

$\overline{M}_{0,n}$ IS NOT A MORI DREAM SPACE

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ABSTRACT. Building on the work of Goto, Nishida and Watanabe on symbolic Rees algebras of monomial primes, we prove that the moduli space of stable rational curves with n punctures is not a Mori Dream Space for $n > 133$. This answers the question of Hu and Keel.

1. INTRODUCTION

We work over an algebraically closed field k . It was argued that $\overline{M}_{0,n}$ should be a Mori Dream Space (MDS for short) because it is “similar to a toric variety” and toric varieties are basic examples of MDS. We suggest to adjust this principle: $\overline{M}_{0,n}$ is similar to the blow-up of a toric variety at the identity element of the torus. Specifically, we prove the following. For any toric variety X , we denote by $\text{Bl}_e X$ the blow-up of X at the identity element of the torus. Let \overline{LM}_n be the Losev–Manin space [LM00]. It is a smooth projective toric variety of dimension $n - 3$.

Theorem 1.1. *There exists a small \mathbb{Q} -factorial projective modification \widetilde{LM}_{n+1} of $\text{Bl}_e \overline{LM}_{n+1}$ and surjective morphisms*

$$\widetilde{LM}_{n+1} \rightarrow \overline{M}_{0,n} \rightarrow \text{Bl}_e \overline{LM}_n.$$

In particular,

- *If $\overline{M}_{0,n}$ is a MDS then $\text{Bl}_e \overline{LM}_n$ is a MDS.*
- *If $\text{Bl}_e \overline{LM}_{n+1}$ is a MDS then $\overline{M}_{0,n}$ is a MDS.*

Next we invoke a beautiful theorem of Goto, Nishida, and Watanabe:

Theorem 1.2 ([GNW94]). *If $(a, b, c) = (7m - 3, 5m^2 - 2m, 8m - 3)$, with $m \geq 4$ and $3 \nmid m$, then $\text{Bl}_e \mathbb{P}(a, b, c)$ is not a MDS when $\text{char } k = 0$.*

We show that

Theorem 1.3. *Let $n = a + b + c + 8$, where a, b, c are positive coprime integers. If $\text{Bl}_e \overline{LM}_n$ is a MDS then $\text{Bl}_e \mathbb{P}(a, b, c)$ is a MDS.*

It immediately follows from these results, answering a question of Hu–Keel [HK00, Question 3.2], that:

Corollary 1.4. *Assume $\text{char } k = 0$. Then $\overline{M}_{0,n}$ is not a Mori Dream Space for $n \geq 134$.*

Understanding the birational geometry of the moduli spaces $\overline{M}_{g,n}$ of stable, n -pointed genus g curves is a problem that received a lot of attention from many authors. Interest in the effective cone originated in the work of Harris and Mumford [HM82] who showed that $\overline{M}_{g,n}$ is a variety of general type for large g . Mumford also raised the question of describing ample divisors, i.e., the nef cone. A long standing conjecture of Fulton and Faber provides a conjectural description, which was reduced to the case of genus 0 by Gibney, Keel, and Morrison [GKM02]. This prompted Hu and Keel [HK00] to raise the question if $\overline{M}_{0,n}$ is a Mori Dream Space. In positive genus, this is known to be typically false. For example, Keel proved in [Kee99] that, in characteristic zero, $\overline{M}_{g,n}$ is not a MDS for $g \geq 3$, $n \geq 1$, by proving that it has a nef divisor that is not semiample. Recently, Chen and Coskun proved in [CC13] that $\overline{M}_{1,n}$ is not a MDS for $n \geq 3$ as it has infinitely many extremal effective divisors. For genus zero, the only previously settled cases were for $n \leq 6$ ($\overline{M}_{0,5}$ is a del Pezzo surface, hence, a MDS by [BP04]; $\overline{M}_{0,6}$ is log-Fano threefold, hence, a MDS by [HK00]; for a direct proof that $\overline{M}_{0,6}$ is a MDS, see [Cas09]. Note more generally that in characteristic zero, log-Fano varieties are MDS by [BCHM10]; however, $\overline{M}_{0,n}$ is not log-Fano for $n \geq 7$). Since [HK00], the question whether $\overline{M}_{0,n}$ is a MDS was raised by several authors, see for example [Cas09], [AGS10], [GM10], [Kie10], [McK10], [Fed11], [BHK12], [GG12], [GHPS12], [GM12], [BGM13], [CT13], [GJM13], [Lar13]. One of the results in [GHPS12] is that $\overline{M}_{0,n}$ is a MDS if and only if the projectivization of the pull-back of the cotangent bundle of \mathbb{P}^{n-3} to \overline{LM}_n is a MDS. In particular, Cor. 1.4 adds to the examples in [GHPS12] of toric vector bundles whose projectivization is not a MDS.

The original motivation for Hu and Keel’s question was coming from Keel and McKernan’s result [KM96] that any extremal ray of the Mori cone of $\overline{M}_{0,n}$ that (1) can be contracted by a map of relative Picard number 1 and (2) the exceptional locus of the map in (1) has dimension at least 2, is generated by a one-dimensional stratum (i.e., the Fulton-Faber conjecture is satisfied for such rays). As in a MDS any extremal ray of the Mori cone can be contracted by a map of relative Picard number 1, a positive answer to the Hu-Keel question “would nearly answer Fulton’s question for $\overline{M}_{0,n}$ ” [HK00]. Implicit in this statement is the expectation that condition (2) should be satisfied. It was a long held belief that the exceptional locus of any map $\overline{M}_{0,n} \rightarrow X$ has all

components of dimension at least 2. We gave counterexamples to this statement in [CT12].

