

CATEGORICAL ASPECTS OF THE KOLLÁR–SHEPHERD-BARRON CORRESPONDENCE

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To Professor Yujiro Kawamata on the occasion of his 70th birthday

ABSTRACT. It is well known that a 2-dimensional cyclic quotient singularity \overline{W} has the same singularity category as a finite dimensional associative algebra \overline{R} introduced by Kalck and Karmazyn. We study the deformations of the algebra \overline{R} induced by the deformations of the surface \overline{W} to a smooth surface. We show that they are Morita-equivalent to path algebras \hat{R} of acyclic quivers for general smoothings within each irreducible component of the versal deformation space of \overline{W} (as described by Kollár and Shepherd-Barron). Furthermore, \hat{R} is semi-simple if and only if the smoothing is \mathbb{Q} -Gorenstein (one direction is due to Kawamata). We provide many applications. For example, we describe strong exceptional collections of length 10 on all Dolgachev surfaces and classify admissible embeddings of derived categories of quivers into derived categories of rational surfaces.

1. INTRODUCTION

In the classical papers from 1974, Pinkham [P2] and Gabriel [G] studied deformations of varieties with \mathbb{G}_m -action and finite dimensional associative algebras, respectively. These papers included famous examples of reducible versal deformation spaces of the cone \overline{W} over a rational normal curve in \mathbb{P}^4 (Figure 1, left) and of the 4-dimensional algebra $\overline{R} = \mathbb{C}[x, y, z]/(x, y, z)^2$ (Figure 1, right), respectively.

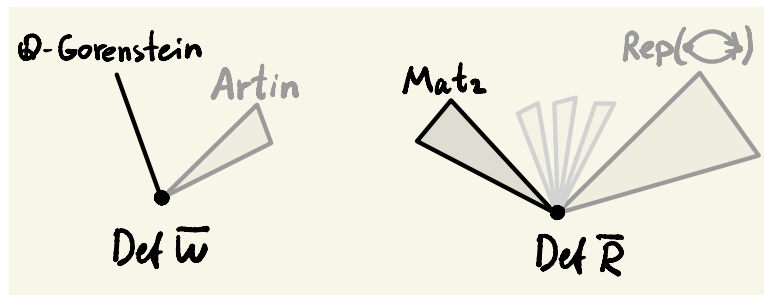


FIGURE 1. Versal deformation spaces of \overline{W} and \overline{R}

We observe that there is a remarkable embedding $\psi : \text{Def } \overline{W} \rightarrow \text{Def } \overline{R}$. The Artin component of $\text{Def } \overline{W}$ (that parametrizes deformations induced by deformations of the minimal resolution of \overline{W}) maps to the deformations of \overline{R} to the path algebra of the Kronecker quiver, whereas the \mathbb{Q} -Gorenstein component (that parametrizes deformations of \overline{W} such that the relative canonical divisor is \mathbb{Q} -Cartier) maps to the deformations of \overline{R} to the matrix algebra $\text{Mat}_2(\mathbb{C})$. We will construct the map ψ for all cyclic quotient singularities \overline{W} and show that ψ sends each irreducible component of $\text{Def } \overline{W}$ to a uniquely determined irreducible component of $\text{Def } \overline{R}$.

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Throughout the paper we fix a cyclic quotient singularity (c.q.s.) of type $\frac{1}{\Delta}(1, \Omega)$. This is a surface germ $P \in \overline{W}$ étale-locally isomorphic to the germ $(0 \in \mathbb{C}^2/\mu_\Delta)$, where a generator $\zeta \in \mu_\Delta$ acts on \mathbb{C}^2 with weights ζ, ζ^Ω for coprime $0 < \Omega < \Delta$. Equivalently, a c.q.s. is a 2-dimensional toroidal singularity, where the toric boundary divisors are the images of the coordinate axes in \mathbb{C}^2 . The exceptional divisor of the minimal resolution of \overline{W} is a chain of rational curves with self-intersections $-e_1, \dots, -e_l$ determined by the Hirzebruch–Jung continued fraction of Δ/Ω .

It is possible to embed the surface germ $P \in \overline{W}$ into a projective surface \overline{W} that satisfies several technical assumptions 1.10. The surface \overline{W} carries a vector bundle \overline{F} of rank Δ , see Section 4 or [KKS, Prop. 6.7]. We call \overline{F} a *Kawamata vector bundle*. It is the maximal iterated extension [K2] of the ideal sheaf $\mathcal{O}_{\overline{W}}(-\overline{A})$ of one of the toric boundaries of \overline{W} . Concretely, we choose \overline{A} so that its proper transform in the minimal resolution of \overline{W} intersects the exceptional curve of self intersection $-e_1$.

The *Kalck–Karmazyn algebra* \overline{R} of the algebra $R := \mathbb{C}[[x, y]]^{\mu_\Delta} = \hat{\mathcal{O}}_{P, \overline{W}}$ is a Δ -dimensional associative algebra which induces an equivalence between the singularity categories of R and \overline{R} [KK]. It was originally introduced by explicit generators and relations. We will use its description as the endomorphism algebra

$$\overline{R} = \text{End}_{\overline{W}}(\overline{F})$$

of the Kawamata bundle. It is non-commutative unless $\Omega = 1$ or $\Delta - 1$. For example, if \overline{W} the cone over the rational normal curve in \mathbb{P}^4 then $\overline{R} = \mathbb{C}\langle x, y, z \rangle / (x, y, z)^2$. In this case \overline{W} is of type $\frac{1}{4}(1, 1)$. The first non-commutative Kalck–Karmazyn algebra is $\overline{R} = \mathbb{C}\langle x, y \rangle / (x^2, y^3, xy, y^2x)$ for the singularity \overline{W} of type $\frac{1}{5}(1, 2)$.

Our results can be informally summarized as follows.

Corollary 1.1 (*Theorem 1.12*). *There is a natural map of versal deformation spaces*

$$\psi : \text{Def}_{P \in \overline{W}} \rightarrow \text{Def}_{\overline{R}}.$$

A general deformation of \overline{W} within a given irreducible component of $\text{Def}_{P \in \overline{W}}$ induces a deformation of \overline{R} to a Δ -dimensional hereditary algebra, which is Morita-equivalent to the path algebra \hat{R} of an acyclic quiver without relations.

Finite-dimensional hereditary algebras are rigid, and each of them corresponds to a dense open subset of an irreducible component of $\text{Def}_{\overline{R}}$ [G]. It follows that ψ induces a map between the sets of irreducible components of $\text{Def}_{P \in \overline{W}}$ and $\text{Def}_{\overline{R}}$.

In order to study how the Kawamata vector bundle \overline{F} and the Kalck–Karmazyn algebra \overline{R} deform under deformations of \overline{W} to a smooth surface Y , we use an interplay of two techniques, one coming from the study of semi-orthogonal decompositions (s.o.d.) of derived categories and another from birational geometry. By [KKS], the surface \overline{W} admits a s.o.d. $D^b(\overline{W}) = \langle \mathcal{A}^{\overline{W}}, \mathcal{B}^{\overline{W}} \rangle$, where $\mathcal{A}^{\overline{W}} \simeq D^b(\overline{R}\text{-mod})$. We show that this s.o.d. deforms to $D^b(Y) = \langle \mathcal{A}^Y, \mathcal{B}^Y \rangle$, where $\mathcal{A}^Y \simeq D^b(\hat{R}\text{-mod})$.

Incidentally, this gives a large amount of admissible embeddings of derived categories $D^b(\hat{R}\text{-mod})$ of acyclic quivers without relations into derived categories of smooth projective surfaces Y (which one can choose to be rational). While Orlov proved [O] that the embedding always exists if $\dim Y$ is sufficiently large, there are strong restrictions in the case of surfaces. In fact, very few examples were known before our work. In particular, Belmans and Raedschelders [BR, Sect.4] ask whether there are bounds on the lengths of paths of realizable quivers and which acyclic quivers $Q_{a,b,c}$ with 3 vertices, where a, b, c are the number of arrows between them, are realizable. Our results show that lengths of paths are unbounded, and we have the following partial answer for the 3 vertices quiver’s question.

Corollary 1.2. (Prop. 6.11) *The quiver $Q_{a,b,c}$ is realizable by the algebra \hat{R} if and only if there exists an extremal P-resolution (see Definition 2.9) with Wahl singularities of indices a and b and with $\delta = c$. See Lemma 6.12 for the list of possible c for fixed values of a, b .*

Our second tool is birational geometry. Let \overline{W} be the total space of a smoothing of \overline{W} to Y over a smooth curve. Special fibers of small birational models $\mathcal{W} \rightarrow \overline{W}$ provide partial resolutions of singularities $W \rightarrow \overline{W}$ that can be deformed to Y via a \mathbb{Q} -Gorenstein deformation. For example, irreducible components of $\text{Def}_{P \in \overline{W}}$ are parametrized by P-resolutions $W^+ \rightarrow \overline{W}$ of Kollár and Shepherd-Barron [KSB]. The algebra \hat{R} is the endomorphism algebra of a strong exceptional collection of vector bundles associated with a smoothing of another partial resolution $W^- \rightarrow \overline{W}$, which we call the *N-resolution*. It is the negative analog of the P-resolution.

Geometric applications can be obtained by considering normal projective surfaces W with $p_g(W) = q(W) = 0$ which contain an N-resolution that contracts to some c.q.s. $P \in \overline{W}$. Assume in addition that \overline{W} is unobstructed in deformations, and that $\overline{W} \setminus \{P\}$ is simply-connected. These surfaces are abundant, and their smoothings Y could be: rational surfaces, Enriques surfaces, proper elliptic surfaces, and surfaces of general type (see e.g. [LP, HP, U]). In Section 8, we consider applications to the following proper elliptic surfaces.

Definition 1.3. A *Dolgachev surface* $D_{p,q}$ is a minimal elliptic fibration $Y \rightarrow \mathbb{P}^1$ with $H^2(Y, \mathcal{O}_Y) = H_1(Y, \mathbb{Z}) = 0$ and with exactly two multiple fibers of coprime multiplicities p and q . It is simply-connected and has Kodaira dimension 1.

In [CL], Lee and Cho construct an exceptional collection of maximum length 12 of line bundles on Dolgachev surfaces $D_{2,3}$. Other Dolgachev surfaces cannot have exceptional collections of length 12 even numerically by results of Vial [V]. On the other hand, our exceptional collections never have full length because they only categorify the Milnor fiber of the smoothing. Our results imply the following.

Corollary 1.4. (Theorem 8.4) *Let $p, q \geq 2$ be coprime integers. Dolgachev surfaces $D_{p,q}$ carry a strong exceptional collection $\overline{E}_9, \dots, \overline{E}_0$ associated with an N-resolution, where*

- (1) $\hat{R} = \text{End}(\overline{E}_9 \oplus \dots \oplus \overline{E}_0)$ is the endomorphism algebra of the quiver with vertices $\overline{P}_0, \dots, \overline{P}_9$ and with $pq - p - q$ arrows connecting each \overline{P}_i to \overline{P}_9 for $i = 0, \dots, 8$.
- (2) The semi-orthogonal complement of $\langle \overline{E}_9, \dots, \overline{E}_0 \rangle$ in $D^b(D_{p,q})$ has Mukai lattice \mathbb{Z}^2 with Euler pairing given by the Gram matrix $\begin{bmatrix} -1 & 3(pq - p - q) \\ 0 & -1 \end{bmatrix}$. This lattice has a full numerical exceptional collection if and only if $p = 3, q = 2$.

We now provide the definitions and the notation that will be used throughout this paper. We work with schemes of finite type over $\text{Spec } \mathbb{C}$. We use notation $\mathcal{W} \rightarrow B$ for a flat deformation of normal surfaces with the special fiber W and the total space \mathcal{W} . We also use notation $Y \rightsquigarrow W$ for a smoothing over a smooth curve germ $0 \in B$ with general fiber Y . A deformation over a smooth curve is called \mathbb{Q} -Gorenstein if $K_{\mathcal{W}}$ is \mathbb{Q} -Cartier. We refer to [H1, S. 3] for a general theory of \mathbb{Q} -Gorenstein deformations. Quotient singularities of dimension 2 admitting a \mathbb{Q} -Gorenstein smoothing are called T-singularities [KSB]. They are either Du Val singularities or c.q.s. of the form $\frac{1}{dn^2}(1, dna - 1)$ for $0 < a < n$ coprime [KSB, Prop. 3.10]. The special case is a *Wahl singularity* $\frac{1}{n^2}(1, na - 1)$.

Definition 1.5. A *c.q.s. surface* $(\Gamma_1 \cup \dots \cup \Gamma_r \subset W)$ is a surface germ that contains a chain of smooth projective rational curves $\Gamma_1, \dots, \Gamma_r$ that are toric boundary divisors at c.q.s. P_0, \dots, P_r , the surface is smooth elsewhere (we also allow P_i to be smooth points). We choose a toric boundary divisor germ Γ_0 at P_0 complementary to Γ_1 and Γ_{r+1} at P_r complementary to Γ_r . A *c.q.s. resolution* of a c.q.s. $P \in \overline{W}$ is a

c.q.s. surface that admits a contraction $(\Gamma_1 \cup \dots \cup \Gamma_r \subset W) \rightarrow (P \in \overline{W})$. A c.q.s. surface is called a *Wahl surface* if P_0, \dots, P_r are Wahl singularities. A *Wahl resolution* is a c.q.s. resolution $W \rightarrow \overline{W}$ such that W is a Wahl surface. In addition, we impose the following minimality assumption: a Wahl resolution W should admit a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W$ that blows down to a smoothing $Y \rightsquigarrow \overline{W}$.

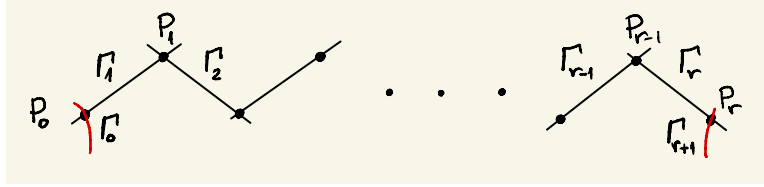


FIGURE 2. Chain of curves in a Wahl (or c.q.s.) surface W

Kollár and Shepherd-Barron proved that irreducible components \mathcal{C} of the reduced versal deformation space $\text{Def}_{P \in \overline{W}}$ are in a one-to-one correspondence with *P-resolutions* [KSB, Th. 3.9] of \overline{W} , i.e. c.q.s. resolutions $W \rightarrow \overline{W}$ with T-singularities and relatively ample canonical class. Concretely, a smoothing $Y \rightsquigarrow \overline{W}$ from \mathcal{C} with total space \overline{W} lifts to a smoothing $Y \rightsquigarrow W$ with total space \mathcal{W} given by the relative canonical model of \overline{W} . We will use an analogous one-to-one correspondence with *M-resolutions* of Behnke and Christophersen [BC]. An M-resolution is a Wahl resolution $W^+ \rightarrow \overline{W}$ such that K_{W^+} is relatively nef. The versal \mathbb{Q} -Gorenstein deformation space $\text{Def}_{\Gamma_1 \cup \dots \cup \Gamma_r \subset W^+}^{\mathbb{Q}G}$ of an M-resolution is smooth. Blowing down deformations [W] gives a map $\text{Def}_{\Gamma_1 \cup \dots \cup \Gamma_r \subset W^+}^{\mathbb{Q}G} \rightarrow \text{Def}_{P \in \overline{W}}$, which is a Galois covering of an irreducible component \mathcal{C} and the Galois group is a reflection group. In particular, \mathcal{C} is also smooth and every deformation in \mathcal{C} is the blow-down of a \mathbb{Q} -Gorenstein deformation in $\text{Def}_{\Gamma_1 \cup \dots \cup \Gamma_r \subset W^+}^{\mathbb{Q}G}$ after a finite base change.

Example 1.6. The *minimal resolution* of singularities of $P \in \overline{W}$ is an example of an M-resolution. The corresponding component of the versal deformation space $\text{Def}_{P \in \overline{W}}$ is called the *Artin component*. It parametrizes deformations of \overline{W} that admit a simultaneous resolution of singularities after a finite base change.

Notation 1.7. Every Wahl surface W , including an M-resolution W^+ , has the following numerical invariants n_i, a_i, δ_i . For $i = 0, \dots, r$, the Wahl singularity $P_i \in W$ has type $\frac{1}{n_i^2}(1, n_i a_i - 1)$, where the Hirzebruch-Jung continued fraction of $\frac{n_i^2}{n_i a_i - 1}$ goes in the direction from Γ_i to Γ_{i+1} . For smooth points, $n_i = a_i = 1$. For $i = 1, \dots, r$, let $\delta_i := n_{i-1} n_i |K_W \cdot \Gamma_i|$ (a non-negative integer).

Definition 1.8. Let W^+ be an M-resolution of a c.q.s. $P \in \overline{W}$ with invariants n_i, a_i, δ_i as in Notation 1.7. The corresponding *N-resolution* W^- is a Wahl resolution of $P \in \overline{W}$ with curves $\bar{\Gamma}_i$ and singularities \bar{P}_i of type $\frac{1}{\bar{n}_i^2}(1, \bar{n}_i \bar{a}_i - 1)$ such that $-K_{W^-}$ is relatively nef, i.e., $K_{W^-} \cdot \bar{\Gamma}_i \leq 0$ for $i = 1, \dots, r$, and

- (1) The singularity \bar{P}_r is the same as P_0 . Moreover, for every $i = 1, \dots, r$, the contraction of the chain $\bar{\Gamma}_{r-i+1} \cup \dots \cup \bar{\Gamma}_r \subset W^-$ is the same c.q.s. as the contraction of the chain $\Gamma_1 \cup \dots \cup \Gamma_i \subset W^+$. We denote that c.q.s. by $\frac{1}{\Delta_i}(1, \Omega_i)$.
- (2) $\bar{\delta}_{r-i+1} = \delta_i$ for $i = 1, \dots, r$.

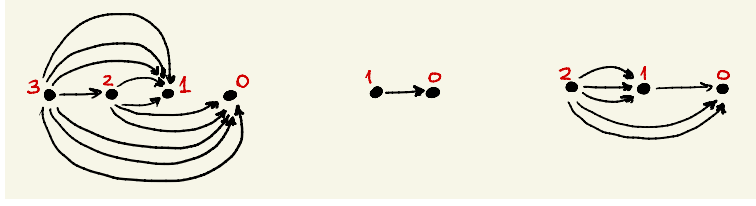
The N-resolution associated with the M-resolution exists and is unique (Section 2).

Example 1.9. By [KSB, Ex. 3.15], the c.q.s. $\frac{1}{19}(1, 7)$ admits three M-resolutions (for the notation see Section 2), where the first M-resolution is the minimal resolution:

$$(3) - (4) - (2) \quad \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} \right] \quad (3) - \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] - (2).$$

We list the corresponding N-resolutions and the quivers for the path algebra \hat{R} . All this information can be computed using the program [Z].

$$\left[\begin{pmatrix} 8 \\ 3 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 8 \\ 3 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] - (1) \quad \left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] \quad \left[\begin{pmatrix} 8 \\ 3 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} \right] - (1)$$



Assumption 1.10. For technical reasons, we need to compactify all surfaces, which requires imposing the following assumptions throughout the paper.

- (1) W is a normal, projective c.q.s. surface smooth outside of $\{P_0, \dots, P_r\}$. If $\Gamma_1 \cup \dots \cup \Gamma_r \subset W$ is a c.q.s. resolution of $P \in \overline{W}$, then the surface \overline{W} is determined as the contraction of the chain $\Gamma_1, \dots, \Gamma_r$ to the point $P \in \overline{W}$.
- (2) $H^1(W, \mathcal{O}_W) = H^2(W, \mathcal{O}_W) = 0$. Since c.q.s. are rational singularities, if W is a c.q.s. resolution of \overline{W} then (2) is equivalent to the same vanishing on \overline{W} or, equivalently, on the minimal resolution of \overline{W} . For example, \overline{W} can be a rational surface. Since rational singularities are Du Bois, (2) is equivalent to the same vanishing on any projective deformation Y of W .
- (3) There is a Weil divisor \bar{A} on \overline{W} that generates the local class group $\text{Cl}(P \in \overline{W})$. By Lemma 3.1, we can choose effective smooth divisors $\bar{A}, \tilde{A} \subset \overline{W}$ such that the germ $P \in (\bar{A} \cup \tilde{A}) \subset \overline{W}$ is étale-locally isomorphic to $0 \in (x=0) \cup (y=0) \subset \frac{1}{\Delta}(1, \Omega)$. Proper transforms Γ_0 (resp. Γ_{r+1}) of \bar{A} (resp. \tilde{A}) in a c.q.s. resolution W of \overline{W} intersect the chain $\Gamma_1 \cup \dots \cup \Gamma_r$ only at P_0 (resp. P_r), where they are equivalent to toric boundaries opposite to Γ_1 (resp. Γ_r) as in Figure 2.
- (4) $H^2(\overline{W}, T_{\overline{W}}) = 0$. By Lemma 3.2, there are no local-to-global obstructions to \mathbb{Q} -Gorenstein deformations of a Wahl resolution W of \overline{W} or the pair (W, Δ) where $\Delta = \Gamma_0 + \Gamma_1 + \dots + \Gamma_r + \Gamma_{r+1}$ if (2) and (3) also hold. A general example satisfying (4) is any surface \overline{W} such that $-K_{\overline{W}}$ is big [HP, Prop. 3.1].

Definition 1.11. Let $Y \rightsquigarrow W$ be a projective \mathbb{Q} -Gorenstein smoothing of a Wahl surface W over a smooth curve germ $0 \in B$ satisfying Assumption 1.10. By [H2], for each $P_i \in W$ and after shrinking B , we have an associated exceptional vector bundle E_i of rank n_i on Y , which we call a *Hacking vector bundle*. Bundles E_r, \dots, E_0 form a *Hacking exceptional collection* on Y , see Section 5.

Our main theorem is the following (see Section 6 for more detailed results).

Theorem 1.12. Let W^+ be an M-resolution of $P \in \overline{W}$ satisfying Assumption 1.10. Fix a projective \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W^+$ which is sufficiently general in its irreducible component of the versal deformation space of \overline{W} . This component also contains a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W^-$, where W^- is the N-resolution associated to W^+ .

- (1) Let $\bar{E}_r, \dots, \bar{E}_0$ be a Hacking exceptional collection on Y associated with the N-resolution W^- . This collection is strong: $\text{Ext}^k(\bar{E}_i, \bar{E}_j) = 0$ for $k > 0$ and $i > j$.
- (2) In contrast, let E_r, \dots, E_0 be a Hacking exceptional collection on Y associated with the M-resolution W^+ . Then we have $\text{Ext}^k(E_i, E_j) = 0$ for $k \neq 1$ and $i > j$.
- (3) For $i = 1, \dots, r$, we have $\text{Hom}(\bar{E}_{r+1-i}, \bar{E}_{r-i}) \simeq \text{Ext}^1(E_i, E_{i-1})^\vee \simeq \mathbb{C}^{\delta_i}$.
- (4) The Kawamata bundle \bar{F} on \bar{W} deforms to a vector bundle $F \simeq \bigoplus_{i=0}^r \bar{E}_i^{n_{r-i}}$ on Y . Since F has rank Δ , we note that $\Delta = n_0 \bar{n}_r + n_1 \bar{n}_{r-1} + \dots + n_r \bar{n}_0$.
- (5) The Kalck–Karmazyn algebra $\bar{R} = \text{End}(\bar{F})$ deforms to the algebra $\text{End}(F)$, which is hereditary and Morita-equivalent to the path algebra $\hat{R} = \text{End}(\bar{E}_r \oplus \dots \oplus \bar{E}_0)$.

Remark 1.13. The algebra \hat{R} is a path algebra of a quiver with vertices $\bar{E}_r, \dots, \bar{E}_0$ and with arrows connecting \bar{E}_i to \bar{E}_j for $i > j$ so that the total number of paths connecting \bar{E}_i to \bar{E}_j is equal to (see Lemma 6.9)

$$\text{hom}(\bar{E}_i, \bar{E}_j) = \bar{n}_j \bar{a}_i - \bar{n}_i \bar{a}_j = \bar{n}_i \bar{n}_j \left(\frac{\bar{\delta}_{j+1}}{\bar{n}_j \bar{n}_{j+1}} + \dots + \frac{\bar{\delta}_i}{\bar{n}_{i-1} \bar{n}_i} \right). \quad (1.1)$$

It follows from this formula that the quiver is connected unless the algebra is semisimple, in which case the N-resolution is the M-resolution of a T-singularity.

Remark 1.14. One of the ingredients of the proof is a result of Kawamata [K1], where Theorem 1.12 was proved in the case when $Y \rightsquigarrow \bar{W}$ is a \mathbb{Q} -Gorenstein smoothing of a cyclic T-singularity $\frac{1}{dn^2}(1, dna - 1)$ (see also [C]). In this case all $\delta_i = 0$, the M-resolution is equal to the N-resolution, Hacking vector bundles are pairwise orthogonal, and $\text{End}(F)$ is a direct sum of d copies of $\text{Mat}_n(\mathbb{C})$.

In Section 7 we illustrate these results in the simplest case of the Artin component, when $W^+ \rightarrow \bar{W}$ is a minimal resolution of singularities. In this case only, the exceptional collection E_r, \dots, E_0 on Y is a collection of line bundles, which is a deformation of an exceptional collection of line bundles on W^+ . However, while the latter has both Hom and Ext^1 in the forward direction, the former has Ext^1 only. As in the general case, the dual collection $\bar{E}_r, \dots, \bar{E}_0$ associated with the N-resolution $W^- \rightarrow \bar{W}$ is a strong exceptional collection of vector bundles.

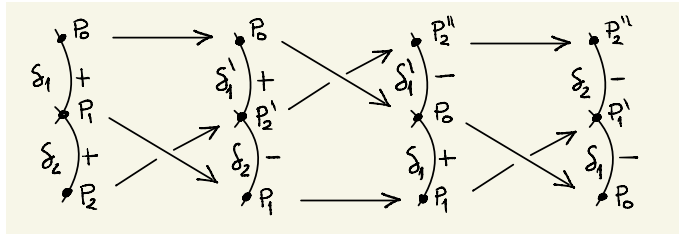


FIGURE 3. Braiding from W^+ to W^- for three Wahl singularities

Remark 1.15. Antiflips of [HTU] are the main tool in the proof of Theorem 1.12. We use them to produce \mathbb{Q} -Gorenstein smoothings $Y \rightsquigarrow W$ of different Wahl resolutions $W \rightarrow \bar{W}$ over different curve germs in a given irreducible component of the versal deformation space of \bar{W} , see Definition 3.3. Antiflips generate a “geometric” braid group action on the infinite set of all Wahl resolutions $W \rightarrow \bar{W}$ compatible with the “categorical” braid group action by mutations of exceptional collections on Y , see Theorem 3.9. After applying finitely many antiflips to the \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W^+$ of the M-resolution, we obtain a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W^-$ of the N-resolution associated to W^+ . Going back from W^-

to W^+ can be done over the same curve B . This decomposes a birational map of total spaces of deformations $\mathcal{W}^- \dashrightarrow \mathcal{W}^+$ into a sequence of $\frac{r(r+1)}{2}$ flips and flops.

We will also describe the derived category of the total space \mathcal{W} of a deformation. We recall that a *semi-orthogonal decomposition* (s.o.d) of a triangulated category \mathcal{T} is a sequence of full triangulated subcategories $\langle \mathcal{A}_0, \dots, \mathcal{A}_r \rangle$ satisfying two conditions: (1) $\text{Hom}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $j > i$, and (2) for every object $T \in \mathcal{T}$, there exist morphisms $0 = T_r \rightarrow \dots \rightarrow T_0 = T$ such that the cone A_i of $T_i \rightarrow T_{i-1}$ belongs to \mathcal{A}_i . The objects A_i are functorial in T , i.e. we have projection functors $\mathcal{T} \rightarrow \mathcal{A}_i$.

Theorem 1.16. *Let $Y \rightsquigarrow W$ be a projective \mathbb{Q} -Gorenstein smoothing of a Wahl surface W satisfying Assumption 1.10 (1), (2), (3). After possibly shrinking B , $D^b(\mathcal{W})$ admits a B -linear¹ s.o.d. $\langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle$ compatible with respect to restrictions to W and Y*

$$\langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle \xleftarrow{Li_W^*} \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle \xrightarrow{Li_Y^*} \langle \mathcal{A}_r^Y, \dots, \mathcal{A}_0^Y, \mathcal{B}^Y \rangle.$$

Each \mathcal{A}_i^Y is generated by the Hacking bundle E_i and each $\mathcal{A}_i^{\mathcal{W}} \simeq D^b(R_i\text{-mod})$, where R_i is the Kalck-Karmazyn algebra associated to $P_i \in W$. Furthermore, $\mathcal{B}^{\mathcal{W}} \subset D^{\text{perf}}(\mathcal{W})$.

