underbrace{\quad 3g-2 \qquad 3g-5 \qquad 3g-8 \qquad \dots \qquad 1 \quad}_{\text{number of copies}}$$

$L\zeta^*$

$$\overbrace{\quad 2 \qquad 2 \qquad 2 \qquad \dots \qquad 1 \quad}$$

$$D^b(N) = \langle D^b(pt), D^b(C), D^b(\mathrm{Sym}^2 C), \dots, D^b(\mathrm{Sym}^{g-1} C) \rangle$$

tensor vector bundles

- These SOD's are incompatible via $L\zeta^*$.
- $\zeta : M_{g-1} \longrightarrow N$

$$\begin{tikzcd} M_{g-1} \arrow[r, "\zeta"] & N \\ Z \arrow[u, phantom, "\cup"] \arrow[r] & B \arrow[u, phantom, "\cup"] \\ & M_{g-2}(2g-3) \arrow[u, "\text{resolution of singularities}"'] \end{tikzcd}$$

B: codim 2 Brill-Noether locus

- Koseki-Toda: $D^b(N)^\perp \simeq D^b(M_{g-2}(2g-3))$