Remarks 1.5. (1) By the Kapranov description, $\overline{M}_{0,n}$ is the iterated blow-up of \mathbb{P}^{n-3} along proper transforms of linear subspaces spanned by $n - 1$ points in linearly general position. The Losev-Manin space \overline{LM}_n is the iterated blow-up of \mathbb{P}^{n-3} along proper transforms of linear subspaces spanned by $n - 2$ points in linearly general position. We denote by X_n the intermediate toric variety obtained by blowing-up only linear subspaces of codimension ≥ 3 . By Cor. 5.4, $\text{Bl}_e X_{n+1}$ is a small modification of a certain \mathbb{P}^1 -bundle over $\overline{M}_{0,n}$. In particular, $\text{Bl}_e X_{n+1}$ is not a Mori Dream Space if $\text{char } k = 0$ and $n \geq 134$.

(2) Thm. 1.2 is stated slightly differently in [GNW94]. In Section 4 we translate into a geometric proof the arguments in [GNW94]. They are based on reduction to positive characteristic and a version of Max Noether’s “AF+BG” theorem that holds for weighted projective planes.

(3) Several arguments in this paper involve elementary transformations of vector bundles, for example the second part of Thm. 1.1 follows by doing elementary transformations of rank 2 bundles on $\overline{M}_{0,n}$. We give a general criterion for being able to iterate elementary transformations (Prop. 5.3), which can be of independent interest.

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2. PRELIMINARIES

We briefly recall some basic properties of MDS from [HK00].

Let X be a normal projective variety. A *small \mathbb{Q} -factorial modification* (SQM for short) of X is a small (i.e., isomorphic in codimension one) birational map $X \dashrightarrow Y$ to another normal \mathbb{Q} -factorial projective variety Y .

Definition 2.1. A normal projective variety X is called a *Mori Dream Space (MDS)* if the following conditions hold:

- (1) X is \mathbb{Q} -factorial and $\text{Pic}(X)_{\mathbb{Q}} \cong N^1(X)_{\mathbb{Q}}$;
- (2) $\text{Nef}(X)$ is generated by finitely many semi-ample line bundles;
- (3) There is a finite collection of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (1), (2) and $\text{Mov}(X)$ is the union of $f_i^*(\text{Nef}(X_i))$.

2.2. In what follows, we will often make use of the following facts:

- If X is a MDS, any normal projective variety Y which is an SQM of X , is also a MDS. This follows from the fact that the f_i of Def. 2.1 are the only SQMs of X [HK00, Prop. 1.11].
- ([Oka11, Thm. 1.1]) Let $X \rightarrow Y$ be a surjective morphism of projective normal \mathbb{Q} -factorial varieties. If X is a MDS then Y is a MDS. Note, we only use this for maps f with connected fibers, in which case the statement follows from [HK00].

Definition 2.3. For a semigroup Γ of Weil divisors on X , consider the Γ -graded ring:

$$R(X, \Gamma) := \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}(D)).$$

where $\mathcal{O}(D)$ is the divisorial sheaf associated to the Weil divisor D . Suppose that the divisor class group $\text{Cl}(X)$ is finitely generated. If Γ is a group of Weil divisors such that $\Gamma_{\mathbb{Q}} \cong \text{Cl}(X)_{\mathbb{Q}}$, the ring $R(X, \Gamma)$ is called a *Cox ring* of X and is denoted $\text{Cox}(X)$.

The definition of $\text{Cox}(X)$ depends on the choice of Γ , but finite generation of $\text{Cox}(X)$ does not. Def. 2.3 differs from [HK00, Def. 2.6], in that $\text{Cl}(X)$ replaces $\text{Pic}(X)$. However, for us X will always be \mathbb{Q} -factorial; hence, finite generation of $\text{Cox}(X)$ is not affected. The following is an algebraic characterization of MDS:

Theorem 2.4. [HK00, Prop. 2.9] *Let X be a \mathbb{Q} -factorial projective variety with $\text{Pic}(X)_{\mathbb{Q}} \cong N^1(X)_{\mathbb{Q}}$. Then X is a MDS if and only if $\text{Cox}(X)$ is a finitely generated k -algebra.*

3. PROOF OF THEOREM 1.3

Proposition 3.1. *Let $\pi : N \rightarrow N'$ be a surjective map of lattices (finitely generated free \mathbb{Z} -modules) with kernel of rank 1 spanned by a primitive vector $v_0 \in N$. Let $\Gamma \subset N_{\mathbb{Q}}$ be a finite set of rays such that the rays $\pm R_0$ spanned by $\pm v_0$ are not in Γ . Let $\mathcal{F}' \subset N'_{\mathbb{R}}$ be a complete simplicial fan with rays given by $\pi(\Gamma)$. Suppose that the corresponding toric variety X' is projective (notice that it is also \mathbb{Q} -factorial because \mathcal{F}' is simplicial). Then*

(A) *There exists a complete simplicial fan $\mathcal{F} \subset N_{\mathbb{R}}$ with rays given by $\Gamma \cup \{\pm R_0\}$ and such that*

- *the corresponding toric variety X is projective;*
- *the rational map $p : X \dashrightarrow X'$ induced by π is regular;*
- *each cone of \mathcal{F} maps onto a cone of \mathcal{F}' .*

(B) *There exists an SQM Z of $\text{Bl}_e X$ such that the rational map $Z \dashrightarrow \text{Bl}_e X'$ induced by p is regular. In particular, if $\text{Bl}_e X$ is a MDS then $\text{Bl}_e X'$ is a MDS.*