See Section 5 for results about deformations of a c.q.s. surface over any smooth base B , for example for the whole versal \mathbb{Q} -Gorenstein deformation of W . The categories $\mathcal{A}_i^{\mathcal{W}}$, \mathcal{A}_i^Y and $\mathcal{A}_i^{\mathcal{W}}$ categorify Wahl singularities $P_i \in W$, their Milnor fibers in Y and terminal singularities $P_i \in \mathcal{W}$, respectively. The categories $\mathcal{B}^{\mathcal{W}}$ and \mathcal{B}^Y categorify the complement of the chain $\Gamma_1 \cup \dots \cup \Gamma_r \subset W$ and the complement of its Milnor fiber in Y , which are topologically equivalent.

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2. N-RESOLUTION ASSOCIATED WITH AN M-RESOLUTION

Let $0 < \Omega < \Delta$ be coprime integers, and let $P \in \overline{W}$ be a c.q.s. of type $\frac{1}{\Delta}(1, \Omega)$. We fix an M-resolution $W^+ \rightarrow \overline{W}$. We will construct its N-resolution $W^- \rightarrow \overline{W}$ in

¹I.e. preserved by tensoring with a pullback of any object $T \in D^{\text{perf}}(B)$.

Lemma 2.7 and prove its uniqueness in Corollary 2.11. In this section we do not consider deformations of surfaces or derived categories. We write

$$\frac{\Delta}{\Omega} = [e_1, \dots, e_\ell] = e_1 - \frac{1}{\dots - \frac{1}{e_\ell}}.$$

If $\frac{\Delta}{\Omega} = [e_1, \dots, e_\ell]$ and $\frac{\Delta}{\Delta - \Omega} = [b_1, \dots, b_s]$, then the Hirzebruch–Jung continued fraction $[b_s, \dots, b_1, 1, e_1, \dots, e_\ell]$ is equal to 0 (and in particular is well-defined, i.e. there is no division by 0). This follows from the Riemenschneider’s diagram [R] and can be interpreted as the consecutive contraction of (-1) -curves in the chain of \mathbb{P}^1 ’s of self-intersections $-b_s, \dots, -b_1, -1, -e_1, \dots, -e_\ell$. The contraction process terminates with a single \mathbb{P}^1 of self-intersection 0. Since $\frac{\Delta}{\Omega'} = [e_\ell, \dots, e_1]$ implies $\Omega' \Omega \equiv 1 \pmod{\Delta}$, then we also have $[e_1, \dots, e_\ell, 1, b_s, \dots, b_1] = 0$.

Notation 2.1. Various operations with Hirzebruch–Jung continued fractions will include the notation $[\binom{n}{a}]$, which abbreviates the Hirzebruch–Jung continued fraction of $\frac{n^2}{na-1}$ of a Wahl singularity. For example, for our M-resolution we have

$$[b_s, \dots, b_1] - (1) - [\binom{n_0}{a_0}] - (c_1) - [\binom{n_1}{a_1}] - (c_2) - \dots - (c_r) - [\binom{n_r}{a_r}] = 0. \quad (2.1)$$

Here $[\binom{n_i}{a_i}]$ represents the Wahl singularity P_i and (c_i) represents the curve Γ_i , so that its proper transform in the minimal resolution of W^+ has self-intersection $-c_i$.

By [Ch, S], there is a bijection between P-resolutions of $P \in \overline{W}$ and the following set of zero continued fractions: $K(\Delta/\Omega) = \{[k_1, \dots, k_s] = 0 : 1 \leq k_i \leq b_i\}$. The M-resolution [BC] is constructed by resolving all Du Val singularities of the P-resolution (they are of type A_m for some m ’s), and partially resolving each T-singularity $\frac{1}{dn^2}(1, dna - 1)$ with $d > 1$ by its crepant Wahl resolution, which has $d - 1$ rational curves and d Wahl singularities of type $\frac{1}{n^2}(1, na - 1)$.

Notation 2.2. Suppose the M-resolution $W^+ \rightarrow \overline{W}$ corresponds to a zero-fraction $[k_1, \dots, k_s] \in K(\Delta/\Omega)$. As in Notation 1.7, the surface W^+ contains curves $\Gamma_1, \dots, \Gamma_r$ and Wahl singularities at P_i of type $\frac{1}{n_i^2}(1, n_i a_i - 1)$. We have

$$\delta_i = n_{i-1} n_i K_{W^+} \cdot \Gamma_i \geq 0$$

for all $i = 1, \dots, r$. Let $d_i := b_i - k_i \geq 0$. Furthermore, $d_1 + \dots + d_s = r + 1$. Let d_{i_1}, \dots, d_{i_e} be the set of nonzero d_i with $i_1 < i_2 < \dots < i_e$.

Proposition 2.3. $\delta_1, \dots, \delta_r$ can be computed as follows: for $k = 1, \dots, e - 1$,

$$\frac{\delta_{d_{i_1} + \dots + d_{i_k}}}{\varepsilon_{d_{i_1} + \dots + d_{i_k}}} = [b_{i_k+1}, \dots, b_{i_{k+1}-1}]$$

if $i_{k+1} > i_k + 1$, or $\delta_{d_{i_1} + \dots + d_{i_k}} = 1$ if $i_{k+1} = i_k + 1$. All other δ_i are equal to 0.

Proof. This is in the algorithm [PPSU, Cor. 10.1] for a P-resolution, adapted to its M-resolution. See also [HTU, Prop. 4.1]. \square

Example 2.4. Consider $\frac{\Delta}{\Omega} = \frac{89}{33} = [3, 4, 2, 2, 4]$. Take the P-resolution W^+ given by

$$[\binom{2}{1}] - (1) - [\binom{3}{1}] - (2) - [\binom{2}{1}] = [4] - (1) - [5, 2] - (2) - [4] \rightarrow [3, 4, 2, 2, 4].$$

We have $\frac{\Delta}{\Delta - \Omega} = [2, 3, 2, 5, 2, 2]$. The element in $K(\Delta/\Omega)$ corresponding to W^+ is $[2, 2, 1, 5, 1, 2] = 0$. Thus $d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 0, d_5 = 1$, and $d_6 = 0$. Therefore, by Proposition 2.3, we have $\delta_1 = 1$ and $\delta_2 = 5$.

Lemma 2.5. *Let $0 < B < A$ be integers with $\gcd(A, B) = 1$. Let*

$$\frac{A}{A-B} = [x_1, \dots, x_p] \text{ and } \frac{A}{B} = [y_1, \dots, y_q].$$

Then

- (1) $\frac{A^2}{AB-1} = [y_1, \dots, y_{q-1}, y_q + x_p, x_{p-1}, \dots, x_1]$,
- (2) $\frac{A^2}{A^2-(AB-1)} = [x_1, \dots, x_p, 2, y_q, \dots, y_1]$, and
- (3) $[x_p, \dots, x_1] - (1) - [y_1, \dots, y_{q-1}, y_q + x_p, x_{p-1}, \dots, x_1]$ contracts to $[x_p, \dots, x_1]$.

Proof. See [HP, Lem.8.5] or [PSU, Cor.2.1 and 2.2]. \square

We use the geometric procedure in [PPSU, Cor.10.1], which interprets the zero continued fraction of the Wahl resolution (2.1) as follows:

- (1) At the beginning of (2.1) we have d_{i_1} Wahl chains $[\binom{n_0}{a_0}]$ as follows:

$$[b_s, \dots, b_1] - (1) - \underbrace{[\binom{n_0}{a_0}] - (1) - [\binom{n_0}{a_0}] - (1) - \dots - (1) - [\binom{n_0}{a_0}]}_{d_{i_1}} - (c_{d_{i_1}}) - \dots$$

We can blow-down the (-1) -curves and new (-1) -curves consecutively until we obtain the new chain

$$[b_s, \dots, b_{i_1+1}, b_{i_1} - d_{i_1}, b_{i_1-1}, \dots, b_1] - (c_{d_{i_1}}) - [\binom{n_{d_{i_1}}}{a_{d_{i_1}}}] - \dots - (c_r) - [\binom{n_r}{a_r}].$$

- (2) If $b_{i_1} - d_{i_1} = 1$, then we contract this (-1) -curve and all new (-1) -curves in the subchain $[b_s, \dots, b_{i_1+1}, b_{i_1} - d_{i_1}, b_{i_1-1}, \dots, b_1]$ until there are none.
- (3) Then the original $(-c_{d_{i_1}})$ -curve becomes a (-1) -curve, and we have

$$\frac{n_{d_{i_1}}}{n_{d_{i_1}} - a_{d_{i_1}}} = [b_1, \dots, b_{i_1-1}, b_{i_1} - d_{i_1}, b_{i_1+1}, \dots, b_{i_2-1}].$$

- (4) We now repeat starting in (1) with the d_{i_2} .
- (5) We end with $[\dots, b_{i_e} - d_{i_e}, \dots, b_{i_1} - d_{i_1}, \dots] = 0$, which is the zero continued fraction corresponding to the M-resolution.

Proposition 2.6. *For every $j = 0, \dots, r$, we have that*

$$[\binom{n_0}{a_0}] - (c_1) - [\binom{n_1}{a_1}] - (c_2) - \dots - (c_j) - [\binom{n_j}{a_j}] - (1) - [\dots, b_{i_e} - d_{i_e}, \dots, b_{i_p} - d_{i_p}, \dots, b_{i_{p-1}} - k, \dots, b_{i_{p-2}}, \dots, b_{i_1}, \dots] = 0,$$

for some p and some k depending on j .

Proof. As explained above, the blowing-down process of the M-resolution produces the zero continued fraction $[\dots, b_{i_e} - d_{i_e}, \dots, b_{i_1} - d_{i_1}, \dots]$ by subtracting 1 from the b_{i_p} for each of the Wahl singularities from n_0, a_0 to n_r, a_r . Let us consider

$$[\binom{n_0}{a_0}] - (c_1) - [\binom{n_1}{a_1}] - (c_2) - \dots - (c_r) - [\binom{n_r}{a_r}] - (1) - [\dots, b_{i_e}, \dots, b_{i_1}, \dots] = 0.$$

Then we do the same process of subtracting 1 but now we start with b_{i_e} and we finish with b_{i_1} . This proves the claim via stopping at the Wahl singularity $[\binom{n_j}{a_j}]$ during this process. \square

Lemma 2.7. *An N -resolution $W^- \rightarrow \overline{W}$ associated to the M-resolution $W^+ \rightarrow \overline{W}$ can be constructed as follows. It has Wahl singularities \overline{P}_i of type $\frac{1}{\tilde{n}_i}(1, \tilde{n}_i \tilde{a}_i - 1)$ for $i = 0, \dots, r$, which we will describe from the bottom up via \tilde{n}_p, \tilde{a}_p such that $\tilde{n}_{r-i} = \tilde{n}_i$, $\tilde{a}_{r-i} = \tilde{a}_i$ for $i = 0, \dots, r$. The algorithm is as follows.*

- If $i_1 = 1$ (i.e. $d_1 \neq 0$), then $\tilde{n}_p = \tilde{a}_p = 1$ for $p = 0, \dots, d_1 - 1$. In other words, we start with d_1 smooth points.
- If $i_1 > 1$, then $\frac{\tilde{n}_p}{\tilde{n}_p - \tilde{a}_p} = [b_1, \dots, b_{i_1-1}]$ for $p = 0, \dots, d_{i_1} - 1$.
- Let $q = \sum_{j=1}^k d_{i_j}$. Then $\frac{\tilde{n}_p}{\tilde{n}_p - \tilde{a}_p} = [b_1, \dots, b_{i_{k+1}-1}]$ for $p = q, \dots, q + d_{i_{k+1}} - 1$.

The curves $\bar{\Gamma}_i$ for $i = 1, \dots, r$ are as follows. If $\bar{\Gamma}_i$ passes through one or two Wahl singularities, then its proper transform in the minimal resolution is a (-1) -curve. Otherwise (i.e. no Wahl singularities) it is a (-2) -curve.

Example 2.8. In Example 2.4, $\bar{n}_0 = 35$, $\bar{n}_1 = 5$, $\bar{n}_2 = 2$. The N-resolution is

$$\left[\begin{pmatrix} 35 \\ 13 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] = [3, 4, 2, 2, 7, 2, 3, 2] - 1 - [3, 5, 2] - 1 - [4].$$

Before proving Lemma 2.7 in general, let us do it just for extremal M-resolutions.

Definition 2.9. A Wahl resolution $W \rightarrow \bar{W}$ of $P \in \bar{W}$ is called *extremal* if the exceptional divisor consists of a single curve Γ_1 . We have two Wahl singularities P_0, P_1 (which may be smooth points). The type of P_i is $\frac{1}{n_i^2}(1, n_i a_i - 1)$ and we have

$$\delta_1 = n_0 n_1 |K_W \cdot \Gamma_1| \quad \text{and} \quad -n_0^2 n_1^2 \Gamma_1^2 = \Delta = n_0^2 + n_1^2 \pm \delta_1 n_0 n_1, \quad (2.2)$$

where \pm is the sign of $K_W \cdot \Gamma_1$. If $\delta_1 = 0$, then we have the M-resolution of $\frac{\Delta}{\Omega} = \frac{2n^2}{2na-1}$ for some $0 < a < n$ coprime [BC]. If W is an extremal M-resolution with $\delta_1 > 0$, then W is an extremal P-resolution introduced and studied in [HTU].

Lemma 2.10. *An extremal M-resolution has a unique N-resolution.*

Proof. If $\delta_1 = 0$, then we have the M-resolution of $\frac{\Delta}{\Omega} = \frac{2n^2}{2na-1}$ for some $0 < a < n$ coprime [BC]. Here N-resolution and M-resolution coincide, and there is only one index i_1 and $d_{i_1} = 2$ (as at the end of Lemma 2.5 with $D = 2$). If W^+ is an extremal M-resolution with $\delta_1 > 0$, then we have an extremal P-resolution of [HTU]. Here we have only two indices i_1, i_2 . We have $d_{i_1} = d_{i_2} = 1$, and

$$[b_1, \dots, b_{i_1} - 1, \dots, b_{i_2} - 1, \dots, b_s] = 0.$$

We now prove that the N-resolution proposed in Lemma 2.7 is indeed an N-resolution. By Lemma 2.5 (3), $[b_s, \dots, b_1] - (1) - \left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix} \right]$ can be blown-down to $[b_s, \dots, b_{i_2} - 1, b_{i_2-1}, \dots, b_1] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix} \right]$ and that can be blown-down to $[b_1, \dots, b_{i_1} - 1, \dots, b_{i_2} - 1, \dots, b_s]$, which is zero, and so $\frac{\Delta}{\Omega} = \left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix} \right]$. Hence we do get a Wahl resolution $W \rightarrow \bar{W}$ in this way.

We now check that $K_W \cdot \Gamma < 0$, where Γ is the central curve, and $\bar{\delta}_1 = \delta_1$. Let $\frac{p_k}{q_k} = [b_1, \dots, b_{k-1}]$, $p_1 = 1$, $p_0 = q_1 = 0$, and $q_0 = -1$. Then

$$\frac{p_{i_1} q_{i_2} - p_{i_2} q_{i_1}}{p_{i_1} q_{i_2-1} - p_{i_2-1} q_{i_1}} = [b_{i_2-1}, \dots, b_{i_1+1}] = \frac{\delta_1}{\varepsilon'_1}$$

by [HTU, Lemma 4.2]. But, by definition, we have $p_{i_1} = n_0$, $q_{i_1} = n_0 - a_0$, $p_{i_2} = \bar{n}_1$, and $q_{i_2} = \bar{n}_1 - \bar{a}_1$. Therefore $\delta_1 = \bar{n}_1 a_0 - n_0 \bar{a}_1$. On the other hand, a toric computation shows that $K_W \cdot \Gamma = -1 + \left(1 - \frac{\bar{n}_1 - \bar{a}_1}{\bar{n}_1}\right) + \left(1 - \frac{a_0}{n_0}\right) = -\frac{\bar{n}_1 a_0 - n_0 \bar{a}_1}{\bar{n}_1 n_0}$, and so $K_W \cdot \Gamma$ is negative and $\bar{\delta}_1 = \delta_1$.

Finally, for uniqueness let us consider some Wahl chain $\left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right]$ such that

$$[b_s, \dots, b_1] - (1) - \left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix} \right] = 0,$$

but then we also have $\left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix} \right] - (1) - [b_s, \dots, b_1] = 0$, and so $\left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right]$ is determined, being dual to the contraction of $\left[\begin{pmatrix} n_0 \\ a_0 \end{pmatrix} \right] - (1) - [b_s, \dots, b_1]$. \square

Proof of Lemma 2.7. Note that the $\bar{\delta}_i$ are determined by the Wahl resolution, but in the proof we will check that $\bar{\delta}_i = \delta_{r+1-i}$ as required by the definition of the N-resolution. We will also need to prove property (1) in the definition.

The strategy is to consider the chain

$$[b_s, \dots, b_1] - (1) - \left[\begin{pmatrix} \bar{n}_0 \\ \bar{a}_0 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right] - (1) - \dots - (1) - \left[\begin{pmatrix} \bar{n}_r \\ \bar{a}_r \end{pmatrix} \right], \quad (2.3)$$

and to prove that it is contractible and contracts to zero. In this way, we would have that $\left[\begin{pmatrix} \bar{n}_0 \\ \bar{a}_0 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} \bar{n}_1 \\ \bar{a}_1 \end{pmatrix} \right] - (1) - \dots - (1) - \left[\begin{pmatrix} \bar{n}_r \\ \bar{a}_r \end{pmatrix} \right] = [e_1, \dots, e_\ell] = \frac{\Delta}{\Omega}$. Notice that $\left[\begin{pmatrix} \bar{n}_r \\ \bar{a}_r \end{pmatrix} \right]$ could correspond to $[2, \dots, 2]$ if $d_1 \neq 0$. Let us consider d_{i_1}, \dots, d_{i_e} (the set of nonzero d_i) and write $[b_s, \dots, b_1] = [\dots, b_{i_e}, \dots, b_{i_{e-1}}, \dots, b_{i_1}, \dots]$ with $i_1 < i_2 < \dots < i_e$. By definition, we have either $\frac{\bar{n}_0}{\bar{n}_0 - \bar{a}_0} = [b_1, \dots, b_{i_e-1}]$ when $i_e > 1$, or \bar{P}_0 is a smooth point. In any case, by Lemma 2.5 (3), we have

$$[\dots, b_{i_e}, \dots, b_{i_{e-1}}, \dots, b_{i_1}, \dots] - (1) - \left[\begin{pmatrix} \bar{n}_0 \\ \bar{a}_0 \end{pmatrix} \right] = [\dots, b_{i_e} - 1, \dots, b_{i_{e-1}}, \dots, b_{i_1}, \dots].$$

We continue with the following singularities, applying Lemma 2.5 (3) each time since $\frac{\bar{n}_p}{\bar{n}_p - \bar{a}_p} = [b_1, \dots, b_{i_k-1}]$ for some k depending on p . We recall that $i_e > i_{e-1} > \dots > i_1$, and so this contraction process makes sense. In this way, we arrive to

$$[\dots, b_{i_e} - d_{i_e}, \dots, b_{i_{e-1}} - d_{i_{e-1}}, \dots, b_{i_1} - d_{i_1}, \dots],$$

and this is the zero continued fraction of the M-resolution. Therefore the original chain is contractible, and it contracts to zero.

We now show that the $\bar{\delta}_i$'s are indeed the ones from the algorithm. First we know that there are no problems with $\bar{\delta}_i = 0$, they obviously coincide. Let us consider $\bar{\delta}_p$ corresponding to the break $i_k < i_{k+1}$. In the blowing-down process of (2.3), consider the step

$$[\dots, b_{i_{k+1}} - d_{i_{k+1}} + 1, \dots, b_{i_k}, \dots, b_1] - (1) - \left[\begin{pmatrix} \bar{n}_{p-1} \\ \bar{a}_{p-1} \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} \bar{n}_p \\ \bar{a}_p \end{pmatrix} \right] - (1) - \dots$$

This is about computing $\bar{\delta}_p$ for $\left[\begin{pmatrix} \bar{n}_{p-1} \\ \bar{a}_{p-1} \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} \bar{n}_p \\ \bar{a}_p \end{pmatrix} \right] = \frac{\Delta'}{\Omega'}$ for some Δ', Ω' . The strategy is to compute $\bar{\delta}_p$ in this situation, and show that it coincides with our definition. For that we compute the dual continued fraction $\frac{\Delta'}{\Delta' - \Omega'}$. But then

$$[b_{i_{k+1}} - d_{i_{k+1}} + 1, \dots, b_{i_k}, \dots, b_1] - (1) - \frac{\Delta'}{\Omega'}$$

is either 0 or we can complete it as

$$[y_u \dots, y_1, b_{i_{k+1}} - d_{i_{k+1}} + 1, \dots, b_{i_k}, \dots, b_1] - (1) - \frac{\Delta'}{\Omega'}$$

to make it zero, where $y_j \geq 2$, and $b_{i_{k+1}} - d_{i_{k+1}} + 1 \geq 2$. Hence

$$\frac{\Delta'}{\Delta' - \Omega'} = [b_1, \dots, b_{i_k}, \dots, b_{i_{k+1}} - d_{i_{k+1}} + 1, y_1, \dots, y_u].$$

But this is the situation of an extremal N-resolution where we must subtract -1 in positions i_k and i_{k+1} to make it zero. Then the computation of $\bar{\delta}_p$ is identical to what we did in Lemma 2.10.

To prove property (1) in Definition 1.8, we use Proposition 2.6. Note that when we eliminate Wahl singularities in the N-resolution, we subtract 1 from the b_{i_p} from i_e to i_1 , and that is exactly what we have in Proposition 2.6. \square

Corollary 2.11. *Every M-resolution has a unique associated N-resolution.*

Proof. We know this is true for $r = 1$ by Lemma 2.10. For $r \geq 2$ we go by induction on r . We have that $\Gamma_1 \cup \dots \cup \Gamma_{r-1}$ is an M-resolution of $\frac{1}{\Delta_{r-1}}(1, \Omega_{r-1})$, and so we can apply induction for all singularities, deltas, and $\frac{1}{\Delta_i}(1, \Omega_i)$ except for \bar{n}_0, \bar{a}_0 . Let $\frac{\Delta_{r-1}}{\Omega_{r-1}} = [f_1, \dots, f_t]$. Then we have $[b_s, \dots, b_1] - (1) - \left[\binom{n'_0}{a'_0}\right] - (1) - [f_1, \dots, f_t] = 0$, and this implies $\left[\binom{n'_0}{a'_0}\right] - (1) - [f_1, \dots, f_t] - (1) - [b_s, \dots, b_1] = 0$, and so $\left[\binom{n'_0}{a'_0}\right]$ is determined by $\frac{1}{\Delta_{r-1}}(1, \Omega_{r-1})$ and $\frac{1}{\Delta}(1, \Omega)$. \square

Example 2.12. Using the computer program [Z], we find all M-resolutions and N-resolutions of the c.q.s. $\frac{1}{85}(1, 49)$. We have $\frac{85}{49} = [2, 4, 5, 2, 2]$, and $\frac{85}{36} = [3, 2, 3, 2, 2, 4]$. This c.q.s. has a deformation space with 5 irreducible components. For each of them, we list the corresponding: zero continued fraction, dimension of the component, the vector of the δ_i , the M-resolution, and the N-resolution.

[1, 2, 2, 2, 2, 1], dimension is 10, (0, 2, 3, 0, 0)
 (2) - (4) - (5) - (2) - (2) (minimal resolution)
 $\left[\binom{26}{15}\right] - (1) - \left[\binom{26}{15}\right] - (1) - \left[\binom{26}{15}\right] - (1) - \left[\binom{5}{3}\right] - (1) - (2)$

[2, 1, 3, 2, 2, 1], dimension is 8, (1, 7, 0, 0)
 (2) - $\left[\binom{2}{1}\right] - (5) - (2) - (2)$
 $\left[\binom{26}{15}\right] - (1) - \left[\binom{26}{15}\right] - (1) - \left[\binom{26}{15}\right] - (1) - \left[\binom{3}{2}\right] - (1)$

[1, 2, 3, 2, 1, 3], dimension is 6, (0, 8, 1)
 (2) - (4) - $\left[\binom{3}{1}\right] - (2)$
 $\left[\binom{26}{15}\right] - (1) - \left[\binom{19}{11}\right] - (1) - (2)$

[2, 2, 3, 1, 2, 4], dimension is 2, (5)
 (2) - $\left[\binom{7}{2}\right]$ (extremal P-resolution)
 $\left[\binom{12}{7}\right] - (1)$

[3, 1, 3, 2, 1, 4], dimension is 2, (5)
 $\left[\binom{3}{2}\right] - (1) - \left[\binom{4}{1}\right]$ (extremal P-resolution)
 $\left[\binom{19}{11}\right] - (1) - \left[\binom{3}{2}\right]$

3. BRAID GROUP ACTION ON WAHL RESOLUTIONS

Given a c.q.s. \overline{W} , we will show how to connect the M-resolution W^+ and the N-resolution W^- by a sequence of antiflips, which are generators of the braid group B_{r+1} action on the set of Wahl resolutions $W \rightarrow \overline{W}$ with $r + 1$ Wahl singularities. This action comes from natural operations on deformations of Wahl resolutions.

We first describe the action of $B_2 \simeq \mathbb{Z}$ on extremal Wahl resolutions $W \rightarrow \overline{W}$, where either $K_W \cdot \Gamma_1 > 0$ (extremal P-resolutions), $K_W \cdot \Gamma_1 < 0$ (*K-negative resolutions*), or $K_W \cdot \Gamma_1 = 0$ when $\delta_1 = 0$ (*K-trivial resolutions*). We will refer to the action of a generator of B_2 as the *right antiflip* and to its inverse as the *left antiflip*.

A \mathbb{Q} -Gorenstein smoothing $\mathcal{W} \rightarrow B$ of an extremal Wahl resolution W over a smooth curve can be blown-down to a smoothing $\overline{\mathcal{W}} \rightarrow B$ of \overline{W} . This gives a threefold contraction $\mathcal{W} \rightarrow \overline{\mathcal{W}}$, which is $K_{\mathcal{W}}$ -positive, $K_{\mathcal{W}}$ -negative, or $K_{\mathcal{W}}$ -trivial depending on the three cases above. The antiflip is defined differently in each case.

Antiflips: *K*-positive case. Consider a \mathbb{Q} -Gorenstein smoothing $\mathcal{W}^+ \rightarrow B$ of an extremal P-resolution W^+ over a smooth curve. One can ask if the morphism of threefolds $\mathcal{W}^+ \rightarrow \overline{\mathcal{W}}$ given by blowing down the deformation admits an antiflip

(a relative anticanonical model with terminal singularities). This was studied in [HTU] following an earlier work of Mori, Kollár, and Prokhorov [M1], [KM], [M2], [MP]. See [U, S. 2] for a summary of results. A terminal antiflip exists if and only if

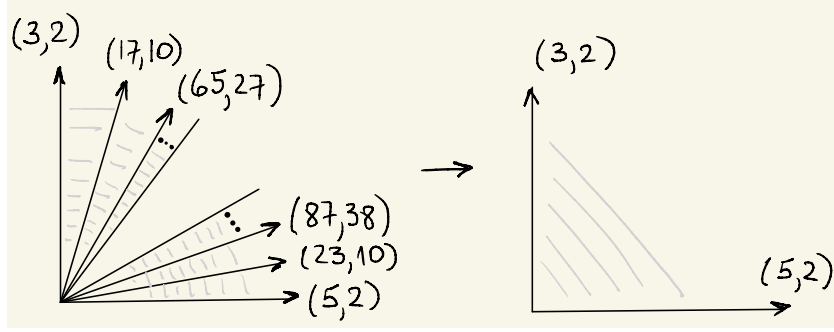


FIGURE 4. Universal family of antiflips (on the left) of the extremal P-resolution (on the right)

the boundary divisor $\Gamma_0 + \Gamma_1 + \Gamma_2 \subset W^+$ deforms in \mathcal{W}^+ and the axial multiplicities at P_0, P_1 satisfy $\alpha_0^2 - \delta\alpha_0\alpha_1 + \alpha_1^2 > 0$. Each antiflip is a \mathbb{Q} -Gorenstein smoothing of a K -negative extremal Wahl resolution $W \rightarrow \overline{W}$ and each of them appears this way for some W^+ . Terminal antiflips admit a universal family (see [HTU] for explicit equations) illustrated in Figure 4 in a concrete example, where $\delta = 4$.

Take an extremal P-resolution $[(\begin{smallmatrix} n_0 \\ a_0 \end{smallmatrix})] - (c) - [(\begin{smallmatrix} n_1 \\ a_1 \end{smallmatrix})]$ (in our example $[(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix})] - (1) - [(\begin{smallmatrix} 5 \\ 2 \end{smallmatrix})]$) and consider its \mathbb{Q} -Gorenstein smoothing \mathcal{W}^+ with a vector of axial multiplicities (α_0, α_1) in the first quadrant (see the right side of Figure 4). The corresponding antiflip \mathcal{W} exists if $\alpha_0^2 - \delta\alpha_0\alpha_1 + \alpha_1^2 > 0$. When (α_0, α_1) is in the interior of a 2-dimensional cone σ of the fan \mathcal{F} on the left side of Figure 4 then the special fiber $W \subset \mathcal{W}$ is a K -negative extremal Wahl resolution. We ignore 1-dimensional cones of \mathcal{F} until Section 6, where they will become crucial in the proof of Prop. 6.1. The fan \mathcal{F} itself depends only on δ but it is decorated with data of Wahl singularities (n, a) determined by certain recurrence relations. Decorations of the cone σ determine Wahl singularities of W . If $\delta_1 > 1$ then σ is infinite and excludes the region $\alpha_0^2 - \delta\alpha_0\alpha_1 + \alpha_1^2 \leq 0$. In our example, we get K -negative extremal Wahl resolutions with singularities with data $[(\begin{smallmatrix} 5 \\ 2 \end{smallmatrix})] - (1) - [(\begin{smallmatrix} 23 \\ 10 \end{smallmatrix})]$, $[(\begin{smallmatrix} 23 \\ 10 \end{smallmatrix})] - (1) - [(\begin{smallmatrix} 87 \\ 38 \end{smallmatrix})]$, etc. (see Figure 6). Expressing (α_0, α_1) as a linear combination of generators of the cone $\sigma \subset \mathcal{F}$ gives axial multiplicities of the anticanonical model (the smoothing \mathcal{W} of W).

There are two particular “initial” K -negative extremal Wahl resolutions W_0^- and W_1^- that correspond to cones adjacent to the boundary of the first quadrant. Each of them preserves one of the Wahl singularities in W^+ (including the case of smooth points). To be precise, let $-c_1$ be the self-intersection of the proper transform of Γ_1 in the minimal resolution of W^+ . The singularities of W_0^- are \overline{P}_0 and $\overline{P}_1 = P_0$ with

$$\bar{n}_0 = \delta_1 n_0 + n_1, \quad \bar{a}_0 = \delta_1 a_0 + a_1 - (c_1 - 1)n_1 \quad \text{and} \quad \bar{n}_1 = n_0, \quad \bar{a}_1 = a_0. \quad (3.1)$$

The singularities of W_1^- are $\overline{P}_0 = P_1$ with $\bar{n}_0 = n_1, \bar{a}_0 = a_1$ and \overline{P}_1 with $\bar{n}_1 = \delta_1 n_1 + n_0, \bar{a}_1 = \delta_1 a_1 + a_0 + (c_1 - 1)n_0$. The proper transform of $\Gamma_{1,i} \subset W_i^-$ for $i = 0, 1$ in the minimal resolution is a (-1) -curve, and $\bar{\delta}_1 = \delta_1$ in both cases. We refer to $W_0^- \rightarrow \overline{W}$ as the *right antiflip* (or just the antiflip) of an extremal P-resolution $W^+ \rightarrow \overline{W}$ and to $W_1^- \rightarrow \overline{W}$ as the *left antiflip*. By Lemma 2.10, the right antiflip is the N-resolution of the extremal P-resolution.

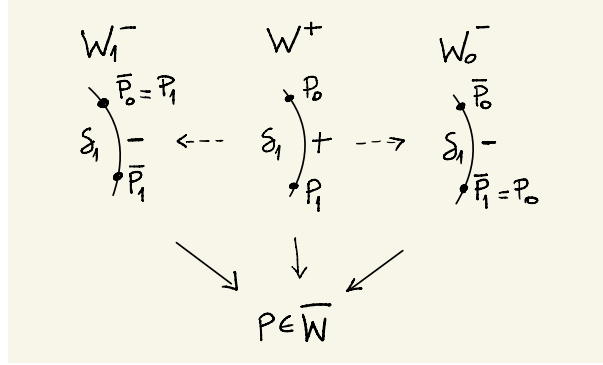


FIGURE 5. Initial negative extremal Wahl resolutions

Given \mathbb{Q} -Gorenstein smoothings \mathcal{W}_i^- of W_i^- ($i = 0, 1$) over smooth curves B_i , the blow-down deformations $(\Gamma_{1,i} \subset \mathcal{W}_i^-) \rightarrow (P \in \overline{W}_i)$ are birational contractions of $K_{\mathcal{W}_i^-}$ -negative curves of flipping type (k2A extremal neighborhoods [HTU]). The flips $\mathcal{W}_i^- \dashrightarrow \mathcal{W}_i^+$ give \mathbb{Q} -Gorenstein smoothings $\mathcal{W}_i^+ \rightarrow B_i$ of the same central fiber W^+ . In particular, general fibers of \mathcal{W}_i^- and \mathcal{W}_i^+ are isomorphic for each i . The curves B_0, B_1 are from the same component of $\text{Def}_{P \in \overline{W}^2}$.

Antiflips: K -negative case. Antiflips of K -negative extremal resolutions W correspond to counter-clockwise and clockwise rotations in Figure 4 through the sequence of two-dimensional cones of the fan \mathcal{F} . The *right* (resp. *left*) *antiflip* of a K -negative extremal resolution $[(\binom{n_0}{a_0}) - (1) - [(\binom{n_1}{a_1})]$ different from W_1^- (resp. W_0^-) is a K -negative extremal resolution $[(\binom{n'_0}{a'_0}) - (1) - [(\binom{n'_1}{a'_1})]$ such that $[(\binom{n_0}{a_0})] = [(\binom{n'_1}{a'_1})]$ (resp. $[(\binom{n'_0}{a'_0})] = [(\binom{n_1}{a_1})]$). Their \mathbb{Q} -Gorenstein smoothings blow-down to smoothings of \overline{W} from the same irreducible component of $\text{Def}_{P \in \overline{W}}$. For example, universal family of Figure 4 gives a sequence of right antiflips in Figure 6 infinite in both directions. The right (resp. left) antiflip of W_1^- (resp. W_0^-) is actually a flip.

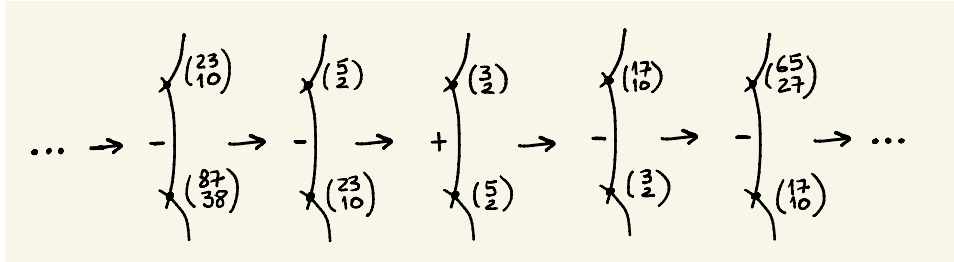


FIGURE 6. Right antiflips of extremal Wahl resolutions.

Antiflips: K -trivial case. Here W^+ is $[(\binom{n}{a}) - (1) - [(\binom{n}{a})]$ (unless $n = 1$ in which case it is a (-2) -curve with two smooth points) and $\overline{W} = \frac{1}{2n^2}(1, 2na - 1)$ [BC]. The blow-down \overline{W} of a \mathbb{Q} -Gorenstein smoothing \mathcal{W}^+ of W^+ is a \mathbb{Q} -Gorenstein smoothing of \overline{W} . The contraction $\mathcal{W}^+ \rightarrow \overline{W}$ is crepant and can be flopped giving a threefold \mathcal{W}^- (which we call an antiflip of \mathcal{W}^+) with a central fiber $W^- \simeq W^+$ (see [BC] or [K1, S. 5]), which we call an antiflip of W^+ in the K -trivial case.

²The threefolds \mathcal{W}_i^- are anticanonical models of \mathcal{W}_i^+ for $i = 0, 1$. Since anticanonical models are unique (whenever they exist), we see that curves B_0 and B_1 are necessarily different.

Before we consider the action of the braid group B_{r+1} on $r + 1$ strands, we address some global questions, namely existence of “good” divisors Γ_0 and Γ_{r+1} and vanishing of local-to-global obstructions to deformations of Wahl resolutions.

Lemma 3.1. *Let $P \in \overline{W}$ be a c.q.s. surface satisfying Assumption 1.10 (1), (3). We can choose effective, smooth divisors \bar{A} and \tilde{A} such that the germ $P \in (\bar{A} \cup \tilde{A}) \subset \overline{W}$ is étale-locally isomorphic to the germ $0 \in (x = 0) \cup (y = 0) \subset \mathbb{C}^2/\mu_\Delta$. Proper transforms Γ_0 of \bar{A} and Γ_{r+1} of \tilde{A} in any Wahl resolution W of \overline{W} intersect the chain $\Gamma_1 \cup \dots \cup \Gamma_r$ only at the end-points P_0 and P_r , where they give toric boundaries opposite to Γ_1 (resp. Γ_r).*

Proof. We start by choosing \bar{A} to be a Weil divisor generating the local class group of $P \in \overline{W}$. We can make it effective by adding a sufficiently ample Cartier divisor H . Then $\Delta \cdot \bar{A}$ is Cartier and effective. We can add another multiple of H to make $\Delta \cdot \bar{A}$ base-point-free. By Bertini theorem, we can find a smooth divisor $D \in |\Delta \cdot \bar{A}|$ which does not pass through P . Consider a cyclic cover $\pi : \hat{W} = \text{Spec}_{\overline{W}} \bigoplus_{k=0}^{\Delta-1} \mathcal{O}_{\overline{W}}(-k\bar{A}) \rightarrow \overline{W}$, where we use the canonical section $s_D \in H^0(\overline{W}, \mathcal{O}_{\overline{W}}(D))$ to define the algebra structure using the map $\mathcal{O}_{\overline{W}}(-\Delta \cdot \bar{A}) \xrightarrow{s_D} \mathcal{O}_{\overline{W}}$ (see [KK] for a theory of cyclic covers). Then \hat{W} is smooth and $\overline{W} = \hat{W}/\mu_n$. More precisely, $\pi_* \mathcal{O}_{\hat{W}} = \bigoplus_{k=0}^{\Delta-1} \mathcal{O}_{\overline{W}}(-k\bar{A})$ is an eigenvalue decomposition with respect to a primitive root $\zeta \in \mu_\Delta$, where ζ acts on $\mathcal{O}_{\overline{W}}(-k\bar{A})$ with weight ζ^{-k} . The cover is branched over D , where it is locally given by $(x, y) \mapsto (x, y^\Delta)$, where $y = 0$ is a local equation of D , and over P , where $\hat{P} = \pi^{-1}(P)$ is a smooth point and ζ acts on the tangent space of \hat{P} with eigenvalues $(\zeta^a, \zeta^{\bar{a}})$, where a and \bar{a} are coprime to Δ . Then μ_n acts on $H^0(\hat{W}, \pi^*H)$ with eigenspaces $H^0(\overline{W}, \mathcal{O}_{\overline{W}}(H - k\bar{A}))$ for $k = 0, \dots, \Delta - 1$. The μ_Δ -action is free everywhere else.

Required divisors A, \tilde{A} on \overline{W} can be found as equivariant sections of $H^0(\hat{W}, \pi^*H)$ with eigenvalues $(\zeta^a, \zeta^{\bar{a}})$. We require them to be smooth (in particular they restrict to given μ_Δ -eigenspaces in the tangent space of \hat{P}) and be transversal to the branch divisor $\pi^{-1}(D)$. We claim that these divisors exist if H is sufficiently ample. Since these geometric conditions on global sections are open (in fixed μ_Δ -eigenspaces of $H^0(\hat{W}, \pi^*H)$), it suffices to prove that there exist sections as above having required geometric properties in finitely many equivariant affine charts U_α that cover \hat{W} and equivariantly trivialize $\mathcal{O}(\pi^*H)$. Since π^*H is ample, any local section of $\mathcal{O}_{\hat{W}}(U_\alpha)$ is a restriction of a global section of $\pi^*(kH)$ for a sufficiently large k . So it suffices to prove that there exist equivariant sections of $\mathcal{O}_{\hat{W}}(U_\alpha)$ which have required geometric properties, which is clear. For example, in the equivariant affine chart containing \hat{P} , we can take any equivariant regular function that restricts to the given eigenspace in the tangent space of \hat{P} , and in particular is smooth at \hat{P} , and then shrink the affine neighborhood to make it smooth everywhere in it. \square

Lemma 3.2. *Let $\pi : W \rightarrow \overline{W}$ be a c.q.s. resolution satisfying Assumption 1.10.*

- (1) *We can choose divisors Γ_0 and Γ_{r+1} as in Lemma 3.1 so that there are no local-to-global obstructions to deformations of a pair (W, Δ) , where Δ is the boundary $\Gamma_0 + \Gamma_1 + \dots + \Gamma_r + \Gamma_{r+1}$, i.e. the morphism $\text{Def}_{(W, \Delta)} \rightarrow \prod_{P_i \in W} \text{Def}_{P_i \in (W, \Delta)}$ is smooth.*
- (2) *If W is a Wahl resolution then there are no local-to-global obstructions to \mathbb{Q} -Gorenstein deformations of W or (W, Δ) , for example there exists a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W$ with a lifting of Δ for any choice of axial multiplicities $\alpha_0, \dots, \alpha_r$.*

Proof. The versal \mathbb{Q} -Gorenstein deformation a Wahl singularity $\frac{1}{n^2}(1, na - 1)$ is $(xy = z^n + t) \subset \frac{1}{n}(1, -1, a, n)$, where t is a deformation parameter and $xy = 0$ is

the local equation of Δ . So the morphism $\text{Def}_{P_i \in (W, \Delta)}^{\mathbb{Q}G} \rightarrow \text{Def}_{P_i \in W}^{\mathbb{Q}G}$ is obviously smooth and we only need to prove the first statement.

We have a subsheaf $T_W(-\log \sum_{i=1}^r \Gamma_i) \subset T_W$ of derivations that preserve ideal sheaves of $\Gamma_1, \dots, \Gamma_r$. It is well-known that $R\pi_* T_W(-\log \sum_{i=1}^r \Gamma_i)$ is a subsheaf of $T_{\overline{W}}$ with the quotient sheaf supported at $P \in \overline{W}$ ³. By Assumption 1.10 (4), it follows that $H^2(W, T_W(-\log \sum_{i=1}^r \Gamma_i)) = H^2(\overline{W}, T_{\overline{W}}) = 0$. Since the sheaf $T_W(-\log \sum_{i=1}^r \Gamma_i)$ is the sheaf of infinitesimal automorphisms of the pair $(W, \Gamma_1 + \dots + \Gamma_r)$, it follows that there are no local-to-global obstructions to deformations of the pair.

We need to find Γ_0 and Γ_{r+1} that satisfy Lemma 3.1 and cohomology vanishing $H^2(W, T_W(-\log \Delta)) = 0$. Let $q : \underline{W} \rightarrow W$ be an orbifold stack associated to W . An exact sequence $0 \rightarrow T_{\underline{W}}(-\log \sum_{i=0}^{r+1} \Gamma_i) \rightarrow T_{\underline{W}}(-\log \sum_{i=1}^r \Gamma_i) \rightarrow N_{\Gamma_0/\underline{W}} \oplus N_{\Gamma_{r+1}/\underline{W}} \rightarrow 0$ on \underline{W} pushes forward to $0 \rightarrow T_W(-\log \Delta) \rightarrow T_W(-\log \sum_{i=1}^r \Gamma_i) \rightarrow F \oplus \tilde{F} \rightarrow 0$, where F (resp. \tilde{F}) is a sheaf supported on Γ_0 (resp. Γ_{r+1}). It suffices to prove that $H^1(\Gamma_0, N_{\Gamma_0/W}) = H^1(\Gamma_{r+1}, N_{\Gamma_{r+1}/W}) = 0$. Pushing forward an exact sequence $0 \rightarrow \mathcal{O}_{\underline{W}} \rightarrow \mathcal{O}_{\underline{W}}(\Gamma_0) \rightarrow N_{\Gamma_0/\underline{W}} \rightarrow 0$, and using Assumption 1.10 (2), it suffices to prove that $H^1(\underline{W}, \mathcal{O}_{\underline{W}}(\Gamma_0)) = 0$. This holds after twisting Γ_0 with a sufficiently ample Cartier divisor H in Lemma 3.1. The same proof works for Γ_{r+1} . \square

We will define the action of generators of B_{r+1} on Wahl resolutions with $r+1$ singularities by treating every irreducible curve in its exceptional divisor as an extremal Wahl resolution. Relations of the braid group are checked in Theorem 3.9.

Definition 3.3. Let $W \rightarrow \overline{W}$ be a Wahl resolution with exceptional divisor $\Gamma_1 \cup \dots \cup \Gamma_r$ and toric boundaries Γ_0 and Γ_{r+1} as in Lemmas 3.1 and 3.2. The neighborhood of $\Gamma_i \subset W$ contains a subchain $[\binom{n_{i-1}}{a_{i-1}}] - (c_i) - [\binom{n_i}{a_i}]$ of an extremal Wahl resolution. The contraction $W \rightarrow W_i$ of $\Gamma_i \subset W$ gives a c.q.s. surface. The image of Γ_i is a c.q.s. $\frac{1}{\Delta_{\Gamma_i}}(1, \Omega_{\Gamma_i})$, which has as toric boundary the image of Γ_{i-1} and Γ_{i+1} . By Lemma 3.2, we can choose two deformations of W_i (the same ones if the extremal resolution is a P-resolution or a K -trivial resolution) which (1) are equisingular at singularities of W_i other than $\frac{1}{\Delta_{\Gamma_i}}(1, \Omega_{\Gamma_i})$, (2) lift the boundary of W_i , and (3) smoothen $\frac{1}{\Delta_{\Gamma_i}}(1, \Omega_{\Gamma_i})$ as in the discussion of antiflips of extremal Wahl resolutions in the beginning of this section. These deformations of W_i are blow-downs of \mathbb{Q} -Gorenstein deformations of W and another Wahl resolution $R_i(W) \rightarrow \overline{W}$, respectively. We call $R_i(W)$ the right antiflip of $W \rightarrow \overline{W}$ at Γ_i . The left antiflip is defined in a similar way. The singularities of $R_i(W)$ and $L_i(W)$ are the same as for W except at the positions $i-1$ and i , where we have the singularities produced by the antiflip of an extremal Wahl resolution $[\binom{n_{i-1}}{a_{i-1}}] - (c_i) - [\binom{n_i}{a_i}]$.

Corollary 3.4. *Given a sequence of Wahl resolutions $W_0, W_1, \dots, W_k \rightarrow \overline{W}$ with Wahl chains $\Gamma_0^j, \dots, \Gamma_{r+1}^j$ for $j = 0, \dots, k$, suppose $W_i = R_{l_i}(W_{i-1})$ for $i = 1, \dots, k$.*

³It suffices to prove that $R^k \pi_* T_W(-\log \sum_{i=1}^r \Gamma_i) = 0$ for $k > 0$. We work in a toric étale neighborhoods of $P \in \overline{W}$ and its partial resolution $\Gamma_1 + \dots + \Gamma_r \subset W$. Then $R^k \pi_* T_W(-\log \Delta) = 0$ for $k > 0$ because $T_W(-\log \Delta) \simeq \mathcal{O}_W \otimes \mathbb{C}^2$ in this étale neighborhood. We have an exact sequence $0 \rightarrow T_W(-\log \Delta) \rightarrow T_W(-\log(\sum_{i=1}^r \Gamma_i)) \rightarrow F \rightarrow 0$, where F is supported on $\Gamma_0 \cup \tilde{\Gamma}_{r+1}$. Then $R^k \pi_* F = 0$ for $k > 0$ and we have a required vanishing.

- (1) There is a sequence of \mathbb{Q} -Gorenstein smoothings $Y_i \rightsquigarrow W_i$ for $i = 0, 1, \dots, k$ over smooth curve germs B_i that belong to the same component of $\text{Def}_{P \in \overline{W}}$.
- (2) If $K_{W_{i-1}} \cdot \Gamma_{i-1}^{i-1} \geq 0$ for $i = 1, \dots, k$, i.e. on every step we antiflip an extremal P -resolution or a K -trivial resolution, then we can assume that $B_1 = \dots = B_k = B$ is the same curve in $\text{Def}_{P \in \overline{W}}$ and $(W_{i-1} \subset \mathcal{W}_{i-1}) \rightarrow (0 \in B)$ is the flip (or flop) of $(W_i \subset \mathcal{W}_i) \rightarrow (0 \in B)$ for all $i = 1, \dots, k$ with respect to the contraction of $\Gamma_{i-1}^{i-1} \subset W_{i-1}$. In particular, the smooth fibers Y_i of these families are isomorphic.

Proof. (1) is clear. To prove (2), choose a \mathbb{Q} -Gorenstein smoothing $(W_k \subset \mathcal{W}_k) \rightarrow (0 \in B)$ over a smooth curve germ B with all axial multiplicities equal to 1, which exists by Lemma 3.2. Then we apply a sequence of flips (or flops if $\delta_{i-1}^{i-1} = 0$) to contractions of $\Gamma_{i-1}^{i-1} \subset W_{i-1}$ for $i = k, k-1, \dots, 1$. \square

Proposition 3.5. *Let $W \rightarrow \overline{W}$ be a Wahl resolution with a chain of 3 curves $\Gamma_1, \Gamma_2, \Gamma_3$, and singularities P_0, P_1, P_2, P_3 where the type of P_i is $\frac{1}{n_i}(1, n_i a_i - 1)$. Consider $W' := R_2(W)$ the right antiflip of $W \rightarrow \overline{W}$ at Γ_2 . Hence we have a Wahl resolution $W' \rightarrow \overline{W}$ with a chain of 3 curves $\Gamma'_1, \Gamma'_2, \Gamma'_3$, and singularities $P'_0 = P_0, P'_1, P'_2 = P_1, P'_3 = P_3$. Let $\delta'_i = n'_{i-1} n'_i |K_{W'} \cdot \Gamma'_i|$. Then we have the following three situations:*

(-/-): $\Gamma_2 \cdot K_W < 0$ and $\Gamma'_2 \cdot K_{W'} < 0$.

- (1) $n'_1 = \delta_2 n_1 - n_2$, $a'_1 = \delta_2 a_1 - a_2$, $n'_2 = n_1$, $a'_2 = a_1$, $\delta'_2 = \delta_2$.
- (2) $n'_0 n'_1 \Gamma'_1 \cdot K_{W'} = \frac{\pm \delta_1 (\delta_2 n_1 - n_2) + \delta_2 n_0}{n_1}$, where \pm is the sign of $K_W \cdot \Gamma_1$.
- (3) $n'_2 n'_3 \Gamma'_3 \cdot K_{W'} = \frac{\pm \delta_3 n_1 - \delta_2 n_3}{n_2}$, where \pm is the sign of $K_W \cdot \Gamma_3$.

(-/+): $\Gamma_2 \cdot K_W < 0$ and $\Gamma'_2 \cdot K_{W'} > 0$. Let $-c'_2$ be the self intersection of the proper transform of Γ'_2 in the minimal resolution of W' .

- (1) $n'_1 = n_2 - \delta_2 n_1$, $a'_1 = a_2 - \delta_2 a_1 - (c'_2 - 1)n'_1$, $n'_2 = n_1$, $a'_2 = a_1$, $\delta'_2 = \delta_2$, and $\delta'_2 = (c'_2 - 1)n'_1 n_1 + n_1 a'_1 - n'_1 a_1$.
- (2) $n'_0 n'_1 \Gamma'_1 \cdot K_{W'} = \frac{\pm \delta_1 (n_2 - \delta_2 n_1) - \delta_2 n_0}{n_1}$, where \pm is the sign of $K_W \cdot \Gamma_1$.
- (3) $n'_2 n'_3 \Gamma'_3 \cdot K_{W'} = \frac{\pm \delta_3 n_1 - \delta_2 n_3}{n_2}$, where \pm is the sign of $K_W \cdot \Gamma_3$.

(+/-): $\Gamma_2 \cdot K_W \geq 0$ and $\Gamma'_2 \cdot K_{W'} \leq 0$. Let $-c_2$ be the self intersection of the proper transform of Γ_2 in the minimal resolution of W .

- (1) $n'_1 = \delta_2 n_1 + n_2$, $a'_1 = \delta_2 a_1 + a_2 - (c_2 - 1)n_2$, $n'_2 = n_1$, $a'_2 = a_1$, $\delta'_2 = \delta_2$.
- (2) $n'_0 n'_1 \Gamma'_1 \cdot K_{W'} = \frac{\pm \delta_1 (\delta_2 n_1 + n_2) + \delta_2 n_0}{n_1}$, where \pm is the sign of $K_W \cdot \Gamma_1$.
- (3) $n'_2 n'_3 \Gamma'_3 \cdot K_{W'} = \frac{\pm \delta_3 n_1 + \delta_2 n_3}{n_2}$, where \pm is the sign of $K_W \cdot \Gamma_3$.

In particular, we have in all cases that

$$K_W \cdot \Gamma_1 = K_{W'} \cdot (\Gamma'_1 + \Gamma'_2) \quad \text{and} \quad K_{W'} \cdot \Gamma'_2 = K_W \cdot (\Gamma_1 + \Gamma_2).$$

Proof. We only prove the case (+/-), the others follow similar computations. The new singularities P'_1, P'_2 in (1) are computed by the formulas (3.1). In particular $\delta'_2 = \delta_2$. To find δ'_1 and δ'_3 , we contract $\Gamma_2 \subset W$ and $\Gamma'_2 \subset W'$, and do intersection theory on the singular surfaces involved. We will only compute δ'_1 .

Let $\pi: W \rightarrow \tilde{W}$ and $\pi': W' \rightarrow \tilde{W}$ be the contractions of $\Gamma_2 \subset W$ and $\Gamma'_2 \subset W'$ respectively. We have $K_W \equiv \pi^*(K_{\tilde{W}}) - \varepsilon_+ \Gamma_2$, and $K_{W'} \equiv \pi'^*(K_{\tilde{W}}) + \varepsilon_- \Gamma'_2$. Intersection with Γ_2 and Γ'_2 gives $\varepsilon_+ = \frac{\delta_2 n_1 n_2}{\tilde{\Delta}}$ and $\varepsilon_- = \frac{\delta_2 n_1 n'_2}{\tilde{\Delta}}$, where $\tilde{\Delta} = n_1^2 + n_2^2 + \delta_2 n_1 n_2$. As $K_{W'} \cdot \Gamma'_1 = \frac{\pm \delta'_1}{n_0 n'_1}$, we use both of the previous equations to intersect with Γ'_1 , and we find (2). We note that $K_W \cdot \Gamma_1 > 0$ implies $K_{W'} \cdot \Gamma'_1 > 0$. \square

Lemma 3.6. *Let $W_0 \rightarrow \overline{W}$ be a Wahl resolution. Then we have the braid relation $R_2 R_1 R_2(W_0) = R_1 R_2 R_1(W_0)$.*

Proof. We have in principle 7 distinct surfaces in this proof: $W_0, R_2(W_0) = W_1, R_1(W_1) = W_2, R_2(W_2) = W_3, R_1(W_0) = W'_1, R_2(W'_1) = W'_2, \text{ and } R_1(W'_2) = W'_3$. We want to show that $W_3 = W'_3$. For each $W_i \rightarrow \overline{W}$ we have a chain of 2 curves $\Gamma_{1,i}, \Gamma_{2,i}$, singularities $P_{0,i}, P_{1,i}, P_{2,i}$, and $\delta_{j,i}$. We have the analogue notation with ' for the Wahl resolutions $W'_i \rightarrow \overline{W}$.

By Proposition 3.5, we have that

$$K_{W_0} \cdot \Gamma_{1,0} = K_{W_1} \cdot (\Gamma_{1,1} + \Gamma_{2,1}) = K_{W_2} \cdot \Gamma_{2,2}.$$

We also have that $P_{0,0} = P_{1,2}$ and $P_{1,0} = P_{2,2}$. Therefore the antiflipping of $\Gamma_{1,0}$ is equal to the antiflipping of $\Gamma_{2,2}$, and so $P_{1,3} = P'_{1,3}$ and $P_{2,3} = P'_{2,3}$. Moreover, by Proposition 3.5 again, we have

$$K_{W'_1} \cdot \Gamma'_{1,1} = K_{W'_2} \cdot (\Gamma'_{1,2} + \Gamma'_{2,2}) = K_{W_3} \cdot \Gamma'_{2,3},$$

and $\Gamma_{1,1}$ is the antiflip of $\Gamma_{1,0}$. Hence the contraction of $\Gamma'_{2,3}$ and $\Gamma_{2,3}$ define the same c.q.s and have same δ .