Proof. We first prove (A). We argue by induction on $|\Gamma| - |\pi(\Gamma)|$. Suppose that this number is zero, and in particular we have a bijection between Γ and $\pi(\Gamma)$. Then we define \mathcal{F} as follows: for any subset $J \subset \Gamma$ (maybe empty) such that rays spanned by vectors in $\pi(J)$ form a cone, \mathcal{F} contains a cone spanned by rays in J , a cone spanned by rays in $J \cup \{R_0\}$, and a cone spanned by rays in $J \cup \{-R_0\}$. It follows from the fact that \mathcal{F}' is a complete simplicial fan that \mathcal{F} is also a complete simplicial fan $\mathcal{F} \subset N_{\mathbb{R}}$ with rays in $\Gamma \cup \{\pm R_0\}$. Moreover, the rational map $p : X \dashrightarrow X'$ induced by π is regular and in fact each cone of \mathcal{F} maps onto a cone of \mathcal{F}' .

Next we show that X is projective. It follows from the description of the map of fans that all fibers of p are \mathbb{P}^1 's (only set-theoretically because fibers are not necessarily reduced), and moreover D_0 , the torus invariant \mathbb{Q} -Cartier divisor corresponding to the ray R_0 , is a section of p and therefore is f -ample. It follows that p is projective and therefore that X is projective because X' is projective. For a purely toric proof of projectivity, let A be an ample Cartier divisor on X' . Let $D = D_0 + mp^*(A)$. We argue that the \mathbb{Q} -Cartier divisor D is ample for large $m > 0$ by using the Toric Kleiman Criterion [CLS11, Thm. 6.3.13], i.e., we prove that $D \cdot C > 0$ for every torus invariant curve C in X . Torus invariant curves have the form $V(\tau)$, for τ a cone in \mathcal{F} of dimension $n - 1$ ($n = \dim X$). There are two cases: (1) τ is spanned by rays R_1, \dots, R_{n-1} in Γ ; and (2) τ is spanned by R_0 and rays R_1, \dots, R_{n-2} in Γ . In Case (1), $p(C)$ is a point in X' ; hence, $D \cdot C = D_0 \cdot C$. Note that $\tau = \sigma \cap \sigma'$, where σ is the cone spanned by τ and R_0 and σ' is the cone spanned by τ and $-R_0$. Then by [CLS11, Lemma 6.4.2] $D_0 \cdot C = \frac{\text{mult}(\tau)}{\text{mult}(\sigma)} > 0$, where $\text{mult}(\sigma)$ denotes the multiplicity of a simplicial cone σ . In Case (2), $p(C)$ is the torus invariant curve $V(\bar{\tau})$ in X' , where $\bar{\tau} = \langle \pi(R_1), \dots, \pi(R_{n-2}) \rangle$. Let $M \geq 0$ be an integer such that $D_0 \cdot C' > -M$, for all the torus invariant curves C' in X . By the projection formula, $D \cdot C = D_0 \cdot C + mA \cdot p_*(C) > 0$ if $m \geq M$.

Now we do an inductive step. Let $R' \in \pi(\Gamma)$ and let $Z \subset \Gamma$ be the set of all rays $R \in \Gamma$ such that $\pi(R) = R'$. Without loss of generality we can suppose that $|Z| > 1$. Choose $R \in Z$. Let $\tilde{\Gamma} = \Gamma \setminus \{R\}$. Since the rays of \mathcal{F}' are given by $\pi(\tilde{\Gamma}) = \pi(\Gamma)$, by inductive assumption, the theorem is true for $\tilde{\Gamma}$. Let $\mathcal{G} \subset N_{\mathbb{R}}$ be the corresponding fan and \tilde{X} be the corresponding toric variety. Then $\pi^{-1}(R')$ is the union of cones in \mathcal{G} spanned by pairs of rays:

$$\{R_0 = U_0, U_1\}, \{U_1, U_2\}, \dots, \{U_{k-1}, U_k\}, \{U_k, U_{k+1} = -R_0\},$$

where $\{U_1, \dots, U_k\} = Z \setminus \{R\}$. Choose an index i such that R belongs to the relative interior of the angle spanned by U_i and U_{i+1} . Then the fan \mathcal{F} is obtained as a star subdivision on \mathcal{G} centered at R . By [CLS11, Prop. 11.1.6] the map $X \rightarrow \tilde{X}$ is projective. All properties in (A) are clearly satisfied.

Now we prove (B). Notice that the map $p : X \rightarrow X'$ over an open torus $T' \subset X'$ is a trivial \mathbb{P}^1 bundle $\text{pr}_1 : T' \times \mathbb{P}^1 \rightarrow T'$. We do a linear change of variables to identify

$$T' \simeq \mathbb{A}^k \setminus \bigcup_i \{x_i = -1\}, \quad e \mapsto 0$$

and

$$\mathbb{P}^1 \simeq \mathbb{P}^1, \quad 1 \mapsto 0.$$

Thus we identify $p|_{p^{-1}(T')}$ with the restriction of the toric projection map $\text{pr}_1 : \mathbb{A}^k \times \mathbb{P}^1 \rightarrow \mathbb{A}^k$ (for a different choice of the toric structure) to an open set $T' \subset \mathbb{A}^k$. Blow-ups of X and X' at the identity elements of their tori now correspond to blow-ups Y (resp. Y') of $\mathbb{A}^k \times \mathbb{P}^1$ (resp. \mathbb{A}^k) in torus fixed points. The fans are as follows: the fan of Y' is the star subdivision of the positive octant $\langle e_1, \dots, e_k \rangle$ in the vector $e_0 := e_1 + \dots + e_k$. Its top-dimensional cones are spanned by e_0 and $\{e_i\}_{i \in I}$, where $I \subset \{1, \dots, k\}$ is a subset of cardinality $k - 1$. The fan of Y contains an octant $\tau = \langle e_1, \dots, e_k, -e_{k+1} \rangle$ and the star subdivision of the positive octant $\langle e_1, \dots, e_k, e_{k+1} \rangle$ in the vector $f_0 := e_1 + \dots + e_{k+1}$. In particular, the fan of Y contains a cone $\tau' = \langle e_1, \dots, e_k, f_0 \rangle$. We construct a small modification \tilde{Y} of Y as follows: We remove cones τ and τ' from the fan of Y and instead add k top-dimensional cones spanned by f_0 , $-e_{k+1}$, and $\{e_i\}_{i \in I}$, where $I \subset \{1, \dots, k\}$ is a subset of cardinality $k - 1$. To see this geometrically, consider the trivial bundle $\mathbb{P} := Y' \times \mathbb{P}^1 \rightarrow Y'$ with its sections $s_0 = Y' \times \{0\}$ and $s_\infty = Y' \times \{\infty\}$. If E denotes the exceptional divisor of $Y' \rightarrow \mathbb{A}^k$, let $Z = s_0(E)$. Let $\tilde{\mathbb{P}}$ of \mathbb{P} be the blow-up of \mathbb{P} along Z . Let $D = E \times \mathbb{P}^1 \subset \mathbb{P}$ and let \tilde{D} be its proper transform in $\tilde{\mathbb{P}}$. There are two ways to blow-down $\tilde{D} \cong \mathbb{P}^{k-1} \times \mathbb{P}^1$:

$$\alpha : \tilde{\mathbb{P}} \rightarrow \tilde{Y}, \quad \alpha(\tilde{D}) = \tilde{s}_\infty(E) \cong \mathbb{P}^{k-1},$$

$$\beta : \tilde{\mathbb{P}} \rightarrow Y, \quad \beta(\tilde{D}) = \tilde{F} \cong \mathbb{P}^1, \quad F = \{0\} \times \mathbb{P}^1$$

where \tilde{s}_∞ is the proper transform of the section s_∞ under the rational map $\mathbb{P} \dashrightarrow \tilde{Y}$ and \tilde{F} is the proper transform of F in Y . Notice that the rational map $\tilde{Y} \dashrightarrow Y'$ is regular, and in fact it is the \mathbb{P}^1 -bundle $\mathbb{P}_{Y'}(\mathcal{O} \oplus \mathcal{O}(-E))$ (an elementary transformation, see Section 5). Note

that over $Y' \setminus E \cong (\mathbb{A}^k \setminus \{0\}) \times \mathbb{P}^1$ all above birational maps are isomorphisms.

To construct Z and the morphism $f : Z \rightarrow \text{Bl}_e X'$ that factors through $\text{Bl}_e X$, we glue $\tilde{Y} \rightarrow Y'$ (with preimages of hyperplanes $\{x_i = -1\}$ removed) to

$$p : X \setminus p^{-1}\{e\} \rightarrow X' \setminus \{e\}$$

along the \mathbb{P}^1 -bundle $\text{pr}_1 : (T' \setminus \{e\}) \times \mathbb{P}^1 \rightarrow (T' \setminus \{e\})$. Clearly, Z is \mathbb{Q} -factorial, since \tilde{Y} and X are \mathbb{Q} -factorial.

It remains to show that Z is projective and it would suffice to show that the morphism f is projective. This morphism is clearly projective in both charts of Z , but since projectivity is not local on the base, we have to give a global argument. It is enough to construct an f -ample divisor on Z . Let A be an irreducible very ample divisor on X and let \tilde{A} be its proper transform in Z . We claim that \tilde{A} is f -ample. Indeed, it is obviously f -ample in the second chart of Z . But the first chart is a \mathbb{P}^1 -bundle and \tilde{A} surjects onto the base, and so it is f -ample. \square

Proof of Thm. 1.3. The toric data of \overline{LM}_n is as follows, see [LM00]. Fix general vectors $e_1, \dots, e_{n-2} \in \mathbb{R}^{n-3}$ such that $e_1 + \dots + e_{n-2} = 0$. The lattice N is generated by e_1, \dots, e_{n-2} . The rays of the fan of \overline{LM}_n are spanned by the primitive lattice vectors $\sum_{i \in I} e_i$, for each subset I of $S := \{1, \dots, n-2\}$ with $1 \leq |I| \leq n-3$. Notice that rays of this fan come in opposite pairs. We are not going to need cones of higher dimension of this fan. The main idea is to choose a sequence of projections from these rays to get a sequence of (generically) \mathbb{P}^1 -bundles

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow \dots,$$

where X_1 is an SQM of \overline{LM}_n which is different from the standard tower of forgetful maps

$$\overline{LM}_n \rightarrow \overline{LM}_{n-1} \rightarrow \overline{LM}_{n-2} \rightarrow \dots$$

Specifically, we partition

$$S = S_1 \coprod S_2 \coprod S_3$$

into subsets of size $a+2, b+2, c+2$ (so $n = a+b+c+8$). We also fix some indices $n_i \in S_i$, for $i = 1, 2, 3$. Let $N'' \subset N$ be a sublattice spanned by the following vectors:

$$e_{n_i} + e_r \quad \text{for } r \in S_i \setminus \{n_i\}, \quad i = 1, 2, 3. \quad (3.1)$$

Let $N' = N/N''$ be the quotient group and let π be the projection map. Then we have the following:

- (1) N' is a lattice;

- (2) N' is spanned by the vectors $\pi(e_{n_i})$, for $i = 1, 2, 3$;
- (3) $a\pi(e_{n_1}) + b\pi(e_{n_2}) + c\pi(e_{n_3}) = 0$.