Recall that $W_3 \rightarrow \overline{W}$ and $W'_3 \rightarrow \overline{W}$ contract the Wahl chain to the same c.q.s. On the other hand, as $\Gamma'_{2,3}$ and $\Gamma_{2,3}$ contract to the same c.q.s., we also have that $\Gamma'_{1,3}$ and $\Gamma_{1,3}$ contract to the same c.q.s. They are both extremal Wahl resolutions over the same c.q.s., and so, since c.q.s. at most have two extremal P-resolutions with the same δ [HTU], we have that $\delta_{1,3} = \delta'_{1,3}$. Moreover, since $P_{1,3} = P'_{1,3}$, we have that $P_{0,3} = P'_{0,3}$. Therefore we obtain $R_2 R_1 R_2(W_0) = R_1 R_2 R_1(W_0)$. \square

We will describe the sequence of right antiflips $W_0, W_1, W_2 \dots$ that terminates with a N-resolution of a given M-resolution $W_0 := W^+ \rightarrow \overline{W}$ with $r + 1$ singularities. For $r = 1$, the N-resolution is equal to the right antiflip of an extremal M-resolution by Lemma 2.10.

Proposition 3.7. *If $r = 2$, then we arrive to the N-resolution after applying three right antiflips as shown in Figure 7, where $R_2(W_0) = W_1, R_1(W_1) = W_2, \text{ and } R_2(W_2) = W_3$. In this way, by Lemma 3.6, we have $W^- = R_2 R_1 R_2(W_0) = R_1 R_2 R_1(W_0)$.*

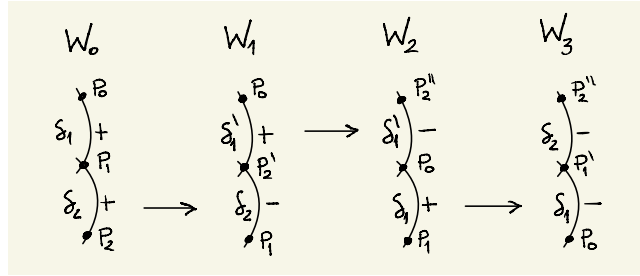


FIGURE 7. The antiflipping process for 3 singularities

Proof. We will do the explicit computation following Prop. 3.5. We first antiflip at $\Gamma_{2,0} \subset W_0$ as in Figure 7. By Prop. 3.5 we have that $\Gamma_{1,1} \cdot K_{W_1} = \frac{\delta_1 \delta_2 n_1 + \delta_1 n_2 + \delta_2 n_0}{n_1 n_0 \bar{n}_2} \geq 0$ using the notation in Figure 7. Next we antiflip at $\Gamma_{1,1} \subset W_1$. By using Prop. 3.5, one can verify that $\Gamma_{2,2} \cdot K_{W_2} = \frac{\delta_1}{n_0 n_1} \geq 0$. Finally, we antiflip at $\Gamma_{2,2} \subset W_2$. We compute using Prop. 3.5 that $\Gamma_{1,3} \cdot K_{W_3} = -\frac{\delta_2}{\bar{n}_2 \bar{n}_1}$. Moreover the c.q.s. contraction of $\Gamma_{1,0}$ in W_0 is the c.q.s. of the contraction of $\Gamma_{2,2}$ in W_2 , which is the c.q.s. of the contraction of $\Gamma_{2,3}$ in W_3 . Hence $R_2 R_1 R_2(W_0)$ is the N-resolution by Definition 1.8. \square

Theorem 3.8. *After applying $r(r+1)/2$ right antiflips of curves contained in the Wahl resolutions starting with $W^+ \rightarrow \overline{W}$, we get the corresponding N-resolution $W^- \rightarrow \overline{W}$. On every step, we antiflip either an extremal P-resolution or a curve with $\delta = 0$.*

Proof. The proof goes by induction on r . By Lemma 2.10, Prop. 3.7, we have it for $r = 1, 2$. Let us say we have it for r . Consider an M-resolution $W^+ \rightarrow \overline{W}$ with the exceptional divisor $\Gamma_1 \cup \dots \cup \Gamma_{r+1}$. We know that there is a unique N-resolution $W^- \rightarrow \overline{W}$ of $W^+ \rightarrow \overline{W}$ with curves $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_{r+1}$ and singularities \tilde{n}_i, \tilde{a}_i for $i = 0, \dots, r$ with $\tilde{\delta}_i$.

We note that the N-resolution for the contraction of the chain $\Gamma_1 \cup \dots \cup \Gamma_r \subset W$ to some c.q.s. has the same \tilde{n}_i, \tilde{a}_i for $i = 0, \dots, r-1$, and same $\tilde{\delta}_i$ for $i = 1, \dots, r-1$ as the N-resolution W^- . This is just part of the algorithm to find the N-resolution.

Let us first antiflip the curves $\Gamma_{r+1}, \dots, \Gamma_1$ in that order, starting with $W_0 = W^+ \rightarrow \overline{W}$ an ending with $W_{r+1} \rightarrow \overline{W}$. Then by using Proposition 3.7 for each consecutive pair Γ_i, Γ_{i+1} , where we have only the first two flips in Figure 7 for each pair, we have

- $n_{i+1, r+1} = n_i, a_{i+1, r+1} = a_i$ for $i = 0, \dots, r$,
- $\delta_{i+1, r+1} = \delta_i$ for $i = 1, \dots, r$,
- The Wahl-chain $\Gamma_{2, r+1}, \dots, \Gamma_{r+1, r+1}$ is an M-resolution, and
- $\Gamma_{1, r+1} \cdot K_{W_{r+1}} \leq 0$.

So we have that $\Gamma_{2, r+1}, \dots, \Gamma_{r+1, r+1}$ is an M-resolution isomorphic to $\Gamma_1, \dots, \Gamma_r$. We now apply induction on this chain, to obtain via $r(r-1)/2$ antiflips its unique N-resolution. As it was said above, the \tilde{n}_i, \tilde{a}_i coincide with the ones of $W^- \rightarrow \overline{W}$, and the only missing part is the first singularity and the intersection with with canonical class of the first curve. In this way, we have a new Wahl-resolution with $r+1$ curves $\tilde{W} \rightarrow \overline{W}$, and we want to prove it is indeed the N-resolution.

At this point, we do not know about \tilde{n}_0, \tilde{a}_0 and $\tilde{\delta}_0$. The corresponding curve $\tilde{\Gamma}_1$ may be positive or negative for canonical class. But we now reverse the continued fractions for $\tilde{W} \rightarrow \overline{W}$ and for $W^- \rightarrow \overline{W}$, and after contracting we get a

$$\begin{aligned} & \left[\begin{pmatrix} \tilde{n}_0 \\ \tilde{n}_0 - \tilde{a}_0 \end{pmatrix} \right] - (c) - \left[\begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_1 - \tilde{a}_1 \end{pmatrix} \right] - (1) - [\text{some c.q.s.}] = 0 \\ & \left[\begin{pmatrix} \tilde{n}_0 \\ \tilde{n}_0 - \tilde{a}_0 \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_1 - \tilde{a}_1 \end{pmatrix} \right] - (1) - [\text{some c.q.s.}] = 0 \end{aligned}$$

over the same c.q.s. But then δ 's must be equal, as we are in the case of an extremal P-resolution and/or extremal N-resolution over the same c.q.s., and so we can apply [HTU]. Moreover, they share a Wahl singularity in the same position, then they must be equal (singularity and sign of intersection with canonical class). \square

Theorem 3.9. *The operations of right antiflips R_i on Wahl resolutions $W \rightarrow \overline{W}$ with $r+1$ singularities satisfy braid relations $R_i R_j = R_j R_i$ for $i > j+1$ and $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$. In particular, they give the action of the braid group B_{r+1} .*

Proof. For the relation $R_i R_j = R_j R_i$ we just note that $i-j \geq 2$, and so the birational operations R_i, R_j on W commute. Indeed, the only scenario to consider is when $i = j+2$, in which case we still need to show that the $(j+1)$ -st curves in $R_i R_j(W)$ and $R_j R_i(W)$ have the same self-intersections in the minimal resolutions. But since all other data is the same, this follows from the fact that both of them resolve \overline{W} .

For the other relation, we first note that the curves at i and $i+1$ positions of $R_i R_{i+1} R_i(W)$ and $R_{i+1} R_i R_{i+1}(W)$ have the same subWahl chain by Lemma 3.6. Hence for the whole Wahl chains in $R_i R_{i+1} R_i(W)$ and $R_{i+1} R_i R_{i+1}(W)$ we have the same singularities. We only need to check the effect on the curves in positions $i-1$ and $i+2$, since in their complement we have equal Wahl subchains. Let $[A]$

be the continued fraction for the minimal resolution of the chain from 0 to $i - 2$, let $[B]$ be the one for the chain from i to $i + 1$, and let $[C]$ be continued fraction for the chain from $i + 3$ to r . Then we are in the situation

$$[A] - (u) - [B] - (v) - [C] = [A] - (u') - [B] - (v') - [C]$$

as continued fractions for the Wahl chains in $R_i R_{i+1} R_i(W)$ and $R_{i+1} R_i R_{i+1}(W)$, since both contract to $\frac{1}{\Delta}(1, \Omega)$. We want $u = u'$ and $v = v'$. Using the representation of continued fractions as multiplications of 2×2 matrices, we simplify that equation into $(u) - [B] - (v) = (u') - [B] - (v')$. But as continued fractions this means that $u - \frac{1}{[B,v]} = u' - \frac{1}{[B,v']}$, and if $u \geq u'$, then $(u - u') - \frac{1}{[B,v]} = -\frac{1}{[B,v]}$. But $\frac{1}{[B,v]} > 0$, and so $(u - u') - \frac{1}{[B,v]} < 0$, but then $u = u'$. Hence $v = v'$. Therefore Wahl chains are equal. \square

Corollary 3.10. *Every Wahl resolution $W \rightarrow \overline{W}$ is in the braid group orbit of a unique M -resolution $W^+ \rightarrow \overline{W}$.*

Proof. By definition of a Wahl resolution, it has a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W$ that blows down to a smoothing $Y \rightsquigarrow \overline{W}$. Let \mathcal{W} be its total space. If $K_W \cdot \Gamma_i < 0$ for one of the curves $\Gamma_i \subset W$ then its contraction $\mathcal{W} \rightarrow \mathcal{W}_i$ is a k2A extremal neighborhood of flipping type and can be flipped to $\mathcal{W}' \rightarrow \mathcal{W}_i$. By the discussion from the beginning of this section, the special fiber $W' \subset \mathcal{W}'$ can be obtained from W via a sequence of antiflips at the same i -th curve.

The indices of the Wahl singularities corresponding to $\Gamma'_i \subset W'$ are always less than or equal to the indices of the Wahl singularities corresponding to $\Gamma_i \subset W$, and one of these new indices is strictly smaller. Indeed, as in Figure 5, the first right or left antiflip behaves like that. If $\delta_i = 1$, then there are no more k2A antiflips. If $\delta_i > 1$, then there are infinitely many antiflips with indices $n_k > n_{k-1}$ which satisfy the recurrence $n_{k-1} + n_{k+1} = \delta_i n_k$, and so indices form an increasing sequence.

Then we can pick another K -negative curve in W' and iterate the process. If we arrive to a surface with no Wahl singularities, then it must be the minimal resolution of \overline{W} , as any (-1) -curve in the Wahl chain would produce a divisorial contraction blow-down deformation of \overline{W} , and this is not allowed by the minimality assumption on Wahl resolutions.

Therefore, as the set of indices in the Wahl chain strictly decreases at least one of them for each flip, we have that this process eventually gives a M -resolution W^+ . (Alternatively, any sequence of flips in dimension 3 terminates.) Since W^+ has a \mathbb{Q} -Gorenstein smoothing \mathcal{W}^+ in the same irreducible component of $\text{Def}_{P \in \overline{W}}$ as \mathcal{W} , the M -resolution W^+ is uniquely determined. \square

4. DERIVED CATEGORY OF A C.Q.S. SURFACE FOLLOWING [KKS]

Notation 4.1. We fix a c.q.s. surface W that satisfies Assumption 1.10 (1), (2), (3). We do not need the singularities to be Wahl or the chain $\Gamma_1, \dots, \Gamma_r$ to be contractible. In fact, a weaker condition than Assumption 1.10 (3), is sufficient in Sections 4 and 5 except in Corollaries 4.7 and 5.6: there exists a Weil divisor $A \subset W$, which is Cartier outside of P_0 and generates the local class group $\text{Cl}(P_0 \in W)$. Then $\tilde{A} = -K_W - \Gamma_1 - \dots - \Gamma_r - A$ is Cartier outside of P_r and generates $\text{Cl}(P_r \in W)$.

Consider the following Weil divisors on W :

$$D_0 = A, \quad D_1 = A + \Gamma_1, \quad \dots, \quad D_r = A + \Gamma_1 + \dots + \Gamma_r. \quad (4.1)$$

If all points P_0, \dots, P_r are smooth then this gives a well-known exceptional collection $\mathcal{O}_W(D_0), \mathcal{O}_W(D_1), \dots, \mathcal{O}_W(D_r)$. But if some points are singular then this

collection of divisorial sheaves is typically not exceptional. Indeed, if $i > j$ then

$$R\Gamma\mathcal{H}om(\mathcal{O}_W(D_i), \mathcal{O}_W(D_j)) = R\Gamma\mathcal{O}_W(-\Gamma_{j+1} - \dots - \Gamma_i) = 0 \quad (4.2)$$

because of the short exact sequence

$$0 \rightarrow \mathcal{O}_W(-\Gamma_{j+1} - \dots - \Gamma_i) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_{\Gamma_{j+1} \cup \dots \cup \Gamma_i} \rightarrow 0.$$

However, if $i > k > j$ and P_k is not Gorenstein (i.e. not an A_s singularity) then $\text{Ext}^1(\mathcal{O}_W(D_i), \mathcal{O}_W(D_j)) \neq 0$. This follows from (4.2), the local-to-global spectral sequence for Ext and the fact that $\text{Ext}_{\mathcal{O}_{W, P_k}}^1(\mathcal{O}_{W, P_k}(\Gamma_k + \Gamma_{k+1}), \mathcal{O}_{W, P_k}) \neq 0$:

Lemma 4.2. *Let $(p \in Z)$ be a germ of a non-Gorenstein c.q.s. with toric boundary divisors $\Gamma, \Gamma' \in \text{Cl}(p \in Z)$. Then $\text{Ext}^1(\mathcal{O}_{Z, p}(\Gamma + \Gamma'), \mathcal{O}_{Z, p}) \neq 0$.*

Proof. Let $R = \mathcal{O}_{Z, p}$. Note that $\Gamma + \Gamma' \sim -K_Z$ in $\text{Cl}(p \in Z)$. Since $-K_Z$ is not Cartier, $M = \mathcal{O}_{Z, p}(\Gamma + \Gamma')$ is not a projective R -module. Thus we have a non-split Auslander–Reiten exact sequence $[A] 0 \rightarrow \tau(M) \rightarrow N_M \rightarrow M \rightarrow 0$, where $\tau(M) = (M \otimes \omega_Z)^{\vee\vee} = R$. Thus $\text{Ext}^1(M, R) \neq 0$. \square

Fortunately, orthogonality holds for the “reversed” collection

$$\mathcal{O}_W(-D_r), \dots, \mathcal{O}_W(-D_1), \mathcal{O}_W(-D_0), \quad (4.3)$$

which therefore is essentially a unique choice that works for singular surfaces. Indeed, the next result appears almost verbatim in [KKS, Th. 2.12].

Definition 4.3. Let Z be a projective variety. A s.o.d. $D^b(Z) = \langle \mathcal{A}_0, \dots, \mathcal{A}_r, \mathcal{B} \rangle$ is called a *Kawamata decomposition* if every subcategory \mathcal{A}_i for $i = 0, \dots, r$ is classically generated by a maximal Cohen-Macaulay sheaf and $\mathcal{B} \subset D^{\text{perf}}(Z)$.

Proposition 4.4. *For $i = 0, \dots, r$, let $\mathcal{A}_i^W \subset D^b(W)$ be a triangulated subcategory classically generated by $\mathcal{O}_W(-D_i)$. Then we have a Kawamata decomposition*

$$D^b(W) = \langle \mathcal{A}_r^W, \dots, \mathcal{A}_0^W, \mathcal{B}^W \rangle, \quad (4.4)$$

so that $\mathcal{B}^W \subset D^{\text{perf}}(W)$, as well as an s.o.d.

$$D^b(W) = \langle \tilde{\mathcal{B}}^W, \mathcal{A}_r^W, \dots, \mathcal{A}_0^W \rangle \quad (4.5)$$

with the property that $\mathbb{D}_W(B)$ is perfect⁴ for every $B \in \tilde{\mathcal{B}}^W$.

Remark 4.5. A coherent sheaf \mathcal{F} on a noetherian scheme \mathcal{X} is called maximal Cohen-Macaulay (mCM) if $\text{depth}_{\mathcal{O}_{\mathcal{X}, x}} \mathcal{F}_x = \dim \mathcal{O}_{\mathcal{X}, x}$ for every $x \in \mathcal{X}$ [BH]. If \mathcal{F} is mCM and $\mathcal{O}_{\mathcal{X}, x}$ is regular then \mathcal{F}_x is locally free (since it has projective dimension 0 by the Auslander–Buchsbaum formula). If \mathcal{F} is mCM then \mathcal{F} is reflexive, which implies that \mathcal{F} is torsion-free (these notions are equivalent to mCM in dimension 2 and 1, respectively). If \mathcal{F} is mCM and $i : \mathcal{D} \hookrightarrow \mathcal{X}$ is an effective Cartier divisor then $i^*\mathcal{F}$ is mCM on \mathcal{D} . If \mathcal{X} is CM with a dualizing sheaf ω then \mathcal{F} is mCM if and only if $\mathcal{E}xt^p(\mathcal{F}, \omega) = 0$ for $p > 0$. And then $\mathcal{H}om(\mathcal{F}, \omega)$ is mCM.

Proof of Prop. 4.4. First we check orthogonality. Let $i > j$ and $p > 0$. For $k > j$,

$$\mathcal{E}xt^p(\mathcal{O}_{W, P_k}(-D_j), \mathcal{O}_{W, P_k}(-D_i)) \simeq \mathcal{E}xt^p(\mathcal{O}_{W, P_k}, \mathcal{O}_{W, P_k}(-D_i)) = 0$$

by obvious reasons and for $k < i$,

$$\mathcal{E}xt^p(\mathcal{O}_{W, P_k}(-D_j), \mathcal{O}_{W, P_k}(-D_i)) \simeq \mathcal{E}xt^p(\mathcal{O}_{W, P_k}(-D_j), \omega_{W, P_k}) = 0$$

because $\mathcal{O}_W(-D_j)$ is reflexive and therefore mCM. By (4.2), this implies $\mathcal{O}(-D_i) \subset \langle \mathcal{O}(-D_j) \rangle^\perp$ and therefore $\mathcal{A}_W^i \subset (\mathcal{A}_W^j)^\perp$. Next we borrow analysis from [KKS].

Let X be the resolution of singularities of W with exceptional divisors $E_i^1 + \dots + E_i^{m_i}$ over each $P_i \in W$. Let $\Gamma_i^X \subset X$ be the proper transform of $\Gamma_i \subset W$

⁴We denote by $\mathbb{D}_{\mathcal{X}} = R\mathcal{H}om(\cdot, \omega_{\mathcal{X}}^\bullet)$ the duality functor on a noetherian scheme \mathcal{X} .

for $i = 1, \dots, r$ and let D_0^X be the proper transform of A . Define divisors D_i^X for $i = 1, \dots, r$ inductively as follows: $D_i^X = D_{i-1}^X + E_i^1 + \dots + E_i^{m_i} + \Gamma_i^X$. This gives a s.o.d. $D^b(X) = \langle \mathcal{B}^X, \mathcal{A}_r^X, \dots, \mathcal{A}_0^X \rangle$, where

$$\mathcal{A}_i^X = \langle \mathcal{O}(-D_i^X - E_i^1 - \dots - E_i^{m_i}), \dots, \mathcal{O}(-D_i^X - E_i^1), \mathcal{O}(-D_i^X) \rangle.$$

The pushforward of this s.o.d. to W was studied in [KKS]:

(1) \mathcal{A}_i^X is classically generated by $\mathcal{O}(-D_i^X)$ and sheaves $\mathcal{O}_{E_i^l}(-1)$, $1 \leq l \leq m_i$. Indeed, $D_i^X \cdot E_i^1 = 1$ and $D_i^X \cdot E_i^j = 0$, so for $k = 1, \dots, m_i$ we have exact sequences $0 \rightarrow \mathcal{O}(-D_i^X - E_i^1 - \dots - E_i^k) \rightarrow \mathcal{O}(-D_i^X - E_i^1 - \dots - E_i^{k-1}) \rightarrow \mathcal{O}_{E_i^k}(-1) \rightarrow 0$.

(2) By [KKS, Th. 2.12], we have a s.o.d.

$$D^b(W) = \langle R\pi_* \mathcal{B}^X, R\pi_* \mathcal{A}_r^X, \dots, R\pi_* \mathcal{A}_0^X \rangle, \quad (4.6)$$

where $R\pi_* \mathcal{B}^X$ is equivalent to \mathcal{B}^X and $R\pi_* \mathcal{A}_i^X$ is equivalent to the Verdier quotient $\mathcal{A}_i^X / \langle \mathcal{O}_{E_i^l}(-1) \rangle_{1 \leq l \leq m_i}$. In particular, it is classically generated by $R\pi_* \mathcal{O}(-D_i^X)$.

(3) Set $\bar{D}_i^X = D_i^X + E_i^1 + \dots + E_i^{m_i}$. Then $R\pi_* \mathcal{O}(-D_i^X) = R\pi_* \mathcal{O}(-\bar{D}_i^X)$ and

$$-\bar{D}_i^X \cdot E_j^k = \begin{cases} 0, & j > i \\ -1 - (E_i^{m_i})^2, & j = i, k = m_i \\ -2 - (E_j^k)^2, & \text{otherwise.} \end{cases}$$

By [KKS, Cor. 6.2], $R\pi_* \mathcal{O}(-\bar{D}_i^X) = \pi_* \mathcal{O}(-\bar{D}_i^X)$ is a reflexive sheaf $\mathcal{O}_W(-D_i)$. This gives an s.o.d. (4.5) with $\tilde{\mathcal{B}}^W = R\pi_* \mathcal{B}^X$.

(4) Let $B \in \tilde{\mathcal{B}}^W$ and write $B = R\pi_* B^X$, $B^X \in \mathcal{B}^X$. Let $i : E \hookrightarrow X$ be a component of the exceptional divisor of π . By coherent duality, $\mathbb{D}_X(Ri_* \mathcal{O}_E(-1)) = Ri_* \mathbb{D}_{\mathbb{P}^1}(\mathcal{O}_E(-1)) = Ri_* \mathcal{O}_E(-1)[-1]$. Since $\mathrm{RHom}(Ri_* \mathcal{O}_E(-1), B^X) = 0$ by (1) above, it follows that $\mathrm{RHom}(\mathbb{D}_X(B^X), Ri_* \mathcal{O}_E(-1)) = 0$. By [KKS, Lemma 2.5], this implies that $\mathbb{D}_W(B) = R\pi_* \mathbb{D}_X(B^X)$ is perfect.

(5) To finish the proof, consider divisors $K_W + D_i = -\tilde{A} - \Gamma_r - \dots - \Gamma_{i+1}$ for $i = 0, \dots, r$, where \tilde{A} is as in Assumption 1.10. Arguing as above, we get an s.o.d.

$$D^b(W) = \langle \mathcal{B}, \langle \omega_W(D_0) \rangle, \dots, \langle \omega_W(D_r) \rangle \rangle, \quad (4.7)$$

where $\mathbb{D}_W(B)$ is perfect for every $B \in \mathcal{B}$. Applying the duality anti-equivalence \mathbb{D}_W to the s.o.d. (4.7) gives the Kawamata decomposition (4.4). \square

Remark 4.6. The proof shows that duality functor \mathbb{D}_W takes the s.o.d. (4.4) to the s.o.d. (4.7), where $\omega_W(D_i) \simeq \mathcal{O}_W(-\tilde{A} - \Gamma_r - \dots - \Gamma_{i+1})$ for every $r = 0, \dots, r$.

Corollary 4.7. *Let $\pi : W \rightarrow \bar{W}$ be a c.q.s. resolution of a surface with a single c.q.s. P satisfying Assumption 1.10 (1), (2), (3). Then s.o.d.'s (4.4) and (4.5) for W and \bar{W} are compatible via $R\pi_*$, that is $R\pi_* \langle \mathcal{A}_r^W, \dots, \mathcal{A}_0^W \rangle = \mathcal{A}^{\bar{W}}$ and \mathcal{B}^W (resp. $\tilde{\mathcal{B}}^W$) is equivalent to $\mathcal{B}^{\bar{W}}$ (resp. $\tilde{\mathcal{B}}^{\bar{W}}$) via $R\pi_*$. Also, $\mathcal{B}^W = L\pi^* \mathcal{B}^{\bar{W}}$.*

Assumption 1.10 (3) implies that proper transforms Γ_0 (resp. Γ_{r+1}) of $\bar{A} \subset \bar{W}$ (resp. $\tilde{A} \subset \bar{W}$) in W of \bar{W} intersect the chain $\Gamma_1 \cup \dots \cup \Gamma_r$ only at P_0 (resp. P_r), where they are equivalent to toric boundaries opposite to Γ_1 (resp. Γ_r). So the corollary follows from the proof of Prop. 4.4 since the minimal resolution of W is also a resolution of \bar{W} and the s.o.d.'s (4.4) and (4.5) defined using (4.6) are compatible. It generalizes [KKS, Th. 2.12] (which is the special case of Corollary 4.7 when $W = X$ is a smooth resolution of \bar{W}). It shows that we can view the category $\langle \mathcal{A}_r^W, \dots, \mathcal{A}_0^W \rangle$ as a partial resolution of singularities of the category $\mathcal{A}^{\bar{W}}$.

Definition 4.8. Let $\mathcal{Z} \rightarrow B = \mathrm{Spec} \mathbb{B}$ be a morphism to an affine scheme, let \mathcal{F} be a coherent sheaf on \mathcal{Z} , and let $\mathbb{R} \subset \mathrm{End}(\mathcal{F})$ be a finite \mathbb{B} -algebra. There is a functor of abelian categories $\otimes_{\mathbb{R}} \mathcal{F} : \mathbb{R}\text{-Mod} \rightarrow \mathrm{Qcoh}(\mathcal{Z})$, which takes a \mathbb{R} -module

$M = \text{Coker}(\mathbb{R}^I \xrightarrow{\phi} \mathbb{R}^J)$ to $M \otimes_{\mathbb{R}} \mathcal{F} := \text{Coker}(\mathcal{F}^I \xrightarrow{\phi} \mathcal{F}^J)$. We denote its left derived functor by $\otimes_{\mathbb{R}}^L \mathcal{F} : D(\mathbb{R}\text{-Mod}) \rightarrow D(\text{Qcoh}(\mathcal{Z}))$ and the corresponding homological functors by $\text{Tor}_{\mathbb{R}}^j(\cdot, \mathcal{F}) : \mathbb{R}\text{-Mod} \rightarrow \text{Qcoh}(\mathcal{Z})$. We call $\otimes_{\mathbb{R}} \mathcal{F}$ *bounded* if, for every f.g. \mathbb{R} -module M , $\text{Tor}_{\mathbb{R}}^j(M, \mathcal{F}) = 0$ for $j \gg 0$. In this case $\otimes_{\mathbb{R}}^L \mathcal{F}$ induces a functor $\otimes_{\mathbb{R}}^L \mathcal{F} : D^b(\mathbb{R}\text{-mod}) \rightarrow D^b(\mathcal{Z})$. We denote its essential image by $\{\mathcal{F}\} \subset D^b(\mathcal{Z})$ ⁵.