It follows at once that the toric surface with lattice N' and rays spanned by $\pi(e_{n_i})$ for $i = 1, 2, 3$, is a weighted projective plane $\mathbb{P}(a, b, c)$.

To finish the proof of the theorem, we apply Prop. 3.1 inductively to a sequence of lattices N_j , $j = 1, \dots, n - 4$, obtained by taking the quotient of N by a sublattice spanned by the first $j - 1$ vectors of the sequence (3.1) (arranged in any order) and sets of rays Γ_j obtained by projecting the rays of the fan of \overline{LM}_n . More precisely, we do an induction backwards, by starting with a canonical simplicial structure on the fan of a complete (hence, projective) toric surface X_{n-4} with data $N' = N_{n-4}$, Γ_{n-4} . It remains to notice that we have a regular map $X_{n-4} \rightarrow \mathbb{P}(a, b, c)$ obtained by dropping all vectors in Γ_{n-4} except for $\pi(e_{n_i})$ for $i = 1, 2, 3$. Clearly, the map is an isomorphism on the open torus; hence, there is a birational morphism $\text{Bl}_e X_{n-4} \rightarrow \text{Bl}_e \mathbb{P}(a, b, c)$. The result of applying induction is a sequence of toric morphisms

$$X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{n-4},$$

such that the rational map $\text{Bl}_e X_i \rightarrow \text{Bl}_e X_{i-1}$ factors through a projective \mathbb{Q} -factorial small modification Z_i of $\text{Bl}_e X_i$, followed by a surjective regular map $Z_i \rightarrow \text{Bl}_e X_{i-1}$. The first toric variety in the sequence X_1 is a small modification of \overline{LM}_n (having the same rays) which is an isomorphism on the open torus. Hence, $\text{Bl}_e X_1$ is a small modification of $\text{Bl}_e \overline{LM}_n$. The result now follows from Thm. 1.2 and 2.2. \square

4. PROOF OF THEOREM 1.2

The results in [GNW94] are stated in a slightly different form than Thm. 1.2. We first explain how our formulation is equivalent to [GNW94, Cor. 1.2]. For the reader's convenience, we also translate the arguments in [GNW94] into a geometric proof of Thm. 1.2.

Let $a, b, c > 0$ be pairwise coprime integers. Let $\mathbb{P} := \mathbb{P}(a, b, c)$ be the weighted projective space $\text{Proj } k[x, y, z]$, with $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$. Then \mathbb{P} is a toric variety which is smooth outside the three torus invariant points. Consider the torus invariant divisors:

$$D_1 = V_+(x), \quad D_2 = V_+(y), \quad D_3 = V_+(z).$$

Let m_i ($i = 1, 2, 3$) be integers such that $m_1 a + m_2 b + m_3 c = 1$ and let $H = \sum m_i D_i$. Then $\text{Cl}(\mathbb{P}) = \mathbb{Z}\{H\}$, H is \mathbb{Q} -Cartier and $H^2 = 1/(abc)$.

Let $\mathfrak{p} := \mathfrak{p}(a, b, c)$ be the kernel of the k -algebra homomorphism:

$$\phi : k[x, y, z] \rightarrow k[t], \quad \phi(x) = t^a, \quad \phi(y) = t^b, \quad \phi(z) = t^c.$$

The identity of the open torus in \mathbb{P} is the point $e = V_+(\mathfrak{p})$. Let $X = \text{Bl}_e \mathbb{P}$ denote the blow-up of \mathbb{P} at e ; let E denote the exceptional divisor. As $e \notin D_i$, we can pull-back to X the Weil divisors D_i and let $A = \sum m_i \pi^{-1}(D_i)$. Then $\text{Cl}(X) = \mathbb{Z}\{A, E\}$. A Cox ring of X is:

$$\text{Cox}(X) = \bigoplus_{d,l \in \mathbb{Z}} \text{H}^0(X, \mathcal{O}(dA - lE)).$$

Note that since a, b, c are pairwise coprime, $\mathcal{O}(dH) \cong \mathcal{O}(d)$.

It was observed by Cutkosky [Cut91] that finite generation of $\text{Cox}(X)$ is equivalent to the finite generation of the symbolic Rees algebra $R_s(\mathfrak{p})$ (here we follow the exposition in [KM09]). Recall that for a prime ideal \mathfrak{p} in a ring R , the l -th *symbolic power* of \mathfrak{p} is the ideal:

$$\mathfrak{p}^{(l)} = \mathfrak{p}^l R_{\mathfrak{p}} \cap R.$$

The subring of the polynomial ring $R[T]$ given by

$$R_s(\mathfrak{p}) := \bigoplus_{l \geq 0} \mathfrak{p}^{(l)} T^l,$$

is called the *symbolic Rees algebra* of \mathfrak{p} .