Definition 4.9. Let \mathcal{D} be a coherent sheaf on a projective variety Z . An iterated extension of \mathcal{D} is defined recursively as either \mathcal{D} or a non-trivial extension of an iterated extension by \mathcal{D} . A universal iterated extension \mathcal{D}^p for $p \geq 0$ is a coherent sheaf defined inductively as $\mathcal{D}^0 = \mathcal{D}$ and \mathcal{D}^p for $p > 0$ is the universal extension

$$0 \rightarrow \mathcal{D} \otimes_{\mathbb{C}} \text{Ext}^1(\mathcal{D}^{p-1}, \mathcal{D})^{\vee} \rightarrow \mathcal{D}^p \rightarrow \mathcal{D}^{p-1} \rightarrow 0. \quad (4.8)$$

An iterated extension $\hat{\mathcal{D}}$ is called *maximal* if $\text{Ext}^1(\hat{\mathcal{D}}, \mathcal{D}) = 0$, see [K2]. If the maximal iterated extension exists then it is unique and equal to \mathcal{D}^p for some $p \geq 0$.

Definition 4.10. For every $i = 0, \dots, n$, let F_i be the maximal iterated extension of $\mathcal{O}_W(-D_i)$, where the Weil divisor D_i on W was defined in (4.1).

Lemma 4.11. *The sheaf F_i exists and is locally free at every $p \in W$ except at P_j for $j < i$, where $\mathbb{D}_W(F_i)$ is locally free. The functor $\otimes_{R_i} F_i$ is bounded, where $R_i = \text{End}(F_i)$ is isomorphic to the Kalck-Karmazyn algebra of the germ $(P_i \in W)$. In notation of Prop. 4.4,*

$$D^b(R_i\text{-mod}) \simeq A_i^W = \langle \mathcal{O}_W(-D_i) \rangle = \{F_i\} \subset D^b(W).$$

Proof. This is a summary of results from [KKS, K1]. □

Lemma 4.12. *For $k > 0, i \leq j$,*

- (1) $\mathcal{E}xt^k(F_i, F_j) = \mathcal{E}xt^k(F_i, \mathcal{O}_W(-D_j)) = 0$;
- (2) $\text{Ext}^k(F_i, F_j) = \text{Ext}^k(F_i, \mathcal{O}_W(-D_j)) = 0$.

Proof. F_i and $\mathcal{O}_W(-D_j)$ are locally isomorphic to ω_W tensored with a free sheaf at P_k for $k < i$. Since F_i is locally free elsewhere, $\mathcal{E}xt^k(F_i, \mathcal{O}_W(-D_j)) = 0$ for $k > 0$. This implies the same vanishing for iterated extensions of $\mathcal{O}_W(-D_j)$ including F_j .

Equation (2) for $i < j$ follows from Prop. 4.4 (the s.o.d.). Let $\mathcal{D} = \mathcal{O}_W(-D_i)$, $\mathcal{D}^p = F_i$. From (1), $\text{Ext}^k(\mathcal{D}^p, \mathcal{D}) = H^k(W, \text{Hom}(\mathcal{D}^p, \mathcal{D}))$, which is equal to 0 if $k > 2$. Also, $\text{Ext}^1(\mathcal{D}^p, \mathcal{D}) = 0$ by the definition of the maximal iterated extension.

The vanishing of $\text{Ext}^2(\mathcal{D}^p, \mathcal{D}) = H^2(W, \text{Hom}(\mathcal{D}^p, \mathcal{D}))$ is equivalent, by Serre duality and reflexivity of $\text{Hom}(\mathcal{D}^p, \mathcal{D})$, to vanishing of $\text{Hom}(\text{Hom}(\mathcal{D}^p, \mathcal{D}), \omega_W)$. We argue by induction on p using (4.8). The base of induction is vanishing of $\text{Hom}(\text{Hom}(\mathcal{D}, \mathcal{D}), \omega_W) = H^0(W, \omega_W)$, which holds by Assumption 1.10 (2). Applying $\text{Hom}(\cdot, \mathcal{D})$ to (4.8) gives an exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{D}^{j-1}, \mathcal{D}) \rightarrow \text{Hom}(\mathcal{D}^j, \mathcal{D}) \rightarrow \mathcal{O}_W \otimes \text{Ext}^1(\mathcal{D}^{j-1}, \mathcal{D}) \rightarrow A \rightarrow 0, \quad (4.9)$$

where A is a sheaf supported at P_i . Since W is a CM surface, $\text{Ext}^1(A, \omega_W) = 0$. Splitting (4.9) into two short exact sequences and using again that $H^0(W, \omega_W) = 0$ shows that $\text{Hom}(\text{Hom}(\mathcal{D}^{j-1}, \mathcal{D}), \omega_W) = 0$ implies $\text{Hom}(\text{Hom}(\mathcal{D}^j, \mathcal{D}), \omega_W) = 0$.

Finally, vanishing of $\text{Ext}^k(\mathcal{D}^p, \mathcal{D})$ for $k > 0$ implies vanishing of $\text{Ext}^k(\mathcal{D}^p, \mathcal{D}^j)$ by induction on j , including vanishing of $\text{Ext}^k(F_i, F_i)$ for $k > 0$. □

Finally, we address “locality” of subcategories $\mathcal{A}_i^W = \langle \mathcal{O}_W(-D_i) \rangle \subset D^b(W)$, i.e. there dependence on the divisor D_i .

⁵Note that $\{\mathcal{F}\} \neq \langle \mathcal{F} \rangle$ in general because $\{\mathcal{F}\}$ is not necessarily classically generated by \mathcal{F} .

Lemma 4.13. *There is a short exact sequence*

$$0 \rightarrow \text{Pic}(W) \rightarrow \text{Cl}(W) \xrightarrow{\gamma} \bigoplus_{i=0}^r \text{Cl}(P_i \in W) \rightarrow 0,$$

where $\text{Cl}(P_i \in W)$ is the local class group and γ is the restriction.

Proof. It suffices to prove surjectivity of γ . Each $\text{Cl}(P_i \in W)$ is a finite cyclic group generated by any one of the toric boundaries. The divisor D_i restricts to a generator in $\text{Cl}(P_i \in W)$ and to 0 in $\text{Cl}(P_j \in W)$ for $j > i$. Arguing by induction on i , some linear combination $D_i + \sum_{j < i} n_j D_j$ with integer coefficients restricts to a generator in $\text{Cl}(P_i \in W)$ and to 0 in $\text{Cl}(P_j \in W)$ for $j \neq i$. \square

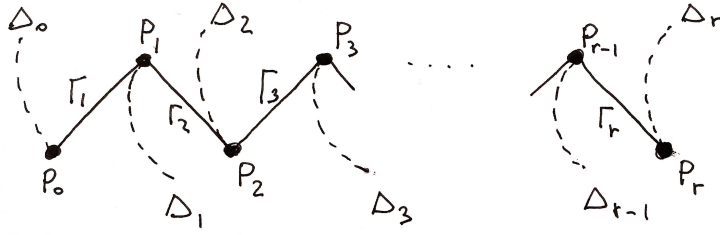


FIGURE 8. Divisors $\Delta_0, \dots, \Delta_r$ on W

Definition 4.14. Fix $q = 0, \dots, r$. By Lemma 4.13, there exists a Weil divisor Δ_q which is equivalent to D_q in $\text{Cl}(P_q \in W)$ and is Cartier at P_i for $i \neq q$, see Figure 8. Let $\omega_W^q = \mathcal{O}_W(-D_q + \Delta_q)$. For $i = q, \dots, r$, let $D_{i,q} = \Delta_q + \Gamma_{q+1} + \dots + \Gamma_i$. Let $\mathcal{A}_{i,q}^W \subset D^b(W)$ be a full triangulated subcategory classically generated by $\mathcal{O}_W(-D_{i,q})$.

Corollary 4.15. *There is a s.o.d. $D^b(W) = \langle \bar{\mathcal{B}}_q^W, \bar{\mathcal{A}}_{r,q}^W, \dots, \bar{\mathcal{A}}_{q,q}^W \rangle$ such that $\mathbb{D}_W(B)$ is perfect at P_i for $i \geq q$ for every $B \in \bar{\mathcal{B}}_q^W$ and $\mathcal{A}_{i,q}^W = \{F_{i,q}\} \simeq D^b(R_{i,q} - \text{mod})$, where $F_{i,q}$ is the maximal iterated extension of $\mathcal{O}_W(-D_{i,q})$ and $R_{i,q} = \text{End}(F_{i,q})$ is isomorphic to the Kalck-Karmazyn algebra of $(P_i \in W)$. Furthermore, categories $\langle \mathcal{A}_{r,q}^W, \dots, \mathcal{A}_{q,q}^W \rangle$ and $\langle \mathcal{A}_r^W, \dots, \mathcal{A}_q^W \rangle$ are equivalent via an adjoint pair of functors $\bullet \otimes^L \omega_W^q$ and $R\mathcal{H}om(\omega_W^q, \bullet)$.*

Proof. Existence of the s.o.d. is proved as in Prop. 4.4. We can ignore singularities at P_0, \dots, P_{q-1} , where the sheaves $\mathcal{O}_W(-\Delta_q - \Gamma_{q+1} - \dots - \Gamma_i)$ are Cartier.

Note that $\omega_{W,P_i}^q \simeq \omega_{W,P_i}$ for $i < q$ and is an invertible sheaf elsewhere. In particular, $R\mathcal{H}om(\omega_W^q, \omega_W^q) \simeq \mathcal{O}_W$. Since $\mathcal{O}_W(-D_{i,q})$ and ω_W^q are non-Cartier at disjoint subsets of points, we have $\mathcal{O}_W(-D_{i,q}) \otimes^L \omega_W^q = \mathcal{O}_W(-D_{i,q}) \otimes \omega_W^q = \mathcal{O}_W(-D_i)$. Likewise, we have $R\mathcal{H}om(\omega_W^q, \mathcal{O}_W(-D_i)) = \mathcal{H}om(\omega_W^q, \mathcal{O}_W(-D_i)) = \mathcal{O}_W(-D_{i,q})$. Indeed, for $p > 0$, $\mathcal{E}xt^p(\omega_W^q, \mathcal{O}_W(-D_i)) = 0$ at P_k for $k \geq q$ because ω_W^q is invertible there but also for $k < q$ because $\mathcal{O}_{W,P_k}(-D_i) \simeq \omega_{W,P_k}$ there and ω_W^q is reflexive.

Let $F_{i,q}$ be the maximal iterated extension of $\mathcal{O}_W(-D_{i,q})$. Adjointness of $\bullet \otimes^L \omega_W^q$ and $R\mathcal{H}om(\omega_W^q, \bullet)$ implies $\text{Ext}^1(\mathcal{O}_W(-D_i), \mathcal{O}_W(-D_i)) = \text{Ext}^1(\mathcal{O}_W(-D_{i,q}), \mathcal{O}_W(-D_{i,q}))$. Arguing by induction on iterated extensions, this gives $F_{i,q} \otimes^L \omega_W^q = F_i$ and $R\mathcal{H}om(\omega_W^q, F_i) = \mathcal{H}om(\omega_W^q, F_i) = F_{i,q}$. In particular, $\mathcal{A}_{i,q}^W$ is equivalent to \mathcal{A}_i^W and the algebras $R_{i,q} = \text{End}(F_{i,q})$ and $R_i = \text{End}(F_i)$ are isomorphic. \square

5. DERIVED CATEGORY OF A DEFORMATION OF A C.Q.S. SURFACE

We use notation of Section 4: W is a c.q.s. surface satisfying Assumption 1.10 (1), (2), (3). Weaker conditions of Notation 6.3 will be sufficient except in Corollary 5.6. For every $i = 0, \dots, r$, let F_i be the Kawamata sheaf on W , see Definition 4.10.

Lemma-Definition 5.1. *Let \mathcal{W} be a projective deformation of W over a smooth affine variety $B = \text{Spec } \mathbb{B}$. For $i = 0, \dots, r$, possibly after shrinking B , there exists a unique coherent sheaf \mathcal{F}_i on \mathcal{W} flat over B such that $\mathcal{F}_i|_W \simeq F_i$. We call \mathcal{F}_i a Kawamata sheaf.*

Proof. Existence and uniqueness of \mathcal{F}_i follow from Lemma 4.12 and deformation theory of coherent sheaves. \square

Lemma 5.2. *Kawamata sheaves have the following properties (after shrinking B):*

- (1) \mathcal{F}_i is locally free at every $p \in W$ except at P_j for $j < i$, where $\mathbb{D}_{\mathcal{W}}(\mathcal{F}_i)$ is locally free. In particular, the restriction of \mathcal{F}_i to every smooth fiber Y of $\mathcal{W} \rightarrow B$ is locally free.
- (2) $\mathcal{E}xt^p(\mathcal{F}_i, \mathcal{F}_j) = 0$ for $p > 0, i \leq j$;
- (3) $\text{Ext}^p(\mathcal{F}_i, \mathcal{F}_i) = 0$ for $p > 0$ and $\text{Ext}^p(\mathcal{F}_i, \mathcal{F}_j) = 0$ for all p and $i < j$.
- (4) $\mathbb{R}_i = \text{End}(\mathcal{F}_i)$ is a finite projective \mathbb{B} -module and $R_i = \mathbb{R}_i \otimes_{\mathbb{B}} k(0)$.
- (5) $\otimes_{\mathbb{R}_i} \mathcal{F}_i$ is bounded.

Proof. Locally free sheaves deform to locally free sheaves. Since $R\mathcal{H}om(\omega_W, \omega_W) = \mathcal{O}_W$, the sheaf ω_W deforms locally uniquely. Since $\omega_{\mathcal{W}}$ is flat over B , the deformation of ω_W must be locally isomorphic to $\omega_{\mathcal{W}}$ by adjunction. It follows that \mathcal{F}_i is locally isomorphic to $\omega_{\mathcal{W}}$ tensored with a locally free sheaf at P_j for $j < i$. Equivalently, we have (1), which immediately implies (2).

By (2), $\text{Ext}^p(\mathcal{F}_i, \mathcal{F}_j) = H^p(\mathcal{W}, \mathcal{H}om(\mathcal{F}_i, \mathcal{F}_j))$ for $i < j$ and all p . Thus (3) follows from Lemma 4.12, cohomology and base change [TS, Tag 0AA7] and Nakayama's lemma. This also proves (4).

It remains to prove (5). We drop the index i . By [ELS], the category $D^b(\mathbb{R}\text{-mod})$ has a classical generator, without loss of generality a \mathbb{R} -module \hat{M} . It suffices to prove that $\hat{M} \otimes_{\mathbb{R}}^L \mathcal{F}$ has bounded cohomology. A free \mathbb{R} -resolution $P^\bullet \rightarrow \hat{M}$ can also be viewed as a projective \mathbb{B} -resolution. Since $\mathcal{O}_{B,0}$ is a regular local ring, $\text{Tor}_{\mathbb{B}}^j(\hat{M}, k(0)) = 0$ for $j > \dim B$. Thus $P^\bullet \otimes_{\mathbb{B}} k(0) = P^\bullet \otimes_{\mathbb{R}} R$ has bounded cohomology, i.e. gives an object in $D^b(R\text{-mod})$. Since $\otimes_R F$ is bounded, it follows that $(P^\bullet \otimes_{\mathbb{B}} k(0)) \otimes_{\mathbb{R}}^L F$ has bounded cohomology, and so $\text{Tor}_{\mathbb{R}}^j(\hat{M}, F) = 0$ for $j \geq j_0$. Since \mathcal{F} is flat over B , this implies that $\text{Tor}_{\mathbb{R}}^j(\hat{M}, \mathcal{F}) \otimes_{\mathbb{B}} k(0) = 0$ for $j \geq j_0$ and thus $\text{Tor}_{\mathbb{R}}^j(\hat{M}, \mathcal{F}) = 0$ for $j \geq j_0$ by Nakayama's lemma (after shrinking B). \square

We also need an analogue of Kawamata sheaves for chains of curves in W that start not at P_0 but at some point $P_q, q > 0$. As in Definition 4.14, we take a Weil divisor Δ_q which is equivalent to D_q in $\text{Cl}(P_q \in W)$ and is Cartier at P_i for $i \neq q$ (see Figure 8). Let $F_{i,q}$ be the maximal iterated extension of $\mathcal{O}_W(-\Delta_q - \Gamma_{q+1} - \dots - \Gamma_i)$ for $i = q, \dots, r$ (see Corollary 4.15 for its properties).

Lemma 5.3. *Fix $q = 0, \dots, r$.*

- (1) *There exist unique mCM sheaves $\omega_{\mathcal{W}}^q$ and $\mathcal{F}_{i,q}$ (for $i \geq q$) on \mathcal{W} flat over B that restrict to sheaves ω_W^q and $F_{i,q}$ on W .*
- (2) *The sheaves $\mathcal{F}_{i,q}$ satisfy (1)–(5) of Lemma 5.2 (ignoring singularities at P_0, \dots, P_{q-1}).*
- (3) *We have $M^\bullet \otimes_{\mathbb{R}_i}^L \mathcal{F}_i \simeq (M^\bullet \otimes_{\mathbb{R}_i}^L \mathcal{F}_{i,q}) \otimes^L \omega_{\mathcal{W}}^q$ for every $M^\bullet \in D^b(\mathbb{R}_i\text{-mod})$.*
- (4) *$R\mathcal{H}om(\mathcal{F}_i, M^\bullet \otimes_{\mathbb{R}_i}^L \mathcal{F}_i) \simeq M^\bullet$ for every $M^\bullet \in D^b(\mathbb{R}_i\text{-mod})$.*

Proof. (1) and (2) are proved in the same way as Lemmas 5.1 and 5.2, respectively.

It is clear that $\mathcal{F}_i \simeq \mathcal{F}_{i,q} \otimes^L \omega_{\mathcal{W}}^q \simeq \mathcal{F}_{i,q} \otimes \omega_{\mathcal{W}}^q$ and $\mathbb{R}_i = \text{End}(\mathcal{F}_i) \simeq \text{End}(\mathcal{F}_{i,q})$. Thus (3) follows from associativity of the derived tensor product.

By Lemma 5.2, $R\mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i) = \mathbb{R}_i$ and $R\mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i) = \mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)$. Let P^\bullet be a bounded above complex of free \mathbb{R}_0 -modules quasi-isomorphic to M^\bullet . Suppose first that $i = 0$. Then \mathcal{F}_0 is locally free and

$$R\mathcal{H}om(\mathcal{F}_0, M^\bullet \otimes_{\mathbb{R}_0}^L \mathcal{F}_0) \simeq \mathcal{F}_0^\vee \otimes (P^\bullet \otimes_{\mathbb{R}_0} \mathcal{F}_0) \simeq P^\bullet \otimes_{\mathbb{R}_0} \mathcal{H}om(\mathcal{F}_0, \mathcal{F}_0).$$

Applying $R\Gamma$ to both sides gives a desired isomorphism. The general case $i = q$ follows by the same calculation applied to a locally free sheaf $\mathcal{F}_{q,q}$ using (3). \square

Definition 5.4. Let $\mathcal{A}_i^{\mathcal{W}} = \{\mathcal{F}_i\} \subset D^b(\mathcal{W})$ for $i = 0, \dots, r$.

Theorem 5.5. Let \mathcal{W} be a projective deformation of W over a smooth affine variety B . After shrinking B , $\mathcal{A}_i^{\mathcal{W}} \simeq D^b(\mathbb{R}_i - \text{mod})$ and we have B -linear s.o.d.'s

$$D^b(\mathcal{W}) = \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle = \langle \tilde{\mathcal{B}}^{\mathcal{W}}, \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle \quad (5.1)$$

that restrict to (4.4) and (4.5) on W . Furthermore, $\mathcal{B}^{\mathcal{W}}$ and $\mathbb{D}_{\mathcal{W}}(\tilde{\mathcal{B}}^{\mathcal{W}}) \subset D^{\text{perf}}(\mathcal{W})$.

Proof. We will prove by induction on k the following statements: (1) $D^b(\mathcal{W})$ has a B -linear s.o.d. $\langle \tilde{\mathcal{B}}_k^{\mathcal{W}}, \mathcal{A}_k^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle$; (2) $\mathcal{A}_k^{\mathcal{W}} \simeq D^b(\mathbb{R}_k - \text{mod})$; (3) $\mathcal{A}_k^{\mathcal{W}}$ restricts to \mathcal{A}_k^W on W ; (4) $\mathbb{D}_{\mathcal{W}}(B)$ is perfect at P_0, \dots, P_k for every $B \in \tilde{\mathcal{B}}_k^{\mathcal{W}}$. Suppose this is known for $k < i$, we will prove it for $k = i$ (for $k = -1$ there is nothing to prove).

First we claim that $\mathcal{A}_i^{\mathcal{W}} \subset \tilde{\mathcal{B}}_{i-1}^{\mathcal{W}}$. Fix $q < i$. Since objects in $\mathcal{A}_q^{\mathcal{W}}$ are represented by bounded above complexes with components isomorphic to $\mathcal{F}_q^{\oplus n}$ for some n , by the spectral sequence for Ext [TS, Tag 07A9] it suffices to prove that $\text{Hom}(\mathcal{F}_q, T) = 0$ for every $T \in \mathcal{A}_i^{\mathcal{W}}$. For $q = 0$, we argue as follows. T can be represented by a bounded above complex with bounded cohomology and with components isomorphic to direct sums of \mathcal{F}_i . By Lemma 5.2, $\text{RHom}(\mathcal{F}_q, \mathcal{F}_i) = 0$. So we can use naive truncations to reduce to the case of $T \simeq G[k]$, where G is a sheaf and $k \gg 0$. Since \mathcal{F}_0 is locally free, $\mathcal{E}xt^l(\mathcal{F}_q, G) = 0$ for $l > 0$. Thus $\text{Hom}(\mathcal{F}_q, T) = \text{Ext}^k(\mathcal{F}_q, G) = H^k(\mathcal{W}, \mathcal{H}om(\mathcal{F}_q, G)) = 0$ since $k \gg 0$. For general q , we use Lemma 5.3 and write $\mathcal{F}_q \simeq \mathcal{F}_{q,q} \otimes^L \omega_{\mathcal{W}}^q$ and $T \simeq T_q \otimes^L \omega_{\mathcal{W}}^q$, with $T_q \in \{\mathcal{F}_{i,q}\}$. Thus $\text{RHom}(\mathcal{F}_q, T) = \text{RHom}(\mathcal{F}_{q,q}, T_q) = 0$ as above because $\mathcal{F}_{q,q}$ is locally free.

The functor $\tilde{\Phi} = \otimes_{\mathbb{R}_i}^L \mathcal{F}_i : D(\mathbb{R}_i - \text{Mod}) \rightarrow D(\text{Qcoh}(\mathcal{W}))$ commutes with arbitrary direct sums, and therefore has a right adjoint functor $\tilde{\Psi} : D(\text{Qcoh}(\mathcal{W})) \rightarrow D(\mathbb{R}_i - \text{Mod})$ by [N]. By adjunction, we have

$$\tilde{\Psi}(T) \simeq \text{RHom}(\mathbb{R}_i, \tilde{\Psi}(T)) \simeq \text{RHom}(\tilde{\Phi}(\mathbb{R}_i), T) \simeq \text{RHom}(\mathcal{F}_i, T).$$

We claim that $\tilde{\Psi}$ induces a functor $\mathcal{B}_{i-1}^{\mathcal{W}} \rightarrow D^b(\mathbb{R}_i - \text{mod})$. Indeed, let $T \in \mathcal{B}_{i-1}^{\mathcal{W}}$. It suffices to prove boundedness of $\text{RHom}(\mathcal{F}_i, T)$. After shrinking B , it suffices to prove boundedness at every point $p \in W$. At $p \neq P_0, \dots, P_{i-1}$, \mathcal{F}_i is locally free and boundedness is clear. At one of the remaining points P_j , \mathcal{F}_i is locally a deformation of $\omega_{\mathcal{W}}^{\oplus s}$ for some s , and therefore is locally isomorphic to $\omega_{\mathcal{W}}^{\oplus s}$. On the other hand, $\mathbb{D}_{\mathcal{W}}(T)$ is perfect at P_j , and so T is locally isomorphic to $\omega_{\mathcal{W}} \otimes^L S$, where S is a perfect complex. Thus $\text{RHom}(\mathcal{F}_i, T)$ is bounded at P_j as well.

By Lemma 5.3 (4), we have

$$\Psi(\Phi(M^\bullet)) \simeq \text{RHom}(\mathbb{R}_i, \Psi(\Phi(M^\bullet))) \simeq \text{RHom}(\mathcal{F}_i, M^\bullet \otimes_{\mathbb{R}_i}^L \mathcal{F}_i) \simeq M^\bullet.$$

Thus the adjunction morphism $M^\bullet \rightarrow \Psi(\Phi(M^\bullet))$ is an isomorphism. Therefore, Φ is fully faithful, $\mathcal{A}_i^{\mathcal{W}}$ is a right admissible subcategory of $\mathcal{B}_{i-1}^{\mathcal{W}}$ equivalent to the category $D^b(\mathbb{R}_i - \text{mod})$, and we have a s.o.d. $\langle \tilde{\mathcal{B}}_i^{\mathcal{W}}, \mathcal{A}_i^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle$.

We need to prove that the s.o.d. $\langle \tilde{\mathcal{B}}_i^{\mathcal{W}}, \mathcal{A}_i^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle$ is B -linear and restricts to the s.o.d. $\langle \tilde{\mathcal{B}}_i^W, \mathcal{A}_i^W, \dots, \mathcal{A}_0^W \rangle$ on W . It is enough to show that $\mathcal{A}_i^{\mathcal{W}}$ is B -linear and restricts to \mathcal{A}_i^W . Equivalently, it suffices to show that $D^b(\mathbb{R}_i - \text{mod})$ is \mathbb{B} -linear, which is clear, and also that it restricts to $D^b(R_i - \text{mod})$ at $0 \in B$. Indeed, since B is smooth, a \mathbb{B} -module $k(0)$ is resolved by a finite Koszul complex (after shrinking B). Thus the functor $\otimes_{\mathbb{B}} k(0)$ sends $D^b(\mathbb{R}_i - \text{mod})$ to $D^b(R_i - \text{mod})$. Moreover, the same Koszul complex shows that every $M^\bullet \in D^b(\mathbb{R}_i - \text{mod})$ is a direct summand of $\tilde{M}^\bullet \otimes^L k(0)$, where \tilde{M}^\bullet is the restriction of scalars to \mathbb{R}_i .

Next we claim that $\mathbb{D}_{\mathcal{W}}(B)$ is perfect at P_i for every $B \in \tilde{\mathcal{B}}_i^{\mathcal{W}}$. It suffices to prove that $\mathbb{D}_{\mathcal{W}}(B) \otimes^L k(P_i)$ has bounded cohomology. Since $i_W : W \hookrightarrow \mathcal{W}$ is a regular embedding, $Li_W^* \mathbb{D}_{\mathcal{W}}(B) \in D^b(W)$, so it suffices to prove that $Li_W^* \mathbb{D}_{\mathcal{W}}(B)$ is perfect. Since $i_W : W \hookrightarrow \mathcal{W}$ is a regular embedding, $Li_W^* \mathbb{D}_{\mathcal{W}}(B) = \mathbb{D}_W(Li_W^*(B))$ (up to a shift), so it suffices to prove that the latter is perfect. Since $Li_W^*(B) \in \tilde{\mathcal{B}}_i^{\mathcal{W}}$, this follows from Prop. 4.4. Finally, the fact that $\mathcal{B}^{\mathcal{W}} \subset D^{perf}(\mathcal{W})$ is proved in the same way using Prop. 4.4 and Remark 4.6. \square

The next corollary provides categorification for “blowing down deformations”.

Corollary 5.6. *Let W be a c.q.s. resolution of a surface \overline{W} with a single c.q.s. P satisfying Assumption 1.10 (1), (2), (3). Let \mathcal{W} be a projective deformation of W over a smooth affine variety B . After shrinking B , there is a morphism $\pi : \mathcal{W} \rightarrow \overline{\mathcal{W}}$ to a projective deformation of \overline{W} . The s.o.d.’s (5.1) for \mathcal{W} and $\overline{\mathcal{W}}$ are compatible, namely $R\pi_* \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle = \mathcal{A}^{\overline{\mathcal{W}}}$ and $\mathcal{B}^{\mathcal{W}}$ (resp. $\tilde{\mathcal{B}}^{\mathcal{W}}$) is equivalent to $\mathcal{B}^{\overline{\mathcal{W}}}$ (resp. $\tilde{\mathcal{B}}^{\overline{\mathcal{W}}}$) via $(R\pi_*, L\pi^*)$.*

Proof. For blowing-down deformations, see [W]. Let $j : W \hookrightarrow \mathcal{W}$ be the (regular) embedding. Choose classical generators $A_i \in \mathcal{A}_i^{\mathcal{W}}$, $\bar{A} \in \mathcal{A}^{\overline{\mathcal{W}}}$, $B \in \mathcal{B}^{\mathcal{W}}$, $\bar{B} \in \mathcal{B}^{\overline{\mathcal{W}}}$.