In our situation, for the prime ideal \mathfrak{p} in $S = k[x, y, z]$ defined above, we identify the symbolic Rees algebra $R_s(\mathfrak{p})$ with a subalgebra of $\text{Cox}(X)$. Using the identification $\text{H}^0(\mathbb{P}, \mathcal{O}(d)) = S_d$, we have:

$$\text{H}^0(X, \mathcal{O}(dA - lE)) \cong \text{H}^0(\mathbb{P}, \mathcal{O}(d) \otimes \mathcal{I}_e^l) = S_d \cap \mathfrak{p}^{(l)},$$

where \mathcal{I}_e denotes the ideal sheaf of the point e . It follows that $R_s(\mathfrak{p})$ is isomorphic to the subalgebra of $\text{Cox}(X)$ given by

$$\bigoplus_{d,l \geq 0} \text{H}^0(X, \mathcal{O}(dA - lE)).$$

Moreover, $\text{Cox}(X)$ is isomorphic to the extended symbolic Rees ring:

$$R_s(\mathfrak{p})[T^{-1}] = \dots \oplus ST^{-2} \oplus ST^{-1} \oplus S \oplus \mathfrak{p}T \oplus \mathfrak{p}^{(2)}T^2 \oplus \dots$$

Clearly, $R_s(\mathfrak{p})$ is a finitely generated k -algebra if and only if $\text{Cox}(X)$ is.

Assume now that

$$(a, b, c) = (7m - 3, 5m^2 - 2m, 8m - 3), \quad m \geq 4, \quad m \not\equiv 0 \pmod{3}.$$

By [GNW94, Cor. 1.2], the symbolic Rees algebra $R_s(\hat{\mathfrak{p}})$ of the extended ideal $\hat{\mathfrak{p}}$ in the formal power series ring $\hat{S} = k[[x, y, z]]$ is not Noetherian if $\text{char } k = 0$ (and it is Noetherian if $\text{char } k > 0$). Since $R_s(\mathfrak{p}) \otimes_S \hat{S} \cong R_s(\hat{\mathfrak{p}})$ [GN94, Lemma 2.3], it follows that $R_s(\mathfrak{p})$ is not finitely generated. Indeed, otherwise $R_s(\hat{\mathfrak{p}})$ would be a finitely generated \hat{S} -algebra and hence Noetherian by Hilbert's basis theorem.

We now give a geometric proof of Thm. 1.2. First note the following characterization of X being a MDS in the presence of negative curve:

Lemma 4.1. [Hun87, Cut91] *Assume $X = \text{Bl}_e \mathbb{P}$ contains an irreducible curve $C \neq E$ with $C^2 < 0$. Then X is a MDS if and only if there exists an effective divisor D such that $D \cdot C = 0$ and D does not contain C as a fixed component.*

Proof. Since $C^2 < 0$, it follows that C generates an extremal ray of the Mori cone $\overline{\text{NE}}(X)$ and hence, $\overline{\text{NE}}(X) = \mathbb{R}_{\geq 0}\{C, E\}$. The nef cone is generated by H and the ray R in $\overline{\text{NE}}(X)$ defined by $R \cdot C = 0$, $R \cdot E > 0$. Then X is a MDS if and only if R is generated by a semiample divisor. This proves the “only if” implication. If there is an effective divisor D as in the lemma, we may replace D with a divisor that has no fixed components and D is semiample by Zariski’s theorem ([Laz04, 2.1.32]). \square

Remark 4.2. As observed by Cutkosky [Cut91], if $\text{char } k > 0$ and $X = \text{Bl}_e \mathbb{P}$ contains a negative curve, then X is always a MDS due to Artin’s contractability criterion [Art62].

Let now $(a, b, c) = (7m - 3, 5m^2 - 2m, 8m - 3)$, $m \geq 4$, $m \not\equiv 0 \pmod{3}$. Let C be the proper transform on X of the curve $y^3 = x^m z^m$ in \mathbb{P} . The class of C in $\text{Cl}(X)$ is

$$C = 3(5m^2 - 2m)H - E.$$

Note that C is an irreducible curve with $C^2 < 0$. If $D \in \overline{\text{NE}}(X)$ is such that $D \cdot C = 0$, the class of D equals

$$D_d := d(7m - 3)(8m - 3)H - 3dE,$$

for some positive integer d .

Consider the set \mathcal{I} of effective Weil divisors D on X such that $D \cdot C = 0$ and D does not contain C as a fixed component. A crucial fact is the following:

Proposition 4.3. [GNW94] *The set*

$$I = \{d \in \mathbb{Z}_{\geq 0} \mid \exists D \in \mathcal{I}, [D] = D_d\}$$

equals $\mathbb{Z}_{\geq 0}d_0$ for some non-negative integer d_0 .

We will prove Prop. 4.3 using a version of Max Noether’s “AF+BG” theorem [Ful89, p. 61] that holds for weighted projective planes. Note that \mathcal{I} and I depend on the field k . We will write \mathcal{I}_k whenever we need to specify the field k .

Definition 4.4. Let $f, g \in S$ and \mathfrak{p} be a prime ideal in S which is a minimal prime of the ideal (f, g) . We say that $h \in S$ satisfies Noether's condition at the prime ideal \mathfrak{p} (with respect to f, g) if $h \in (f, g)S_{\mathfrak{p}}$.

Proposition 4.5 (AF+BG theorem). Let $f, g, h \in S$. Assume that the minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of the ideal (f, g) all have height 2. If h satisfies Noether's condition at \mathfrak{p}_i for all $i = 1, \dots, s$, then $h \in (f, g)$.