By Assumption 1.10 (3), proper transforms Γ_0 (resp. Γ_{r+1}) of $\bar{A} \subset \overline{\mathcal{W}}$ (resp. $\bar{A} \subset \overline{W}$) in W of $\overline{\mathcal{W}}$ intersect the chain $\Gamma_1 \cup \dots \cup \Gamma_r$ only at P_0 (resp. P_r), where they are equivalent to toric boundaries opposite to Γ_1 (resp. Γ_r). So we can apply Corollary 4.7. For every $i = 0, \dots, r$, $\mathrm{RHom}(Lj^* L\pi^* \bar{B}, Lj^* A_i) = 0$ by Corollary 4.7 and $\mathrm{RHom}(L\pi^* \bar{B}, A_i)$ has bounded cohomology since $L\pi^* \bar{B}$ is perfect. Thus $\mathrm{RHom}(L\pi^* \bar{B}, A_i) = 0$ by cohomology and base change and Nakayama’s lemma (after shrinking B). It follows that $L\pi^*(\mathcal{B}^{\overline{\mathcal{W}}}) \subset \mathcal{B}^{\mathcal{W}}$ and, by adjunction, $R\pi_* \mathcal{A}_i^{\mathcal{W}} \subset \mathcal{A}^{\overline{\mathcal{W}}}$. Next, $\mathrm{RHom}(Lj^* R\pi_* B, Lj^* \bar{A}) = \mathrm{RHom}(R\pi_* Lj^* B, Lj^* \bar{A}) = 0$ by Corollary 4.7. Furthermore, $\mathrm{RHom}(R\pi_* B, \bar{A}) = \mathrm{RHom}(B, \pi^! \bar{A})$ is bounded above since B is perfect. Thus, $\mathrm{RHom}(R\pi_* B, \bar{A}) = 0$ by cohomology and base change and Nakayama’s lemma (after shrinking B). It follows that $R\pi_*(\mathcal{B}^{\mathcal{W}}) \subset \mathcal{B}^{\overline{\mathcal{W}}}$. Since π has fibers of dimension at most 1, $R\pi_*(D^b(\mathcal{W})) = D^b(\overline{\mathcal{W}})$ by [K, Cor. 2.5]. Therefore, $R\pi_* \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle = \mathcal{A}^{\overline{\mathcal{W}}}$ and $R\pi_*(\mathcal{B}^{\mathcal{W}}) = \mathcal{B}^{\overline{\mathcal{W}}}$. Since $R\pi_* \mathcal{O}_{\mathcal{W}} = \mathcal{O}_{\overline{\mathcal{W}}}$, $R\pi_* L\pi^*$ gives the identity functor on $\mathcal{B}^{\overline{\mathcal{W}}}$ by projection formula. It remains to show that $L\pi^* : \mathcal{B}^{\overline{\mathcal{W}}} \rightarrow \mathcal{B}^{\mathcal{W}}$ is essentially surjective. Indeed, the unit of adjunction $L\pi^* R\pi_* B \rightarrow B$ is an isomorphism for any $B \in \mathcal{B}^{\mathcal{W}}$ by Nakayama’s lemma since its restriction to W is an isomorphism by Corollary 4.7. \square

Remark 5.7. We focus on situations when π induces an isomorphism on generic fibers $Y \simeq \bar{Y}$ of deformations, e.g. when \bar{Y} is a smoothing of \overline{W} . Corollary 5.6 shows that then $\mathcal{A}^{\overline{\mathcal{W}}}$ deforms to a category $\mathcal{A}^{\bar{Y}}$, which has an s.o.d. $\langle \mathcal{A}_r^{\bar{Y}}, \dots, \mathcal{A}_0^{\bar{Y}} \rangle$ related to the fact that $\mathcal{A}^Y = \mathcal{A}^{\bar{Y}}$ is also a deformation of $\langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}} \rangle$.

The following theorem completes the proof of Theorem 1.16.

Theorem 5.8. *In the notation of Theorem 5.5, suppose W has Wahl singularities. Let $B^0 \subset B$ be the locus of smooth fibers Y of the deformation and suppose $B^0 \neq \emptyset$. Let the s.o.d. $D^b(Y) = \langle \mathcal{A}_r^Y, \dots, \mathcal{A}_0^Y, \mathcal{B}^Y \rangle$ be the base change of $D^b(\mathcal{W}) = \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle$. After shrinking B , for $i = 0, \dots, r$, the restriction of the Kawamata bundle $\mathcal{F}_i|_Y$ is isomorphic to $E_i^{\oplus n_i}$ and $\mathcal{A}_i^Y = \langle E_i \rangle$, where E_i is a Hacking exceptional vector bundle with*

$$\mathrm{rk} E_i = n_i, \quad c_1(E_i) = -n_i(A + \Gamma_1 + \dots + \Gamma_i) \in H_2(Y).$$

Remark 5.9. By Riemann–Roch, we also have $c_2(E_i) = \frac{n_i-1}{2n_i}(c_1(E_i)^2 + n_i + 1)$.

A direct corollary is the Hacking exceptional collection (H.e.c.) announced in the introduction, which for dual bundles was stated in [H3] (without proof).

Corollary 5.10. *Hacking vector bundles E_r, \dots, E_0 on Y form an exceptional collection.*

Proof of Theorem 5.8. This is essentially [K1, Th. 4.3] applied to each Kawamata bundle \mathcal{F}_i . More precisely, choose $y \in B^0$ and let $(0 \in \Delta) \rightarrow (0 \in B)$ be a curve passing through y . For inductive reasons, we allow Δ to be singular at 0. By [K1, Th. 4.3], after possibly shrinking Δ , there exists a finite morphism $u : (0 \in \Delta') \rightarrow (0 \in \Delta)$ such that $u^{-1}(0) = 0$, Δ' is smooth, and $\mathcal{W}|_{\Delta' \setminus \{0\}}$ carries a relatively exceptional Hacking vector bundle \mathcal{E}_i such that $\mathcal{F}_i \simeq E_i^{\oplus n_i}$ on $\mathcal{W}|_{\Delta' \setminus \{0\}}$. We claim that the bundle \mathcal{E}_i and this isomorphism descend to $\mathcal{W}|_{\Delta \setminus \{0\}}$. Indeed, otherwise Y carries non-isomorphic exceptional Hacking bundles E_i and E'_i such that $\mathcal{F}_i \simeq E_i^{\oplus n_i} \simeq E'_i{}^{\oplus n_i}$. So $E'_i \in \langle E_i \rangle$, which is impossible, because $\langle E_i \rangle$ contains only one exceptional object (up to a shift). Since E_i is exceptional, it deforms uniquely to relatively exceptional vector bundle \mathcal{E}_i on $\mathcal{W}|_U$ for some open subset $U \subset B^0$. It follows that $F_i|_Y$ deforms to $\mathcal{E}_i^{\oplus n_i}$ over U . But $\text{Ext}^p(F_i|_Y, F_i|_Y) = 0$ for $p > 0$, so this deformation must be equal to \mathcal{F}_i over U . If $U = B^0$ (after shrinking B) then we are done. If not, choose the next curve Δ intersecting B^0 and contained in $B \setminus U$ and argue by Noetherian induction. \square

In the next lemma we consider a “hybrid” situation when the \mathbb{Q} -Gorenstein deformation smoothens some points (just one to simplify notation), is locally isotrivial around others, and the chain of rational curves connecting them also deforms.

Lemma 5.11. *Suppose W has a Wahl singularity at P_i . Let $\mathcal{W} \rightarrow B$ be a \mathbb{Q} -Gorenstein deformation over a smooth curve which is locally isotrivial at P_j for $j \neq i$, smoothens P_i , and such that the chain $A + \Gamma_1 + \dots + \Gamma_r$ on W deforms to the chain $A^Y + \Gamma_1^Y + \dots + \Gamma_{i-1}^Y + \Gamma_{i+1}^Y + \dots + \Gamma_r^Y$ on a general fiber Y . More precisely, $\Gamma_{i-1} + \Gamma_i \subset W$ deforms to $\Gamma_{i-1}^Y \subset Y$ connecting P_{i-1} and P_{i+1} . Take the s.o.d. $D^b(\mathcal{W}) = \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle$ as in Theorem 5.5 and its restriction to Y , $D^b(Y) = \langle \mathcal{A}_r^Y, \dots, \mathcal{A}_0^Y, \mathcal{B}^Y \rangle$. Then $\mathcal{A}_j^Y = \{F_j^Y\}$ for $j \neq i$ and $\mathcal{A}_i^Y = \langle E_i \rangle$, where E_i is the Hacking vector bundle associated with a singularity $P_i \in W$ as in Th. 5.8 and F_j^Y for $j \neq i$ are Kawamata sheaves on Y associated with singularities $P_j \in Y$ as in Prop. 4.4.*

Proof. $\mathcal{A}_i^Y = \langle E_i \rangle$ by the argument of Th. 5.8. We just have to analyze \mathcal{A}_j^Y for $j \neq i$, i.e. prove that the Kawamata sheaf F_j of W deforms to the Kawamata sheaf F_j^Y , i.e. $\mathcal{F}_j|_Y = F_j^Y$. There is a divisorial sheaf \mathcal{D}_j on \mathcal{W} flat over B and such that $\mathcal{D}_j|_W = \mathcal{O}_W(-A - \Gamma_1 - \dots - \Gamma_j)$ and $\mathcal{D}_j|_Y := \mathcal{O}_Y(-A^Y - \Gamma_1^Y - \dots - \Gamma_j^Y)$.

The sheaf F_j (resp., F_j^Y) is a maximal iterated extension of $\mathcal{D}_j|_W$ (resp., $\mathcal{D}_j|_Y$). Furthermore, $\text{Ext}^p(\mathcal{F}_j|_Y, \mathcal{D}_j|_Y) = 0$ for $p > 0$ and $\text{RHom}(\mathcal{F}_j|_Y, \mathcal{D}_j|_Y) = \mathbb{C}$ by Lemma 4.12, cohomology and base change, and Nakayama’s lemma. It follows that $\text{RHom}(\mathcal{F}_j|_Y, F_j^Y) = \mathbb{C}^{\Delta_j}$. Both $\mathcal{F}_j|_Y$ and F_j^Y are vector bundles on Y of rank Δ_j except at points P_k for $k < j$, where they are locally isomorphic to $\omega_Y^{\oplus \Delta_j}$. Since F_j^Y has a filtration with quotients isomorphic to $\mathcal{D}_j|_Y$, and $\text{Hom}(\mathcal{F}_j|_Y, \mathcal{D}_j|_Y) = \mathbb{C}$, it follows that there exists a generically surjective morphism $\psi : \mathcal{F}_j|_Y \rightarrow F_j^Y$. As both are torsion-free sheaves of the same rank, we have $\ker(\psi) = 0$. Thus we have a short exact sequence $0 \rightarrow \mathcal{F}_j|_Y \rightarrow F_j^Y \rightarrow G \rightarrow 0$. But $\mathcal{F}_j|_Y$ and F_j^Y have the same Chern character $\Delta_j \text{ch}(\mathcal{D}_j|_Y)$. Thus $\text{ch}(G) = 0$, and therefore $G = 0$. \square

6. COMPARING BRAID GROUP ACTIONS: MUTATIONS AND ANTIFLIPS

According to Theorem 3.9, the braid group B_{r+1} on $r + 1$ strands acts on the set of Wahl resolutions $W \rightarrow \overline{W}$ of a fixed c.q.s. surface \overline{W} with $r + 1$ singularities by antiflips of curves Γ_i contained in the exceptional divisor. These Wahl resolutions admit \mathbb{Q} -Gorenstein smoothings to the surfaces within the same irreducible component of the versal deformation space of \overline{W} . According to Corollary 5.10, each

Wahl resolution W gives rise to an exceptional collection of Hacking vector bundles on its \mathbb{Q} -Gorenstein smoothing. Since Hacking bundles deform uniquely, in fact W gives a H.e.c. on all sufficiently general smoothings of \overline{W} within a given irreducible component of its versal deformation space. In this section we show that antiflips of Wahl resolutions correspond to mutations of these H.e.c. While the braid group action on exceptional collections is well-known, the special feature of our situation is that mutations of exceptional vector bundles are also exceptional vector bundles (up to a shift) and not more complicated exceptional objects. This has strong consequences for their Hom spaces and clarifies the structure of the deformation of the Kalck–Karmazyn algebra that corresponds to the germ $P \in \overline{W}$.

We recall that from an exceptional collection $\langle F, E \rangle \subset D^b(Y)$ we can obtain two other exceptional collections $\langle E, R_E(F) \rangle$ (right mutation of F over E) and $\langle L_E(F), E \rangle$ (left mutation of F over E), so that $\langle F, E \rangle = \langle E, R_E(F) \rangle = \langle L_E(F), E \rangle$. The objects are defined by distinguished evaluation triangles

$$E \otimes \mathrm{RHom}(E, F) \rightarrow F \rightarrow L_E(F) \rightarrow \quad \text{and} \quad R_E(F) \rightarrow F \rightarrow E \otimes \mathrm{RHom}(F, E)^\vee \rightarrow .$$

For a longer exceptional collection $\langle E_r, \dots, E_0 \rangle$, the action of left and right mutations induces an action of the braid group B_{r+1} of $r+1$ strands on $\langle E_r, \dots, E_0 \rangle$. We will also use mutations of more general s.o.d.'s. Matching braid group actions on Wahl resolutions and exceptional collections on their smoothings relies on a geometric construction (Proposition 6.1), which uses deformations from the universal family of antiflips that correspond to 1-dimensional cones of the fan \mathcal{F} in Figure 4.

Proposition 6.1. *Let W be a Wahl resolution of a c.q.s. surface \overline{W} satisfying Assumption 1.10. Let $W' = R_i(W)$ be the right antiflip of W at the curve Γ_i for some $i = 1, \dots, r$. The Wahl singularities of W' are $P'_j = P_j$ for $j \neq i-1, i$, and $P'_i = P_{i-1}$.*

We consider two geometrically different situations: either (a) $K_W \cdot \Gamma_i \geq 0$ or (b) $K_W \cdot \Gamma_i < 0$. Suppose also that $K_{W'} \cdot \Gamma'_i < 0$ in case (b)⁶

Then there exist \mathbb{Q} -Gorenstein families $\mathcal{W}, \mathcal{W}'$ over a smooth curve B with the following properties. In case (a) \mathcal{W} (resp. \mathcal{W}') has special fiber W (resp. W') over $0 \in B$, the families are isomorphic over $B \setminus \{0\}$. In case (b) we have $B = \mathbb{P}^1$, $\mathcal{W} = \mathcal{W}'$, and the family has fiber W (resp. W') over $0 \in \mathbb{P}^1$ (resp. $\infty \in \mathbb{P}^1$). Let $Z \simeq Z'$ be isomorphic (in both cases) general fibers of these families. These deformations have the following properties:

- (1) $Z \rightsquigarrow W$ is equisingular at P_j for $j \neq i$, P_i is smoothed.
- (2) $Z' \rightsquigarrow W'$ is equisingular at P'_j for $j \neq i-1$, P'_{i-1} is smoothed.
- (3) Boundary divisors $\Gamma_0 + \dots + \Gamma_{r+1} \subset W$ and $\Gamma'_0 + \dots + \Gamma'_{r+1} \subset W'$ lift to $Z \simeq Z'$.
- (4) The surface $Z \simeq Z'$ admits a \mathbb{Q} -Gorenstein smoothing to a surface Y in the fixed irreducible component of the versal deformation space of \overline{W} .

Remark 6.2. By analogy with case (b), in case (a) one can glue families \mathcal{W} and \mathcal{W}' to one family over a non-separated curve with a double origin $\{0, 0'\}$.

Proof. We start with case (a). By Lemma 3.2, there exists a deformation $Z' \rightsquigarrow W'$ over a smooth curve with properties (2) and (3) and total space $\mathcal{W}' \rightarrow B$. (This deformation corresponds to one of the two boundary rays of the first quadrant on the left side of Figure 4.) By [HTU, Prop. 3.16], there exists a contraction $\mathcal{W}' \rightarrow \mathcal{W}'_i$ of the curve Γ'_i only. This contraction is either K -negative or K -trivial, let $\mathcal{W}^+ \rightarrow \mathcal{W}'_i$ be its flip (or flop) with special fiber W^+ . We claim that $W^+ = W$ and the flip (or flop) gives a required deformation $Z \rightsquigarrow W$. Indeed, W^+ has the same singularities as W and the same flipped curve Γ_i by [HTU, Prop. 3.16]. The chains up to P'_{i-2} and from P'_{i+1} are not affected by the flip. The curve from P'_{i-2} to P'_i in Z' degenerates in W^+ to an irreducible curve Γ_{i-1} from P_{i-2} to P_{i-1} while the

⁶If $K_{W'} \cdot \Gamma'_i > 0$ then the left antiflip $W = L_i(W')$ will be in case (a) and the same results hold.

curve from P'_i to P'_{i+1} breaks in W^+ into a union of a curve Γ_{i+1} from P_i to P_{i+1} and a flipped curve Γ_i from P_{i-1} to P_i . Since \mathcal{W}'_i can be further contracted to a deformation \overline{W} of \overline{W} , W^+ is a Wahl resolution of \overline{W} . By Prop. 3.5, the curves Γ_{i-1} and Γ_{i+1} have the same self-intersections in W and W^+ and so the same δ 's.

Geometry in case (b) is different. Take a contraction $(\Gamma_i \subset W) \rightarrow (Q_i \in W_i)$. By Lemma 3.2, there exists a deformation $Z \rightsquigarrow W$ with properties (1) and (3). (This deformation corresponds to one of the non-boundary rays of the fan \mathcal{F} on the left side of Figure 4.) By [HTU, Prop. 2.4], Z is the special fiber of a k1A neighborhood. This means the following: there is a contraction $Z \rightarrow W_i$, which is a (non-c.q.s.!) resolution of $Q_i \in W_i$. Its exceptional divisor $E \simeq \mathbb{P}^1$ passes through the Wahl singularity $P_{i-1} \in W$. The proper transform of E intersects exactly one of the irreducible curves D_s in the chain D_1, \dots, D_p of the minimal resolution $P_{i-1} \in W$, the intersection of E and D_s is transversal in one point. Contracting E gives a chain $D_1, \dots, \tilde{D}_s, \dots, D_p$ with $\tilde{D}_s^2 = D_s^2 + 1$. This chain is a (non-minimal) resolution of the singularity $Q_i \in W_i$. Blowing up a varying point of \tilde{D}_s and contracting the chain $D_1, \dots, \tilde{D}_s, \dots, D_p$ back to $P_{i-1} \in W$ gives an obvious equisingular family with fibers Z over \mathbb{G}_m . Furthermore, by [HTU, Prop. 2.4], the same family arises from deformations $Z' \rightsquigarrow W'$ with properties (2) and (3). This gives the family \mathcal{W} over \mathbb{P}^1 . By the proof of Lemma 3.2 and semicontinuity, we have $H^2(Z, T_Z(-\log \Delta)) = 0$, which again by Lemma 3.2 shows that there are no local-to-global obstructions to its deformations. So we can obtain a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow Z$. Since Z admits a contraction to W_i , it can be further contracted to \overline{W} . By blowing down deformations, the smoothing $Y \rightsquigarrow Z$ blows down to a smoothing $Y \rightsquigarrow \overline{W}$ proving (4). \square

Lemma 6.3. *In notation of Prop. 6.1, consider s.o.d.'s $D^b(\mathcal{W}) = \langle \mathcal{A}_r^{\mathcal{W}}, \dots, \mathcal{A}_0^{\mathcal{W}}, \mathcal{B}^{\mathcal{W}} \rangle$ and $D^b(\mathcal{W}') = \langle \mathcal{A}_r^{\mathcal{W}'}, \dots, \mathcal{A}_0^{\mathcal{W}'}, \mathcal{B}^{\mathcal{W}'} \rangle$ of Theorem 5.5 and their restrictions to $Z \simeq Z'$:*

$$D^b(Z) = \langle \mathcal{A}_r, \dots, \mathcal{A}_0, \mathcal{B} \rangle = \langle \mathcal{A}'_r, \dots, \mathcal{A}'_0, \mathcal{B}' \rangle. \quad (6.1)$$

Then $\mathcal{B} = \mathcal{B}'$, $\mathcal{A}_j = \{F_j\}$ for $j \neq i$, and $\mathcal{A}'_j = \{F_j\}$ for $j \neq i-1$. Furthermore,

$$\langle \mathcal{A}_i, \mathcal{A}_{i-1} \rangle = \langle E_i, \{F_{i-1}\} \rangle = \langle \{F_{i-1}\}, E'_{i-1} \rangle = \langle \mathcal{A}'_i, \mathcal{A}'_{i-1} \rangle.$$

Here E_i and E'_{i-1} are Hacking vector bundles associated with singularities $P_i \in W$ and $P'_{i-1} \in W'$ of central fibers of deformations as in Th. 5.8 and F_j for $j \neq i$ are Kawamata sheaves associated with singularities P_j on the general fiber Z as in Prop. 4.4.

Proof. In case (a) of Prop. 6.1, the deformations \mathcal{W} and \mathcal{W}' blow-down to the same deformation \overline{W} of \overline{W} , so we have $\mathcal{B} = \mathcal{B}'$ by Corollary 5.6. In case (b) the deformation $\mathcal{W} \rightarrow \mathbb{P}^1$ blows down to the constant deformation $\overline{W} \times \mathbb{P}^1$ of \overline{W} . For fibers over $b \neq 0, \infty$, the contraction $\pi : \mathcal{W}_b \rightarrow \overline{W}$ is not a c.q.s. resolution. However, $\mathcal{W}_0 = W \rightarrow \overline{W}$ and $\mathcal{W}_\infty = W' \rightarrow \overline{W}$ are c.q.s. resolutions, so Corollary 5.6 still applies and gives $\mathcal{B} = \mathcal{B}'$ since it is a pullback of the category $\mathcal{B}^{\overline{W}}$ via $L\pi^*$.

So the only thing to check is that subcategories \mathcal{A}_j and \mathcal{A}'_j that correspond to singular points of W and W' where the deformations are equisingular are subcategories of $D^b(Z)$ associated with Kawamata sheaves of these singularities. This follows from Lemma 5.11. \square

Theorem 6.4. *In assumptions of Prop. 6.1, let Y be a general smooth surface within an irreducible component of the versal deformation space of \overline{Y} that contains \mathbb{Q} -Gorenstein smoothings of W and $W' = R_i(W)$. Consider the corresponding H.e.c.*

$$D^b(Y) = \langle E_r, \dots, E_0, \mathcal{B}^Y \rangle = \langle E'_r, \dots, E'_0, \mathcal{B}'^Y \rangle. \quad (6.2)$$

Then $\mathcal{B}^Y = \mathcal{B}^Y$, $E'_j = E_j$ for $j \neq i, i-1$, $E'_i = E_{i-1}$, and $E'_{i-1} = R_{E_{i-1}}(E_i)[k]$, where $k = 0$ in case (a) of Prop. 6.1 and $k = 1$ in case (b). Moreover, in case (a)

$$\mathrm{Hom}(E_i, E_{i-1}) = \mathrm{Ext}^2(E_i, E_{i-1}) = \mathrm{Ext}^1(E'_i, E'_{i-1}) = \mathrm{Ext}^2(E'_i, E'_{i-1}) = 0,$$

$$\mathrm{Ext}^1(E_i, E_{i-1}) \simeq \mathrm{Hom}(E'_i, E'_{i-1})^\vee \simeq \mathbb{C}^{\delta_i}.$$

In case (b), $\mathrm{Ext}^k(E_i, E_{i-1}) = \mathrm{Ext}^k(E'_i, E'_{i-1}) = 0$ for $k > 0$, and

$$\mathrm{Hom}(E_i, E_{i-1}) \simeq \mathrm{Hom}(E'_i, E'_{i-1})^\vee \simeq \mathbb{C}^{\delta_i}.$$

Proof. By Theorem 5.8, the subcategories $\langle E_i \rangle$ and $\langle E'_i \rangle$ are also generated by the Kawamata bundles F_i and F'_i . By Lemma 6.3, we can consider a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow Z$ with the total space \mathcal{Y} inside the \mathbb{Q} -Gorenstein deformation spaces of W and W' and Kawamata sheaves on Z are deformations of Kawamata sheaves on W and W' . Therefore, s.o.d.'s (6.2) are specializations of the s.o.d.'s

$$D^b(\mathcal{Y}) = \langle \mathcal{A}_r^\mathcal{Y}, \dots, \mathcal{A}_0^\mathcal{Y}, \mathcal{B}^\mathcal{Y} \rangle = \langle \mathcal{A}'_r^\mathcal{Y}, \dots, \mathcal{A}'_0^\mathcal{Y}, \mathcal{B}'^\mathcal{Y} \rangle$$

obtained by Theorem 5.5 as deformation of the s.o.d.'s (6.1). Thus $\mathcal{B}^Y = \mathcal{B}^Y$ by Corollary 5.6, $E'_j = E_j$ for $j \neq i, i-1$, $E'_i = E_{i-1}$, and $R_{E_{i-1}}(E_i) \in \langle E'_{i-1} \rangle$, and so $E'_{i-1} = R_{E_{i-1}}(E_i)[k]$ for some k .

Next we consider case (a), where we will show now that $k = 0$. Let $C^\bullet = R_{E_{i-1}}(E_i)$. By definition of mutation we have an exact sequence

$$0 \rightarrow E_{i-1} \otimes \mathrm{Ext}^2(E_i, E_{i-1})^\vee \rightarrow C^{-1} \rightarrow 0 \rightarrow E_{i-1} \otimes \mathrm{Ext}^1(E_i, E_{i-1})^\vee \rightarrow$$

$$C^0 \rightarrow E_i \rightarrow E_{i-1} \otimes \mathrm{Hom}(E_i, E_{i-1})^\vee \rightarrow C^1 \rightarrow 0.$$

So either $k = 0$ or $k = \pm 1$. Since $\mathrm{rk}(E'_{i-1}) = n'_{i-1} = \delta_i n_{i-1} + n_i = \delta_i \mathrm{rk}(E_{i-1}) + \mathrm{rk}(E_i)$ by Prop. 3.5 and $\mathrm{rk}(R_{E_{i-1}}(E_i)) = \delta_i \mathrm{rk}(E_{i-1}) + \mathrm{rk}(E_i)$ by definition of mutation, we have $k = 0$, and so $E'_{i-1} = R_{E_{i-1}}(E_i)$.

Since $C^{-1} = C^1 = 0$, we have $\mathrm{Ext}^2(E_i, E_{i-1}) = 0$. Recall that $E_{i-1} \simeq E'_i$. Applying $\mathrm{RHom}(E_{i-1}, \bullet)$ to the distinguished triangle $E'_{i-1} \rightarrow E_i \rightarrow E_{i-1} \otimes \mathrm{RHom}(E_i, E_{i-1})^\vee$ gives $\mathrm{RHom}(E_i, E_{i-1})^\vee \simeq \mathrm{RHom}(E'_i, E'_{i-1})[1]$. Therefore

$$\mathrm{Hom}(E'_i, E'_{i-1}) = \mathrm{Ext}^1(E_i, E_{i-1})^\vee \quad \text{and} \quad \mathrm{Ext}^1(E'_i, E'_{i-1}) = \mathrm{Hom}(E_i, E_{i-1})^\vee.$$

This also shows that $\mathrm{Ext}^2(E'_i, E'_{i-1}) = 0$. We claim that $\mathrm{Ext}^1(E'_i, E'_{i-1}) = 0$ as well, which will show that $\mathrm{Hom}(E_i, E_{i-1}) = 0$ and therefore that $\mathrm{Ext}^1(E_i, E_{i-1}) \simeq \mathbb{C}^{\delta_i}$ by Lemma 6.5. In order to do this we will analyze the next antiflip $W'' = R_i(W')$, which is an instance of case (b). This calculation will also settle the case (b).

As before, we have $E''_i = E'_{i-1}$ and $E''_{i-1} = R_{E'_{i-1}}(E_i)[k]$ for some k . Let $C^\bullet = R_{E'_{i-1}}(E_i)$. By definition of mutation we have an exact sequence

$$0 \rightarrow C^{-1} \rightarrow 0 \rightarrow E'_{i-1} \otimes \mathrm{Ext}^1(E'_i, E'_{i-1})^\vee \rightarrow$$

$$C^0 \rightarrow E'_i \rightarrow E'_{i-1} \otimes \mathrm{Hom}(E'_i, E'_{i-1})^\vee \rightarrow C^1 \rightarrow 0.$$

So $C^{-1} = 0$ and therefore $k = 0$ or $k = 1$. We claim that $k = 1$. Indeed, $\mathrm{rk}(E''_{i-1}) = n''_{i-1} = \delta n'_{i-1} - n'_i = \delta \mathrm{rk}(E'_{i-1}) - \mathrm{rk}(E'_i)$. On the other hand, $\mathrm{rk}(R_{E'_{i-1}}(E_i)) = -\chi(E'_i, E'_{i-1}) \mathrm{rk}(E'_{i-1}) + \mathrm{rk}(E'_i) = \delta \mathrm{rk}(E'_{i-1}) + \mathrm{rk}(E'_i)$. So $k = 1$. This shows that $\mathrm{Ext}^1(E'_i, E'_{i-1}) = 0$, which proves all claims in case (a).

Finally, to finish case (b), we apply $\mathrm{RHom}(E'_{i-1}, \bullet)$ to the distinguished triangle $E''_{i-1}[-1] \rightarrow E'_i \rightarrow E'_{i-1} \otimes \mathrm{RHom}(E'_i, E'_{i-1})^\vee$, which gives $\mathrm{RHom}(E'_i, E'_{i-1})^\vee \simeq \mathrm{RHom}(E''_i, E''_{i-1})$. So $\mathrm{Ext}^k(E''_i, E''_{i-1}) = 0$ for $k > 0$, and we also have an isomorphism $\mathrm{Hom}(E''_i, E''_{i-1}) \simeq \mathrm{Hom}(E'_i, E'_{i-1})^\vee \simeq \mathbb{C}^{\delta_i}$. \square

Lemma 6.5. *Let Y be a \mathbb{Q} -Gorenstein smoothing of a Wahl resolution W . Let E_r, \dots, E_0 be the corresponding H.e.c. on Y . Then $\chi(E_i, E_{i-1}) = -n_i n_{i-1} \Gamma_i \cdot K_W$ ⁷. In particular, if $K_W \cdot \Gamma_i > 0$ then $\chi(E_i, E_{i-1}) = -\delta_i$ and if $K_W \cdot \Gamma_i < 0$ then $\chi(E_i, E_{i-1}) = \delta_i$.*

Proof. $\chi(E_i, E_{i-1}) = \sum (-1)^j \text{ext}^j(E_i, E_{i-1})$, and so by Riemann-Roch

$$\begin{aligned} \chi(E_i, E_{i-1}) &= n_i \text{ch}_2(E_{i-1}) - c_1(E_i) \cdot c_1(E_{i-1}) + n_{i-1} \text{ch}_2(E_i) \\ &\quad + \frac{1}{2} (n_{i-1} c_1(E_i) - n_i c_1(E_{i-1})) \cdot K_Y + n_i n_{i-1}, \end{aligned}$$

where $\text{ch}_2 = \frac{1}{2}(c_1^2 - 2c_2)$. By Theorem 5.8, we know that

$$c_1(E_i) = -n_i(A + \Gamma_1 + \dots + \Gamma_i) \in H_2(Y),$$

and $c_2(E_i) = \frac{n_i-1}{2n_i}(c_1(E_i)^2 + n_i + 1)$. Hence we do the computation in W . We have $\text{ch}_2(E_i) = \frac{1}{2} \left(n_i(A + \Gamma_1 + \dots + \Gamma_i)^2 - \frac{n_i^2-1}{n_i} \right)$, and so

$$n_i \text{ch}_2(E_{i-1}) - c_1(E_i) \cdot c_1(E_{i-1}) + n_{i-1} \text{ch}_2(E_i) = \frac{1}{2} n_i n_{i-1} \Gamma_i^2 - n_i n_{i-1} + \frac{n_{i-1}}{2n_i} + \frac{n_i}{2n_{i-1}}.$$

But on W we have $\Gamma_i^2 = -\Gamma_i \cdot K_W - \frac{1}{n_i} - \frac{1}{n_{i-1}}$. On the other hand $n_{i-1} c_1(E_i) - n_i c_1(E_{i-1}) = -n_{i-1} n_i \Gamma_i$, and so we plug in the formula for $\chi(E_i, E_{i-1})$ to obtain the formula. \square

Finally, we can finish the proof of Theorem 1.12 with Lemmas 6.6-6.8.

Lemma 6.6. *Let W be a Wahl resolution of \overline{W} satisfying Assumption 1.10 and let Y be a sufficiently general surface from the corresponding versal deformation space of \overline{W} . Let E_r, \dots, E_0 be the corresponding Hacking exceptional collection on Y . Then, for every $i > j$, either $\text{Ext}^k(E_i, E_j) = 0$ for $k \neq 1$ or $\text{Ext}^k(E_i, E_j) = 0$ for $k \neq 0$.*

In particular, suppose E_r, \dots, E_0 (resp. $\bar{E}_r, \dots, \bar{E}_0$) is the H.e.c. that corresponds to the M-resolution W^+ (resp. the N-resolution W^-). Then

- (1) $\bar{E}_r, \dots, \bar{E}_0$ is a strong exceptional collection, i.e. $\text{Ext}^k(\bar{E}_i, \bar{E}_j) = 0$ for $k > 0, i > j$.
- (2) $\text{Ext}^k(E_i, E_j) = 0$ for $k \neq 1, i > j$.
- (3) For $i = 1, \dots, r-1$, we have $\text{Hom}(\bar{E}_{r-i+1}, \bar{E}_{r-i}) \simeq \text{Ext}^1(E_i, E_{i-1})^\vee \simeq \mathbb{C}^{\delta_i}$.
- (4) $\bar{E}_r, \dots, \bar{E}_0$ is a mutation of E_r, \dots, E_0 (no homological shifts).

Proof. Let W be an arbitrary Wahl resolution. We claim that we can bring any two singularities $P_i, P_j, i > j$, in W together (without changing them) via a sequence of right antiflips. If $i = j+1$, then P_i, P_j are already together via Γ_i . Otherwise we have a chain $\Gamma_{j+1}, \dots, \Gamma_i$ connecting them, which we can antiflip from $k = j+1$ to $k = i-1$, after that the new singularities in positions $i-1$ and i are P_j and P_i . By Th. 6.4, the corresponding mutations do not change Hacking bundles E_i and E_j and we have $\text{Ext}^k(E_i, E_j) = 0$ for $k \neq 1$ if $K \cdot \Gamma_i \geq 0$ and or $\text{Ext}^k(E_i, E_j) = 0$ for $k \neq 0$ if $K \cdot \Gamma_i \leq 0$, where Γ_i is the curve connecting the points after the antiflips.

Now suppose E_r, \dots, E_0 (resp. $\bar{E}_r, \dots, \bar{E}_0$) is the H.e.c. that corresponds to the M-resolution W^+ (resp. the N-resolution W^-) Part (3) follows from Th. 6.4 and definition of the N-resolution. For part (2), we argue as follows. By Prop. 3.5, for every antiflip of $\Gamma_k, k = j+1, \dots, k = i-1$ described above, the curves in positions $> k$ retain non-negative intersection with the canonical class, including the last curve Γ_i that will connect P_i to P_j . Thus (2) follows from Th. 6.4.

By Th. 6.4, the sequence of antiflips that connects W^+ to W^- corresponds to the sequence of mutations that takes an exceptional collection E_r, \dots, E_0 literally to the exceptional collection $\bar{E}_r, \dots, \bar{E}_0$, i.e. without homological shifts. This shows (4). Finally, we prove (1). We follow E_i and E_j through a sequence of mutations.

⁷The Euler pairing for $\alpha, \beta \in D^b(Z)$ is the integer $\chi(\alpha, \beta) = \sum_j (-1)^j \dim_{\mathbb{C}} \text{Hom}(\alpha, \beta[j])$.

To simplify notation, we will denote their images after mutations by the same letters. We can decompose this element of the braid group as follows: arrange bundles into three blocks: $A = \langle E_r, \dots, E_{i+1} \rangle$, $B = \langle E_i, \dots, E_j \rangle$, $C = \langle E_{j-1}, \dots, E_0 \rangle$. Rearrange the bundles in A and C in the opposite order (moving E_{i+1} , resp. E_0 all the way to the left without changing it as in the definition of the N-resolution.) This does not change E_i or E_j , so the RHom between them stays the same. Next, rearrange bundles in B in the opposite order by moving E_j all the way to the left. By the analysis in the beginning of the proof of the lemma, this will change non-zero components of RHom from $\mathrm{Ext}^1(E_i, E_j)$ to $\mathrm{Hom}(E_j, E_i)$. Finally, we mutate s.o.d.'s: $\langle A, B, C \rangle \rightarrow \langle A, C, B' \rangle \rightarrow \langle C, A', B' \rangle \rightarrow \langle C, B', A'' \rangle$. This gives an equivalence $B \rightarrow B'$ which does not change RHom 's between its objects, for example between E_i and E_j . \square

Lemma 6.7. *Let $\pi : W \rightarrow \bar{W}$ be a Wahl resolution satisfying Assumption 1.10 and let Y be a sufficiently general surface from the corresponding versal deformation space of \bar{W} . Let E_r, \dots, E_0 be the corresponding Hacking exceptional collection on Y .*

- (1) *The Kawamata vector bundle \bar{F} on \bar{W} deforms uniquely to a vector bundle F on Y .*
- (2) *For every $i = 0, \dots, r$, $\mathrm{Hom}(F, E_i) = \mathbb{C}^{\mathrm{rk} E_i}$ and $\mathrm{Ext}^k(F, E_i) = 0$ for $k > 0$.*
- (3) *$\{F\} = \langle F \rangle = \langle E_r, \dots, E_0 \rangle \subset D^b(Y)$.*

Proof. Let $F = \bar{F}|_Y$ be the unique deformation of \bar{F} given by Lemma 5.1, which is locally free by Lemma 5.2, which proves (1). By Corollary 5.6, Remark 5.7 the admissible subcategory $\{F\} \subset D^b(Y)$ is equal to the subcategory $\langle E_r, \dots, E_0 \rangle$. By Theorem 5.8, the Kawamata bundle $\mathcal{F}_i|_Y$ on Y associated with the singularity $P_i \in W$ is isomorphic to $E_i^{\mathrm{rk} E_i}$. So (2) will follow if we can show that $\mathrm{Hom}(F, \mathcal{F}_i|_Y) = \mathbb{C}^{\mathrm{rk} \mathcal{F}_i|_Y}$ and $\mathrm{Ext}^k(F, \mathcal{F}_i|_Y) = 0$ for $k > 0$. Since $\mathcal{F}_i|_Y$ is a deformation of the Kawamata sheaf F_i on W , by semi-continuity it suffices to show that $\mathrm{RHom}_W(\pi^* \bar{F}, F_i) = \mathbb{C}^{\mathrm{rk} F_i}$. Since F_i is the maximal iterated extension of $\mathcal{O}_W(-D_i)$, it suffices to show that $\mathrm{RHom}_W(\pi^* \bar{F}, \mathcal{O}_W(-D_i)) = \mathbb{C}$. This will follow at once from Lemma 4.12 and adjunction if we can show that $R\pi_* \mathcal{O}_W(-D_i) = \mathcal{O}_{\bar{W}}(-\bar{A})$. In the exact sequence $0 \rightarrow \mathcal{O}_W(-D_i) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_{D_i} \rightarrow 0$, the derived pushforward by π of $\mathcal{O}_{D_i} = \mathcal{O}_{\Gamma_0 \cup \dots \cup \Gamma_i}$ is equal to $\mathcal{O}_{\pi(\Gamma_0)} = \mathcal{O}_{\bar{A}}$ and the derived pushforward of \mathcal{O}_W is $\mathcal{O}_{\bar{W}}$. So $R\pi_* \mathcal{O}_W(-D_i) = \mathcal{O}_{\bar{W}}(-\bar{A})$. In (3), we just need to prove that $\{F\} = \langle F \rangle$, which will follow once we show that $F \simeq \bigoplus_{i=0}^r \bar{E}_i^{n_{r-i}}$ in the next Lemma 6.8 because then $\langle F \rangle$ contains $\langle \bar{E}_r, \dots, \bar{E}_0 \rangle$, which is equal to $\{F\}$. \square

Lemma 6.8. *Let \bar{W} be a c.q.s. surface satisfying Assumption 1.10 and let Y be its sufficiently general smoothing from a fixed component of the versal deformation space. Let E_r, \dots, E_0 (resp. $\bar{E}_r, \dots, \bar{E}_0$) be the H.e.c. on Y that corresponds to the M-resolution W^+ (resp. the N-resolution W^-). Then*

- (1) *These collections are dual: $\mathrm{RHom}(\bar{E}_j, E_{r-i}) = \mathbb{C}$ if $i = j$ and 0 otherwise.*
- (2) *The Kawamata bundle F is isomorphic to $\bigoplus_{i=0}^r \bar{E}_i^{n_{r-i}}$, where $n_j = \mathrm{rk} E_j$. The Kalck-Karmazyn algebra $R = \mathrm{End}(\bar{F})$ deforms to the algebra $\mathrm{End}(F)$ Morita-equivalent to the endomorphism algebra $\hat{R} = \mathrm{End}(\bar{E}_r \oplus \dots \oplus \bar{E}_0)$ of a strong exceptional collection.*
- (3) *\hat{R} is a path algebra of a quiver with $(r+1)$ vertices labeled by $\bar{E}_r, \dots, \bar{E}_0$ and with arrows connecting \bar{E}_i to \bar{E}_j for $i > j$ so that the total number of paths connecting \bar{E}_i to \bar{E}_j is equal to $\mathrm{hom}(\bar{E}_i, \bar{E}_j)$. In particular, the category $\langle F \rangle$ does not depend on Y .*

Proof. We will prove by a down-ward induction on j that if $X \in \langle \bar{E}_r, \dots, \bar{E}_j \rangle$ and $\mathrm{RHom}(X, \bar{E}_i) = \mathbb{C}^{\bar{n}_i}$, $\mathrm{RHom}(X, E_{r-i}) = \mathbb{C}^{n_{r-i}}$ for $i \geq j$ then $X \simeq \bar{E}_j^{n_{r-j}} \oplus \dots \oplus \bar{E}_r^{n_0}$. If $j = r$ then this is clear because $\bar{E}_r \simeq E_0$ and $\bar{n}_r = n_0$. When $j = 0$, this will

give part (2) by Lemma 6.7 (2). To prove the step of induction, let the claim be true for $j + 1$. Take $X \in \langle \bar{E}_r, \dots, \bar{E}_j \rangle$. By definition of s.o.d., we have a distinguished triangle

$$\bar{E}_j \otimes C^\bullet \rightarrow X \rightarrow X' \rightarrow, \quad (6.3)$$

where C^\bullet is a complex of vector spaces and $X' \in \mathcal{A}$, where \mathcal{A} is an admissible subcategory $\langle \bar{E}_r, \dots, \bar{E}_{j+1} \rangle$, which is equal to $\langle E_{r-j-1}, \dots, E_0 \rangle$ by Th. 6.4. Also, \bar{E}_j is the right mutation of E_{r-j} over \mathcal{A} . So we have a mutation triangle

$$\bar{E}_j \rightarrow E_{r-j} \rightarrow T \rightarrow 0 \quad (6.4)$$

with $T \in \mathcal{A}$. This triangle proves part (1). Since $\text{RHom}(\bar{E}_j, \bar{E}_i) = 0$ for $i > j$, we have $\text{RHom}(X', \bar{E}_i) = \text{RHom}(X, \bar{E}_i) = \mathbb{C}^{\bar{n}_i}$ by (6.3). Since $\text{RHom}(\bar{E}_j, E_{r-i}) = 0$ for $i > j$ by (6.4), we have $\text{RHom}(X', E_{r-i}) = \text{RHom}(X, E_{r-i}) = \mathbb{C}^{n_{r-i}}$ by (6.3). This implies that $X' \simeq \bar{E}_{j+1}^{n_{r-j-1}} \oplus \dots \oplus \bar{E}_r^{n_0}$ by inductive assumption. The triangle (6.3) and part (1) imply $C^\bullet = \text{RHom}(X, E_{r-j}) = \mathbb{C}^{n_{r-j}}$. Since $\text{Ext}^1(\bar{E}_i, \bar{E}_j) = 0$ for $i > j$, this implies that $X \simeq \bar{E}_j^{n_{r-j}} \oplus \dots \oplus \bar{E}_r^{n_0}$. This proves (2).

Under the equivalence $\langle F \rangle \rightarrow D^b(\hat{R} - \text{mod})$, $X \mapsto \text{RHom}(\hat{R}, X)$, the vector bundles E_r, \dots, E_0 go to simple modules of \hat{R} by part (1). Since $\text{Ext}^2(E_i, E_j) = 0$ for all i, j by Lemma 6.6 (2), this implies that \hat{R} is hereditary, and therefore \hat{R} is isomorphic to a path algebra of a quiver by a well-known theorem of Gabriel. \square

Lemma 6.9. *The formula (1.1) holds.*

Proof. One can compute this dimension by \mathbb{Q} -Gorenstein smoothing all singularities between \bar{P}_i and \bar{P}_j to obtain a curve Γ through singularities \bar{P}_i and \bar{P}_j which are now consecutive. Now use Theorem 6.4. \square

Remark 6.10. In Definition 1.5 we required Wahl resolutions $W \rightarrow \bar{W}$ to have a \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W$ that blows down to a smoothing $Y \rightsquigarrow \bar{W}$, because otherwise there seems to be no natural way to define the braid group action. For

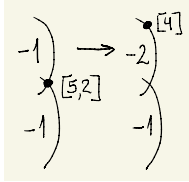


FIGURE 9. Divisorial contraction

example, consider the Wahl resolution W of $\bar{W} = \frac{1}{3}(1, 1)$ singularity on the left side of Figure 9. Its \mathbb{Q} -Gorenstein smoothing $Y \rightsquigarrow W$ blows down to a smoothing $\bar{Y} \rightsquigarrow \bar{W}$ but $\bar{Y} \neq Y$, in fact Y is a blow-up of \bar{Y} at a smooth point. To see this, we can flip W at the top curve producing a resolution W' on the right side of Figure 9, which contains a (-1) -curve E which deforms to a nearby fiber producing a divisorial contraction of the threefold. Accordingly, an antflip of W' at E is not defined. On the level of derived categories, W gives a H.e.c. E_2, E_1, E_0 on Y where E_0 and E_2 are line bundles and E_1 has rank 3. This is mutated to a H.e.c. E'_2, E'_1, E'_0 on W' , where E'_0 has rank 2 and $E'_2 = E'_1(-E)$ are line bundles. One can of course mutate this exceptional collection further but this gives $E'_1, E'_1|_E, E'_0$, where the sheaf in the middle is a torsion sheaf supported on E . By contrast, mutating collections of vector bundles associated with Wahl resolutions always gives collections of vector bundles associated with other Wahl resolutions (up to a homological shift which is easy to compute, see Theorem 6.4).

We end this section by analyzing which quivers show up in Theorem 1.12 when the M-resolution $W^+ \rightarrow \overline{W}$ has two exceptional curves Γ_1, Γ_2 with associated $\delta_1 =: a$ and $\delta_2 =: b$. We have Wahl singularities P_i of type $\frac{1}{n_i}(1, n_i a_i - 1)$ for $i = 0, 1, 2$. For the N-resolution we use the bar notation: Wahl singularities \overline{P}_i of type $\frac{1}{\overline{n}_i}(1, \overline{n}_i \overline{a}_i - 1)$ for $i = 0, 1, 2$, etc. The associated H.e.c. is $\langle \overline{E}_2, \overline{E}_1, \overline{E}_0 \rangle$, and we have $\text{hom}(\overline{E}_2, \overline{E}_0) = ab + c$, where $c := \frac{an_2 + bn_0}{n_1}$. The algebra \hat{R} is the path algebra of the quiver $Q_{a,b,c}$ with vertices $\overline{E}_2, \overline{E}_1, \overline{E}_0$ and a arrows between $\overline{E}_2, \overline{E}_1$, b arrows between $\overline{E}_1, \overline{E}_0$, and c arrows between $\overline{E}_2, \overline{E}_0$, as in Figure 10.

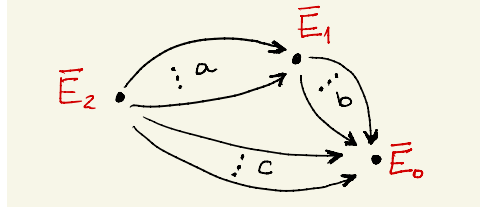


FIGURE 10. The quiver $Q_{a,b,c}$ for an N-resolution with two curves

Proposition 6.11. *The representation algebra of the quiver $Q_{a,b,c}$ can be realized by the categorified Milnor fiber, i.e. the algebra \hat{R} of Theorem 1.12, if and only if there exists an extremal P-resolution with Wahl singularities of indices a and b and $\delta = c$ (see Definition 2.9). See Lemma 6.12 for the list of possible c for fixed values of a, b .*

A direct corollary is that if we realize $Q_{a,b,c}$ by our construction, then all permutations of a, b, c are realizable by our construction. This is because we can reverse the orientation of the c.q.s. to obtain $Q_{b,a,c}$ and we can use the “circular” zero continued fraction of the extremal P-resolution of $Q_{a,b,c}$ to realize zero continued fractions for $Q_{c,a,b}$ and $Q_{b,c,a}$ (see [HTU, Section 4]).

By Example 7.3, $D^b(\hat{R})$ is an admissible subcategory of the derived category of a smooth projective rational surface. This partially answers question (Q5) in [BR].

Proof. If $a = 0$ and $b > 0$, then $b = c$; similarly for $b = 0$ and $a > 0$ we have $c = a$. If $a = b = 0$, then there are no arrows.

Let us assume that $a, b > 0$. Let $\frac{\Delta}{\Delta - \Omega} = [b_1, \dots, b_s]$. As in Section 2, we have that $d_{i_1} = d_{i_2} = d_{i_3} = 1$, and the zero continued fraction corresponding to the N-resolution is $[b_s, \dots, b_{i_3} - 1, \dots, b_{i_2} - 1, \dots, b_{i_1} - 1, \dots, b_1] = 0$. We have that

$$\frac{\overline{n}_{2-k}}{\overline{n}_{2-k} - \overline{a}_{2-k}} = [b_1, \dots, b_{i_{k+1}-1}],$$

for $k = 0, 1, 2$. (If $i_1 = 1$, then we set $\overline{n}_2 = \overline{a}_2 = 1$, and so \overline{P}_2 is a smooth point in that case.) We also have

$$\frac{a}{\varepsilon_a} = [b_{i_2+1}, \dots, b_{i_3-1}] \quad \text{and} \quad \frac{b}{\varepsilon_b} = [b_{i_2-1}, \dots, b_{i_1+1}]$$

for some $\varepsilon_a, \varepsilon_b$ coprime to a, b respectively. (If $i_3 = i_2 + 1$ ($i_2 = i_1 + 1$), then $a = 1$ ($b = 1$).)

Let us consider the Hirzebruch-Jung continued fraction that results from contracting (possibly) 1 and all new 1's from $[b_{i_3-1}, \dots, b_{i_1} - 1, \dots, b_1]$. This corresponds to a rational number $\frac{\alpha}{\beta} > 1$. Then, if $\frac{\alpha}{\alpha - \beta} = [x_1, \dots, x_t]$, we have

$$[x_t, \dots, x_1, 1, b_{i_3-1}, \dots, b_{i_1} - 1, \dots, b_1] = 0,$$

and so the \mathbb{Q} -Gorenstein smoothing of \bar{P}_1 which keeps the singularities at \bar{P}_0 and \bar{P}_2 and a curve between them corresponds to that extremal N-resolution. In particular, we have

$$\frac{ab+c}{\varepsilon_{ab+c}} = [b_{i_3-1}, \dots, b_{i_1+1}]$$

for some ε_{ab+c} .

Lemma 6.12. *Let $\lambda := b_{i_2} \geq 2$.*

- (0) *If $a = b = 1$, then $c = \lambda - 1$,*
- (1) *If $a = 1$ and $b > 1$, then $c = \lambda b - b - \varepsilon_b$; If $b = 1$ and $a > 1$, then $c = \lambda a - a - \varepsilon_a$,*
- (2) *If $a, b > 1$, then $c = (\lambda - 1)ab - \varepsilon_a b - \varepsilon_b a$. In particular $\gcd(a, b)$ always divides c .*

Proof. Part (0) is trivial because $a = b = 1$ implies $i_3 = i_2 + 1 = i_1 + 2$, and so $\frac{ab+c}{\varepsilon_c} = [b_{i_2}]$. For (1) and (2) we use the identity (for (1) we eliminate the “ a or b matrix”)

$$\begin{pmatrix} a & -\varepsilon_a \\ \varepsilon'_a & \frac{1-\varepsilon_a\varepsilon'_a}{a} \end{pmatrix} \cdot \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b & -\varepsilon'_b \\ \varepsilon_b & \frac{1-\varepsilon_b\varepsilon'_b}{b} \end{pmatrix} = \begin{pmatrix} ab+c & -\varepsilon'_{ab+c} \\ \varepsilon_{ab+c} & \frac{1-\varepsilon_{ab+c}\varepsilon'_{ab+c}}{ab+c} \end{pmatrix},$$

where $0 < \varepsilon'_x < x$ is the inverse modulo x of ε_x . \square

Proposition 6.13. *Each of the cases (t) in Lemma 6.12 corresponds to an extremal P-resolution with t singularities and $\delta = c$, represented by the continued fraction (here $*$ is a smooth point):*

- (0) $* - (\lambda + 1) - *$,
- (1) $[(\frac{a}{a-\varepsilon_a})] - (\lambda) - *$ or $* - (\lambda) - [(\frac{b}{\varepsilon_b})]$,
- (2) $[(\frac{a}{a-\varepsilon_a})] - (\lambda - 1) - [(\frac{b}{\varepsilon_b})]$.

On the other hand, each extremal P-resolution in those three cases (with $\delta = c$) produces infinitely many M-resolutions with two exceptional curves, so that $\delta_1 = a$, $\delta_2 = b$ and $n_1c = an_2 + bn_0$.

Proof. First, it is easy to check that each (t) case in Lemma 6.12 corresponds to the described extremal P-resolution, since $\frac{1}{a}(1, \varepsilon'_a) - (\lambda - 1) - \frac{1}{b}(1, \varepsilon_b)$ is contractible. The opposite is trickier for the case $\lambda = 2$, so we first show it for $\lambda \geq 3$. In this case we have that $\frac{1}{a}(1, \varepsilon'_a) - (\lambda - 1) - \frac{1}{b}(1, \varepsilon_b)$ contracts to a c.q.s. Let us choose arbitrarily a Hirzebruch-Jung continued fractions $[x_1, \dots, x_u]$ where $x_i \geq 2$ for all i . Then

$$\frac{\alpha}{\beta} = \frac{1}{a}(1, \varepsilon'_a) - (\lambda - 1) - \frac{1}{b}(1, \varepsilon_b) - [x_1, \dots, x_u]$$

is the continued fraction of some c.q.s., and so if $\frac{\alpha}{\alpha-\beta} = [y_1, \dots, y_v]$ we have

$$[y_v, \dots, y_1] - (1) - \frac{1}{a}(1, \varepsilon'_a) - (\lambda - 1) - \frac{1}{b}(1, \varepsilon_b) - [x_1, \dots, x_u] = 0.$$

Therefore we can consider

$$[y_v, \dots, y_1] - (b_{i_3} = 2) - \frac{1}{a}(1, \varepsilon'_a) - (b_{i_2} = \lambda) - \frac{1}{b}(1, \varepsilon_b) - [b_{i_1} = x_1 + 1, \dots, x_u]$$

as the continued fraction that defines an M-resolution with $d_{i_3} = d_{i_2} = d_{i_1} = 1$.