Proof. As $h \in (f, g)S_{\mathfrak{p}_i}$, there exist $u_i \in S \setminus \mathfrak{p}_i$ such that $u_i h \in (f, g)$. For each i we can find elements $y_i \in \bigcap_{j \neq i} \mathfrak{p}_j \setminus \mathfrak{p}_i$. Then $u := \sum u_i y_i \notin \mathfrak{p}_i$ for any i and $uh \in (f, g)$. Since S is Cohen-Macaulay, by the Unmixedness Theorem [Eis95, Cor. 18.14], all the associated primes of (f, g) are minimal. Hence, the zero divisors of $S/(f, g)$ consist of elements from \mathfrak{p}_i 's. It follows that u is not a zero divisor in $S/(f, g)$, hence $h \in (f, g)$. \square

Corollary 4.6. If $F = V_+(f)$, $G = V_+(g)$ are curves in \mathbb{P} with no common components and $h \in S$ satisfies Noether's condition at each point of $F \cap G$, then $h = Af + Bg$, for some $A, B \in S$.

Lemma 4.7. Assume $F = V_+(f)$ and $G = V_+(g)$ are curves in \mathbb{P} with no common components, $F \cap G$ does not contain any of the torus invariant points, and F is smooth along $F \cap G$. Let $h \in S$ and let $G' = V_+(h)$. Assume that for all $p \in F \cap G$ we have:

$$\text{mult}_p(G', F) \geq \text{mult}_p(G, F).$$

Then h satisfies Noether's condition at each point of $F \cap G$.

Remark 4.8. Note that this lemma includes the "classical" case when F and G intersect transversally (and away from torus fixed points) and G' passes through all points in $F \cap G$.

Proof. Let $p \in F \cap G$ with the corresponding homogeneous prime ideal \mathfrak{p} . By assumption, at least two of x, y, z are not in \mathfrak{p} . Say $x, y \notin \mathfrak{p}$. Since a, b are coprime, let m_1, m_2 be integers such that $m_1 a + m_2 b = 1$. Let $r = x^{m_1} y^{m_2}$. Note that r is a unit in S_{xy} . For $f \in S_d$, denote $f_1 = f/r^d \in S_{(xy)}$. Consider the functions f_1, g_1, h_1 corresponding to f, g, h . Denote by t the generator of the maximal ideal of $\mathcal{O}_{C,p} = \mathcal{O}_{\mathbb{P},p}/(f_1)$. If \bar{g}_1, \bar{h}_1 denote the images of g_1, h_1 in $\mathcal{O}_{C,p}$, we have $\bar{g}_1 = ut^n, \bar{h}_1 = vt^m$, for units $u, v \in \mathcal{O}_{C,p}$ and with $n = \text{mult}_p(G, F)$, $m = \text{mult}_p(G', F)$. As $m \geq n$, it follows that $\bar{h}_1 \in (\bar{g}_1)$, i.e., $h_1 \in (f_1, g_1) \subseteq \mathcal{O}_{\mathbb{P},p} = S_{(\mathfrak{p})}$. Since $x, y \notin \mathfrak{p}$, it follows that $h \in (f, g)S_{\mathfrak{p}}$. \square

Proof of Thm. 4.3. Assume $\mathcal{I} \neq \emptyset$ and let d_0 be the smallest positive integer in I . Let $g \in S$ be such that the proper transform D in X of

$G := V_+(g) \subset \mathbb{P}$ has class D_{d_0} and such that D does not contain C . Let $d \in I$, $d > 0$. Let $h \in S$ be such that the proper transform D' of $G' := V_+(h)$ has class D_d and such that D' does not contain C . Recall that C is the proper transform in X of $F := V_+(c)$. Since $D \cdot C = 0$, D and C are disjoint in X , but G and F intersect only at e in \mathbb{P} and we have:

$$\text{mult}_e(G, F) = \text{mult}_e(G) = 3d_0.$$

Similarly, $\text{mult}_e(G', F) = 3d$. Since $d \geq d_0$, by Lemma 4.7, h satisfies Noether's condition (with respect to c, g). By Cor. 4.6, $h = Ac + Bg$ for some $A, B \in S$. If D_1 denotes the proper transform in X of $V_+(B)$, note that $[D'] = [D] + [D_1]$. It follows that $D_1 \in \mathcal{I}$ and so $d - d_0 \in I$. The statement now follows by induction. \square

Lemma 4.9. [GNW94] *Assume $\text{char } k = p \geq 3$. Then there exists $D \in \mathcal{I}_k$ with class D_p .*

Proof. We recall from [GNW94, p. 390] the construction of a polynomial $h \in \mathfrak{p}^{(3p)}$ of degree $p(7m-3)(8m-3)$ such that $c \nmid h$. The ideal \mathfrak{p} contains polynomials a, b and c , where

$$a = z^{3m-1} - x^{2m-1}y^2, \quad b = x^{3m-1} - yz^{2m-1}$$

(in fact a, b, c generate \mathfrak{p} by the Hilbert–Burch theorem but we don't need this). Let

$$\begin{aligned} d_2 &= x^{m-1}y^5z^{m-1} - 3x^{2m-1}y^2z^{2m-1} + x^{5m-2}y + z^{5m-2}, \\ d_3 &= -x^{3m-2}y^7 + 2x^{m-1}y^5z^{3m-1} + x^{4m-2}y^4z^m - 5x^{2m-1}y^2z^{4m-1} + \\ &\quad + 3x^{5m-2}yz^{2m} - x^{8m-3}z + z^{7m-2}, \\ d'_3 &= y^8z^{2m-2} - 4x^m y^5z^{3m-2} + x^{4m-1}y^4z^{m-1} + 6x^{2m}y^2z^{4m-2} - \\ &\quad - 4x^{5m-1}yz^{2m-1} + x^{8m-2} - xz^{7m-3}. \end{aligned}$$

A direct computation shows:

$$\begin{aligned} x^m d_2 - yb^2 + z^{m-1}ac &= 0, \\ x^{m-1}b^2c + ad_2 - z^{m-1}d_3 &= 0, \\ xd_3 + ybc^2 + zd'_3 &= 0. \end{aligned}$$