The key for a similar construction in the case $\lambda = 2$ is to prove that $\frac{1}{a}(1, \varepsilon'_a) - (1) - \frac{1}{b}(1, \varepsilon_b)$ contracts (to a c.q.s. or a smooth point), which is proved in the next lemma. \square

Lemma 6.14. *Given an extremal P-resolution $[(\frac{a}{a-\varepsilon_a})] - (1) - [(\frac{b}{\varepsilon_b})]$, then $\frac{1}{a}(1, \varepsilon'_a) - (1) - \frac{1}{b}(1, \varepsilon_b)$ contracts.*

Proof. Let us write $\frac{a}{\varepsilon_a} = [x_1, \dots, x_p]$, $\frac{a}{a-\varepsilon_a} = [y_1, \dots, y_q]$, $\frac{b}{\varepsilon_b} = [z_1, \dots, z_u]$, and $\frac{b}{b-\varepsilon_b} = [w_1, \dots, w_v]$. Then by Lemma 2.5 we have

$$\left[\begin{pmatrix} a \\ a - \varepsilon_a \end{pmatrix} \right] - (1) - \left[\begin{pmatrix} b \\ \varepsilon_b \end{pmatrix} \right] = [y_1, \dots, y_q + x_p, \dots, x_1] - (1) - [z_1, \dots, z_u + w_v, \dots, w_1].$$

Hence, if $[x_p, \dots, x_1] - (1) - [z_1, \dots, z_u]$ does not contract, then $[z_1, \dots, z_u] = [y_1, \dots, y_q, t_1, \dots, t_l]$, where $[t_1, \dots, t_l]$ is the Hirzebruch-Jung continued fraction of some c.q.s. This is because of the algorithm that constructs Wahl chains, and we are assuming that x_p is the first curve that becomes 0 (so that it does not contract). But now we can compute that the intersection of the canonical class with the image of the (-1) -curve in the original extremal P-resolution is $-1 + \frac{a-\varepsilon_a}{a} + \frac{b-\varepsilon_b}{b} > 0$. On the other hand, the existence of $[t_1, \dots, t_l]$ gives $\frac{a}{a-\varepsilon_a} > \frac{b}{\varepsilon_b}$, which is a contradiction. Therefore we do have contraction. \square

This finishes the proof of Proposition 6.11 \square

Example 6.15. Let us quickly classify triangles, i.e. $a = b = c = 1$. In this case $\lambda = 2$, and we have that the zero continued fraction is

$$[b_s, \dots, b_{i_3} - 1, 1, b_{i_1} - 1, \dots, b_1] = 0,$$

and so $\frac{\alpha}{\beta} = [b_{i_2} - 1, \dots, b_s]$ and $\frac{\alpha}{\alpha-\beta} = [b_{i_1} - 1, \dots, b_1]$. The smallest case is $\frac{12}{12-7} = [3, 2, 3]$ with M-resolution is $(2) - [4] - (2)$.

Example 6.16. Not any a, b, c with $\gcd(a, b)$ dividing c is possible. For example, if $b = 1$ and $a = 2$, then $c = 2\lambda - 3$ and so c must be odd.

7. N-RESOLUTION OF THE MINIMAL RESOLUTION

Let $0 < \Omega < \Delta$ be coprime integers, and let $P \in \overline{W}$ be a c.q.s. of type $\frac{1}{\Delta}(1, \Omega)$. In this section we specialize our results to the case when the M-resolution W^+ is the minimal resolution of \overline{W} , i.e. all points P_i 's are smooth points. The corresponding smoothing $Y \rightsquigarrow \overline{W}$ is by definition a smoothing from the Artin component.

Example 7.1. Suppose W^+ is an extremal minimal resolution, i.e. $W^+ \rightarrow \overline{W}$ is a contraction of a single smooth rational curve Γ of self-intersection $-\Delta \leq -2$. A concrete example of a surface \overline{W} satisfying Assumption 1.10 is the projective cone $\mathbb{P}(1, 1, \Delta)$ over a rational normal curve of degree Δ . We take a ruling of the cone as a divisor \bar{A} . The surface W^+ is the Hirzebruch surface with a negative curve Γ and a ruling A . Its general smoothing Y is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\text{Bl}_p \mathbb{P}^2$ depending on parity of Δ . The derived category $D^b(W)$ contains an admissible subcategory $\langle \mathcal{A}_1^{W^+}, \mathcal{A}_0^{W^+} \rangle$, which in this case is just generated by an exceptional collection $\langle \mathcal{O}_{W^+}(-A - \Gamma), \mathcal{O}_{W^+}(-A) \rangle$, i.e. Kawamata sheaves on W^+ are line bundles. This subcategory deforms to a subcategory $\langle E_1, E_0 \rangle \subset D^b(Y)$, i.e. in this case Hacking bundles are line bundles equal to the corresponding Kawamata bundles. While $\text{Ext}_{W^+}^k(\mathcal{O}(-A - \Gamma), \mathcal{O}(-A)) \neq 0$ for $k = 0, 1$, we have $\text{Hom}_Y(E_1, E_0) = 0$ because a general smoothing Y does not contain a lift of a negative curve $\Gamma \subset W$. Thus $\text{Ext}^k(E_1, E_0) = 0$ for $k \neq 1$ and $\text{Ext}^1(E_1, E_0) = \mathbb{C}^\delta$ as predicted by Theorem 1.12. Here $\delta = \Delta - 2$. The N-resolution W^- is equal to W^+ if $\delta = 0$, otherwise it is

$$\left[\begin{pmatrix} \delta + 1 \\ 1 \end{pmatrix} \right] - (1) - [*] = [\delta + 3, 2, \dots, 2] - (1) - [*],$$

where $[*]$ is a smooth point. The corresponding exceptional collection $\langle \bar{E}_1, \bar{E}_0 \rangle$ on Y is strong and contains a line bundle $\bar{E}_1 = E_0$ and a vector bundle \bar{E}_0 of rank $\delta + 1 = \Delta - 1$, which is the universal extension of E_0 by E_1 . The Kawamata bundle \bar{F} on \overline{W} is the maximal iterated extension of $\mathcal{O}_{\overline{W}}(-\bar{A})$ by itself. It has

Its determinant is equal to Δ . On the other hand, we can use the linearity of the determinant on its first row $(y_e+2, -1, 0, \dots, 0) = (1, -1, 0, \dots, 0) + (y_e+1, 0, 0, \dots, 0)$, via the sum $M = M_1 + M_2$ where M_1 corresponds to the continued fraction

$$[1, \underbrace{2, \dots, 2}_{x_{e-1}-3}, y_{e-1} + 3, \dots, y_3 + 3, \underbrace{2, \dots, 2}_{x_2-3}, y_2 + 3, \underbrace{2, \dots, 2}_{x_1-3}, y_1 + 2],$$

and $\det(M_2) = (y_e + 1)\bar{n}_e$. But then $\det(M_1)$ is the numerator of the continued fraction

$$[y_{e-1} + 2, \dots, y_3 + 3, \underbrace{2, \dots, 2}_{x_2-3}, y_2 + 3, \underbrace{2, \dots, 2}_{x_1-3}, y_1 + 2],$$

by contracting the 1 and the consecutive 2's in the diagonal of M_1 . Now we use induction on e to write the claimed formula.

Example 7.3. Let us realize Example 7.2 in a projective surface, where we can apply Theorem 1.12. In fact the following construction from [PPSU, Section 3] works for any M-resolution W^+ of $P \in \bar{W}$ and gives a normal rational projective surface W^+ that satisfies Assumption 1.10. Its \mathbb{Q} -Gorenstein smoothing is the compactified Milnor fiber of the corresponding smoothing of $P \in \bar{W}$. Let \mathbb{F}_1 be the blow-up of \mathbb{P}^2 , with (-1) -curve S_0 . We have the fibration $\mathbb{F}_1 \rightarrow \mathbb{P}^1$ where S_0 is a section. Let F be a fiber. Choose another section S_∞ disjoint from S_0 . The configuration S_0, F, S_∞ gives us a chain of rational curves with self-intersections $\{-1, 0, +1\}$.

Let us come back to the notation $\frac{\Delta}{\Omega} = [e_1, \dots, e_\ell]$ and $\frac{\Delta}{\Delta-\Omega} = [b_1, \dots, b_s]$. Then, by doing blow-ups over $S_0 \cap F$, the chain S_0, F, S_∞ can be transformed into a new chain of rational curves with self-intersections

$$\{+1, 1 - b_s, -b_{s-1}, \dots, -b_1, -1, -e_1, -e_2, \dots, -e_\ell\}$$

where the first curve on the left corresponds to the proper transform of S_∞ , and the last curve on the right to the proper transform of S_0 . Let $\pi: X \rightarrow \mathbb{F}_1$ be the corresponding composition of blow-ups. Let G be the reduced total pull-back by π of $S_0 + F + S_\infty$. Then $H^2(X, T_X(-\log G)) = 0$ by [PPSU, Lemma 3.3]. The contraction of $[e_1, \dots, e_s]$ defines our surface \bar{W} where P is the singularity. By blowing-up adequately X over the chain $[e_1, \dots, e_s]$, we obtain a surface \tilde{X} and a contraction $\tilde{X} \rightarrow W^+$, giving the M-resolution $W^+ \rightarrow \bar{W}$ over P . Then W^+ satisfies Assumption 1.10 (we can use as A the ‘‘middle’’ (-1) -curve in the original configuration in X). A \mathbb{Q} -Gorenstein smoothing Y of W^+ is the compactified Milnor fiber. We now can apply Theorem 1.12 to Y .

Example 7.4. In the notation of Example 7.2, if $y_1 = \dots = y_e = 0$ then the algebra \hat{R} of Theorem 1.12 is the representation algebra of a quiver that contains a Dynkin subquiver of type A_{e-1} . It follows by Example 7.3 that the derived category of a smooth projective rational surface can contain as an admissible subcategory the derived category of representations of a quiver with a path of an arbitrary length. This answers the question (Q4) from [BR].

8. S.O.D.'S FOR MAXIMALLY DEGENERATED DOLGACHEV SURFACES

In examples of Section 7, the surface W and its \mathbb{Q} -Gorenstein smoothing Y were rational. In this section, W will be rational but Y will be a Dolgachev surface.

We start with a topological fact which will allow us to guarantee that Assumption 1.10 part (3) is satisfied in our examples.

Lemma 8.1. *Let Z be a surface with only c.q.s. $\{Q_0, \dots, Q_s\}$ of type $\frac{1}{m_i}(1, q_i)$, and with $H^1(Z, \mathcal{O}_Z) = H^2(Z, \mathcal{O}_Z) = 0$. Let $Z^\circ := Z \setminus \{Q_0, \dots, Q_s\}$. If $H_1(Z^\circ, \mathbb{Z}) = 0$, then*

there is a short exact sequence

$$0 \rightarrow \text{Pic}(Z) \rightarrow \text{Cl}(Z) \rightarrow \bigoplus_{i=0}^s \text{Cl}(Q_i \in Z) \rightarrow 0,$$

where $\text{Cl}(Q_i \in Z) \simeq \mathbb{Z}/m_i\mathbb{Z}$ is the local class group of $Q_i \in Z$.

Proof. We have the long exact sequence of the pair $(Z^\circ, \cup L_i)$ for integral homology groups, where L_i is the link of $Q_i \in Z$:

$$\bigoplus_{i=0}^s H_2(L_i, \mathbb{Z}) \rightarrow H_2(Z^\circ, \mathbb{Z}) \rightarrow H_2(Z, \mathbb{Z}) \rightarrow \bigoplus_{i=0}^s H_1(L_i, \mathbb{Z}) \rightarrow H_1(Z^\circ, \mathbb{Z}) \rightarrow H_1(Z, \mathbb{Z})$$

Since $H^i(Z, \mathcal{O}_Z) = 0$ for $i = 1, 2$, we have $H_2(Z^\circ, \mathbb{Z}) = \text{Pic}(Z)$, $H_2(Z, \mathbb{Z}) = \text{Cl}(Z)$, and $H_1(L_i, \mathbb{Z})$ is the local class group of $Q_i \in Z$, see [K4, Prop.4.2 and 4.11]. The claim follows since $H_1(L_i, \mathbb{Z}) = \mathbb{Z}/m_i\mathbb{Z}$, $H_2(L_i, \mathbb{Z}) = 0$, and $H_1(Z^\circ, \mathbb{Z}) = 0$. \square

Corollary 8.2. *If \overline{W}° is simply-connected, then \overline{W} satisfies Assumption 1.10 (3).*

Let $p, q \geq 2$ be coprime integers, where q is not divisible by 3. We first construct a resolution X of \overline{W} by blowing up a rational elliptic fibration S with a section σ_0 and a I_9 fiber (Kodaira notation). There is a unique such elliptic fibration [P1]. We further blow-up X to get a resolution \tilde{X} of an M -resolution W^+ of \overline{W} . Finally, $D_{p,q}$ is a \mathbb{Q} -Gorenstein smoothing of W^+ . The diagram summarizes the construction:

$$\begin{array}{ccccccc} \mathbb{P}^2 & \longleftarrow & S & \longleftarrow & X & \longleftarrow & \tilde{X} & \longrightarrow & W^+ & \longleftarrow & \text{---} & D_{p,q} \\ & & \downarrow & & & & & & \downarrow & & & \\ & & \mathbb{P}^1 & & & & & & \overline{W} & & & \end{array}$$

Here $S \rightarrow \mathbb{P}^2$ is the resolution of the base points of the cubic pencil which defines the elliptic fibration $S \rightarrow \mathbb{P}^1$. Let us choose one of the I_1 fibers of S , and let $\pi: X \rightarrow S$ be the blow-up of S as indicated in Figure 11, where $A_1^2 = B_1^2 = -1$, $A_i^2 = B_i^2 = -2$ for $i > 1$. The curves G_0, \dots, G_9 are proper transforms of irreducible components of I_9 and I_1 fibers. The curves $\sigma_0, \sigma_1, \sigma_2$ in Figure 11 are the 3 sections of $S \rightarrow \mathbb{P}^1$. Note that $\rho(X) = 10 + p - 1 + q - 1 = p + q + 8$. The surface \overline{W} is

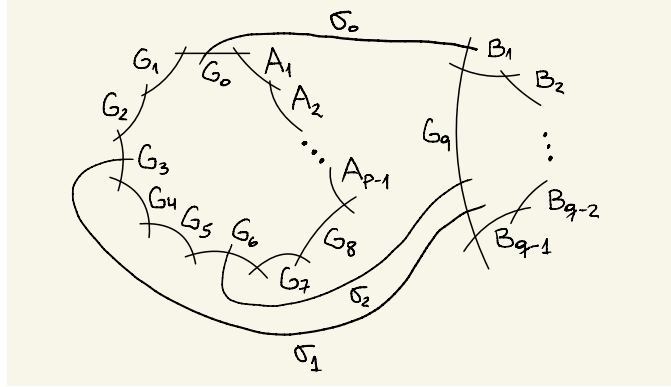


FIGURE 11. X as a blow-up of $S (I_9 + 3I_1)$

obtained by contracting the chain $B_2, \dots, B_{q-1}, G_q, \sigma_0, G_0, G_1, \dots, G_8, A_{p-1}, \dots, A_2 \subset X$, which corresponds to the continued fraction

$$\left[\underbrace{2, \dots, 2}_{q-2}, q+2, 1, p+1, \underbrace{2, \dots, 2}_7, 3, \underbrace{2, \dots, 2}_{p-2} \right].$$

In particular, \overline{W} has Picard number 1.

Proposition 8.3. *The surface \overline{W}^o is simply-connected.*

Proof. We are going to use Mumford's computation [M, p.99] on the resolution $X \rightarrow \overline{W}$, which is not minimal. To show that $\pi_1(\overline{W}^o) = \{1\}$, we use Van-Kampen's Theorem for a neighborhood of P and its complement. As $\pi_1(\overline{W}) = \{1\}$, it suffices to show that a generator loop around the exceptional divisor is homotopically trivial in the complement of the exceptional divisor. We note that loops α and β around the ending (-2) -curves A_2 and B_2 respectively are (each) generators of the fundamental group of the link of $P \in \overline{W}$. Let γ be a loop around σ_0 . By Mumford's computation, we have that α^{9p^2} is conjugate to γ , as well as β^{q^2} . We claim that γ is homotopically trivial in the complement of the exceptional divisor. Given the claim, α^{9p^2} and β^{q^2} are trivial. But because $\gcd(9p^2, q^2) = 1$, we have that both loops α and β are actually trivial as had to be demonstrated.

To prove the claim we use that σ_0 intersects another I_1 fiber transversally at one point. It follows that γ is a loop around a puncture in the nodal cubic punctured

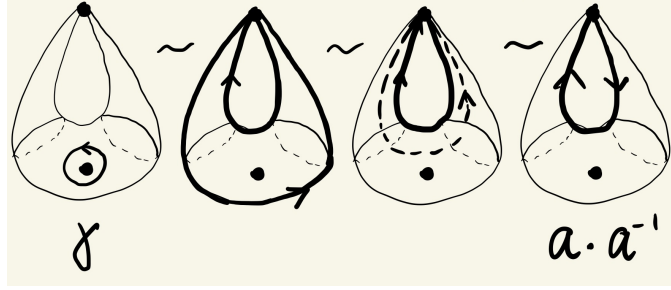


FIGURE 12. Homotopy from γ to identity

in one point. As Figure 12 shows, this loop is homotopically trivial. \square

We now consider a composition of blow-ups $\tilde{X} \rightarrow X$ to obtain 9 Wahl chains of type $[p+2, 2, \dots, 2]$ over the I_9 fiber together with the Wahl chain $[q+2, 2, \dots, 2]$ from the chosen I_1 fiber. The Wahl chains $[p+2, 2, \dots, 2]$ have the proper transform of G_i as $-(p+2)$ -curve for $i = 0, \dots, 8$. The contraction of these Wahl chains defines the surface W^+ with nine $\frac{1}{p^2}(1, p-1)$ and one $\frac{1}{q^2}(1, q-1)$ singularities. The surface W^+ is an M -resolution of \overline{W} with curves $\Gamma_i, i = 1, \dots, 9$ defined as follows: Γ_1 is the image of σ_0 , the rest are the images of the (-1) -curves connecting the Wahl chains $[p+2, 2, \dots, 2]$. The construction of $D_{p,q}$ via \mathbb{Q} -Gorenstein smoothings of surfaces W^+ was considered in [U, Cor. 4.3]. In particular, $H^2(\overline{W}, T_{\overline{W}}) = 0$.

Now we can apply Theorem 1.12 to prove the following.

Theorem 8.4. *A Dolgachev surface $D_{p,q}$ has a H.e.c. E_9, \dots, E_0 associated with W^+ and a strong H.e.c. $\bar{E}_9, \dots, \bar{E}_0$ associated with the N -resolution W^- , where*

- (1) $\delta_1 = pq - p - q$ and $\delta_i = 0$ for $i = 2, \dots, 9$, in particular $\text{End}(\bar{E}_9 \oplus \dots \oplus \bar{E}_0)$ is the endomorphism algebra of the quiver with vertices $\bar{P}_0, \dots, \bar{P}_9$ and with $pq - p - q$ arrows connecting each \bar{P}_i to \bar{P}_9 for $i = 0, \dots, 8$.
- (2) $\bar{n}_i = \text{rank } \bar{E}_i$ and Wahl singularities in the N -resolution can be computed using continued fractions

$$\frac{\bar{n}_k^2}{\bar{n}_k \bar{a}_k - 1} = \underbrace{[2, \dots, 2, q+1, p+1, 2, \dots, 2, 3, 2, \dots, 2, q]}_{q-2} \underbrace{\hspace{1.5cm}}_{p-4} \underbrace{\hspace{1.5cm}}_{q-2}$$
 for $k = 0, \dots, 8$ and

$$\frac{\bar{n}_9^2}{\bar{n}_9 \bar{a}_9 - 1} = \underbrace{[2, \dots, 2, q+2]}_{q-2}.$$

(3) The orthogonal complement of $\langle \bar{E}_9, \dots, \bar{E}_0 \rangle \subset D^b(Y)$ has Mukai lattice \mathbb{Z}^2 with Euler pairing given by the Gram matrix

$$\begin{pmatrix} -1 & 3(pq - p - q) \\ 0 & -1 \end{pmatrix}. \quad (8.1)$$

This lattice has a full numerical exceptional collection if and only if $p = 3, q = 2$.

Example 8.5. For $p = 3, q = 2$, we obtain a M-resolution

$$[4] - (1) - [5, 2] - (1) - [5, 2] - (1) - \dots - (1) - [5, 2],$$

and a N-resolution

$$[3, 5, 2] - (1) - [3, 5, 2] - (1) - \dots - (1) - [3, 5, 2] - (1) - [4].$$

Proof. We first compute the numerical data. Let us assume $p, q \geq 3$. Then the c.q.s. $P \in \bar{W}$ is $\frac{\Delta}{\Omega} = [2, \dots, 2, q+1, p, \underbrace{2, \dots, 2}_7, 3, \underbrace{2, \dots, 2}_{p-2}]$, and so the dual fraction is $\frac{\Delta}{\Delta - \Omega} = [q, \underbrace{2, \dots, 2}_{q-2}, 3, \underbrace{2, \dots, 2}_{p-3}, \bar{10}, p]$, where $\bar{2}$ is the position of $d_{i_1} = 1$, and $\bar{10}$ is the position of $d_{i_2} = 9$. In this way $\frac{\bar{n}_k}{\bar{n}_k - \bar{a}_k} = [q, \underbrace{2, \dots, 2}_{q-2}, 3, \underbrace{2, \dots, 2}_{p-3}]$ for all $k = 0, \dots, 8$, and $\frac{\bar{n}_9}{\bar{n}_9 - \bar{a}_9} = [q]$. Therefore, the Wahl singularities in the N-resolution have continued fractions as in part (2). The only $\delta \neq 0$ is $\delta_1 = qp - p - q$, since it is the numerator of $[\underbrace{2, \dots, 2}_{q-3}, 3, \underbrace{2, \dots, 2}_{p-3}]$. It remains to compute the Mukai lattice.

Recall that if Z is a rational projective normal surface then we have an isomorphism of abelian groups $(r, c_1, \chi): G_0(Z) \rightarrow \mathbb{Z} \oplus \text{Cl } Z \oplus \mathbb{Z}$, where $G_0(Z)$ is the Grothendieck group of $D^b(Z)$, see [KKS, Lemma 4.2, Remark 4.3].

Lemma 8.6. Let X be a resolution of singularities (not necessarily minimal) with the exceptional divisor C_1, C_2, \dots, C_s of a c.q.s. surface \bar{W} satisfying Assumption 1.10. The Mukai lattice of \mathcal{B}^W is isomorphic to a sublattice in $K_0(X)$ formed by elements β that satisfy the equations $\chi(\beta) = 0, r(\beta) = -C_1 \cdot c_1(\beta)$, and $C_i \cdot c_1(\beta) = 0$ for $i > 1$.

Proof. We recall that the Euler pairing $\chi(\alpha, \beta) = \sum (-1)^i \text{ext}^i(\alpha, \beta)$ has a form

$$\begin{aligned} \chi(\alpha, \beta) &= r(\alpha) \text{ch}_2(\beta) - c_1(\alpha) \cdot c_1(\beta) + r(\beta) \text{ch}_2(\alpha) \\ &\quad + \frac{1}{2} (r(\beta)c_1(\alpha) - r(\alpha)c_1(\beta)) \cdot K_X + r(\alpha)r(\beta)\chi(\mathcal{O}_X). \end{aligned}$$

In our case $\chi(\mathcal{O}_X) = 1$. The subcategory \mathcal{B}^W is the orthogonal complement to the exceptional collection of line bundles $\mathcal{O}_X(-C_s - \dots - C_1 - C_0), \mathcal{O}_X(-C_{s-1} - \dots - C_1 - C_0), \dots, \mathcal{O}_X(-C_1 - C_0), \mathcal{O}_X(-C_0)$, where C_0 is the strict transform of \bar{A} in \bar{W} (Assumption 1.10 (3)). As tensoring by $\mathcal{O}_X(C_0)$ is an autoequivalence of $D^b(X)$, for this computation we consider the orthogonal complement of $L_s := \mathcal{O}_X(-C_s - \dots - C_1), L_{s-1} := \mathcal{O}_X(-C_{s-1} - \dots - C_1), \dots, L_1 := \mathcal{O}_X(-C_1), L_0 := \mathcal{O}_X$, which have classes $r(\alpha_i) = 1, c_1(\alpha_i) = L_i, \text{ch}_2(\alpha_i) = \frac{L_i^2}{2}$. It follows that

$$\text{ch}_2(\beta) - L_i \cdot c_1(\beta) + r(\beta) \frac{L_i^2}{2} + \frac{1}{2} (r(\beta)L_i - c_1(\beta)) \cdot K_X + r(\beta) = 0$$

We start with $L_0 = 0$, which gives $\text{ch}_2(\beta) - \frac{1}{2}c_1(\beta) \cdot K_X + r(\beta) = 0$. The remaining equations then become $-2L_i \cdot c_1(\beta) + r(\beta)(K + L_i) \cdot L_i = 0$. Since $p_a(L_i) = 0$ for every $i > 0$, the equation in fact is simply $L_i \cdot c_1(\beta) + r(\beta) = 0$. Finally, notice that $L_i - L_{i-1} = C_i$, which gives equations in the statement. \square

For any rational elliptic fibration S with a section σ_0 , $\text{Pic}(S) = -\mathbb{E}_8 \oplus \langle 1 \rangle \oplus \langle -1 \rangle$, where $\langle -1 \rangle$ is generated by σ_0 and $\langle 1 \rangle$ is generated by $\sigma_0 + F$, where $F \sim -K_S$ is a general fiber. Furthermore, $-\mathbb{E}_8$ contains a sublattice T generated by components of reducible fibers that do not intersect σ_0 .

In our case we consider the basis $G_1, G_2, G_3, G_4, G_5, G_6, G_7, v := \sigma_1 - \sigma_2$ of $-\mathbb{E}_8$. Let $\beta \in \mathcal{B}$. By Lemma 8.6, we have that $(r(\beta), c_1(\beta), \chi(\beta)) = (-B_2 \cdot B, B, 0)$, where B satisfies $C_i \cdot B = 0$ for $i > 1$ and $C_1 := B_2$. Let us write

$$B = z_1 G_1 + z_2 G_2 + z_3 G_3 + z_4 G_4 + z_5 G_5 + z_6 G_6 + z_7 G_7 + \\ xv + y(\sigma_0 + F) + z\sigma_0 + \sum_{i=1}^{q-1} x_i \beta_i + \sum_{i=1}^{p-1} y_i \alpha_i,$$

where $\alpha_i = A_1 + \dots + A_i$ and $\beta_j = B_1 + \dots + B_j$ for $i = 1, \dots, p-1$ and $j = 1, \dots, q-1$. By intersecting B with

$$C_2 := B_3, C_3 := B_4, \dots, B_{q-1}, G_9, \sigma_0, G_0, G_1, G_2, \dots, G_8, A_{p-1}, \dots, C_{p+q+7} := A_2,$$

we obtain the linear system

$$\begin{aligned} 0 &= x_2 - x_3, \quad 0 = x_3 - x_4, \dots, 0 = x_{q-2} - x_{q-1}, \\ 0 &= y + z + x_1 + x_2 + \dots + x_{q-2} + 2x_{q-1}, \quad 0 = -z \\ 0 &= z_1 + y + z + y_1 + y_2 + \dots + y_{p-1}, \quad 0 = -2z_1 + z_2, \quad 0 = z_1 - 2z_2 + z_3, \\ 0 &= z_2 - 2z_3 + z_4 + x, \quad 0 = z_3 - 2z_4 + z_5, \quad 0 = z_4 - 2z_5 + z_6, \\ 0 &= z_5 - 2z_6 + z_7 - x, \quad 0 = z_6 - 2z_7, \quad 0 = z_7 + y_{p-1}, \quad 0 = -y_{p-1} + y_{p-2}, \dots, 0 = -y_2 + y_1, \end{aligned}$$

which has solutions $y_1 = \dots = y_{p-1} = -3z_5$, $x_2 = \dots = x_{q-1}$, $x = -8z_5$, $y = 3pz_5$, $z = 0$, $x_1 = -3pz_5 - (q-1)x_2$, $z_1 = -3z_5$, $z_2 = -6z_5$, $z_3 = -9z_5$, $z_4 = -4z_5$, $z_6 = 6z_5$, and $z_7 = 3z_5$. This solution can be expressed via the \mathbb{Z} -basis

$$v_1 := -3G_1 - 6G_2 - 9G_3 - 4G_4 + G_5 + 6G_6 + 3G_7 - 8v + 3p(\sigma_0 + F) - 3pB_1 - 3 \sum_{i=1}^{p-1} \alpha_i,$$

and $v_2 := -(q-1)B_1 + \sum_{i=2}^{q-1} \beta_i$. Using Lemma 8.6, these vectors can be considered as generators \tilde{v}_1, \tilde{v}_2 of \mathcal{B} by setting $\chi(\tilde{v}_i) = 0$, $\text{rank}(\tilde{v}_1) = 3p$, and $\text{rank}(\tilde{v}_2) = q$. We have the intersection numbers $v_1 \cdot K_X = 3(p-1)$, $v_1 \cdot v_1 = -9p + 1$, $v_2 \cdot K_X = 1$, $v_2 \cdot v_2 = -q^2 + q + 1$, and $v_1 \cdot v_2 = -3p(q-1)$. Therefore $\chi(\tilde{v}_1, \tilde{v}_1) = -1$, $\chi(\tilde{v}_2, \tilde{v}_2) = -1$, $\chi(\tilde{v}_1, \tilde{v}_2) = 3(pq - p - q)$, and $\chi(\tilde{v}_2, \tilde{v}_1) = 0$, and so the Gram matrix is (8.1). \square

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