It follows that $d_2 \in \mathfrak{p}^{(2)}$ and $d_3, d'_3 \in \mathfrak{p}^{(3)}$. Note also that $c \nmid d_3$. Since $\text{char } k = p$, we get from the third equation that

$$x^p d_3^p + y^p b^p c^{2p} + z^p d'_3{}^p = 0.$$

Write $p = 2q + 1$ for some integer $q > 0$. Since

$$x^p d_3^p + y^p b^p c^{2p} \equiv 0 \pmod{(z^p)}, \quad x^m a + y^2 b + z^{2m-1} c = 0,$$

it follows that

$$\begin{aligned}
x^p d_3^p + y^p b^p c^{2p} &= x^p d_3^p + (-1)^q y b^{p-q} c^{2p} (x^m a + z^{2m-1} c)^q = \\
&= x^p d_3^p + (-1)^q \sum_{i=0}^q \binom{q}{i} x^{m(q-i)} y z^{(2m-1)i} a^{q-i} b^{p-q} c^{2p+i} \\
&\equiv 0 \pmod{(z^p)}.
\end{aligned}$$

Notice that either $m(q-i) \geq p$ or $(2m-1)i \geq p$ for each $0 \leq i \leq q$ (use $m \geq 4$). Then

$$x^p d_3^p + (-1)^q \sum_{(2m-1)i < p} \binom{q}{i} x^{m(q-i)} y z^{(2m-1)i} a^{q-i} b^{p-q} c^{2p+i} \equiv 0 \pmod{(z^p)},$$

and therefore,

$$z^p h = d_3^p + (-1)^q \sum_{(2m-1)i < p} \binom{q}{i} x^{m(q-i)-p} y z^{(2m-1)i} a^{q-i} b^{p-q} c^{2p+i},$$

for some $h \in \mathfrak{p}^{(3p)}$. If $c \mid h$, then $c \mid d_3$, which is a contradiction. \square

Proof of Thm. 1.2. Assume that X is a MDS in characteristic 0. By Lemma 4.1, there exists an integer $d > 0$ and a monic polynomial $f \in S$ such that the proper transform D in X of $V_+(f)$ has class D_d and D does not contain C as a fixed component. Since a multiple of D is base-point free and D is big, by eventually replacing d with a multiple, we may assume by Bertini's theorem that D is smooth and connected.

Let R be the \mathbb{Z} -algebra generated by the coefficients of f . Let $\mathbb{P}_R := \text{Proj } R[x, y, z]$ and e_R be the section of $\mathbb{P}_R \rightarrow \text{Spec}(R)$ corresponding to $\mathfrak{p}R[x, y, z]$. Let \mathcal{X}_R be the blow-up of \mathbb{P}_R along e_R , with exceptional divisor \mathcal{E} . Let \mathcal{D} be the proper transform of $V_+(f) \subset \mathbb{P}_R$ in \mathcal{X}_R . Since the geometric generic fiber of $\rho: \mathcal{D} \rightarrow \text{Spec}(R)$ is smooth and connected, by eventually replacing R with a localization, we may assume that ρ is smooth and all its geometric fibers \mathcal{D}_s are connected. Since ρ is flat, $\deg \mathcal{O}(\mathcal{E})|_{\mathcal{D}_s}$ does not depend on s . It follows that all \mathcal{D}_s have class D_d and do not contain the curve C , i.e., for each $s \in \text{Spec}(R)$, we obtain a divisor in $\overline{\mathcal{I}_{k(s)}}$. For each prime p in the image of the dominant map $\text{Spec } R \rightarrow \text{Spec } \mathbb{Z}$, pick some $s_p \in \text{Spec}(R)$. By Prop. 4.3, there are integers d_p such that $\overline{\mathcal{I}_{k(s_p)}} = \mathbb{N}\{d_p\}$. Hence, $d_p \mid d$ for sufficiently large primes p . As by Lemma 4.9, $d_p \mid p$ for all primes $p \geq 3$, we must have that $d_p = 1$ for all sufficiently large p .

But one can see directly that D_1 is not effective in characteristic 0 (and hence, in characteristic p , for p large). To see this, note that the

only monomials in S of degree $(7m - 3)(8m - 3)$ are

$$x^{m-1}y^5z^{3m-2}, x^{4m-2}y^4z^{m-1}, x^{2m-1}y^2z^{4m-2}, x^{5m-2}yz^{2m-1}, x^{8m-3}, z^{7m-3}.$$

No linear combination of these monomials has all six derivatives of order 2 vanishing at $e = (1, 1, 1)$. A direct computation shows that the determinant of the corresponding 6×6 matrix is (up to a sign):

$$4(7m - 3)^2(8m - 3)^2(7m - 4)(8m - 4)(51m^2 - 43m + 9).$$

Q.E.D. □

5. PROOF OF THEOREM 1.1

We recall elementary transformations of Maruyama [Mar82] in the generality that we need. Let X be a scheme of finite type over k , let $i : D \hookrightarrow X$ be an effective Cartier divisor, let \mathcal{F} be a locally free sheaf of rank 2 on X , and let $\mathcal{F}|_D \rightarrow \mathcal{L}$ be a surjection onto an invertible sheaf on D . Then we have a commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & i_*\mathcal{L}' & \longrightarrow & i_*(\mathcal{F}|_D) & \longrightarrow & i_*\mathcal{L} \longrightarrow 0 \\
 & & \pi' \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \xrightarrow{\pi} & i_*\mathcal{L} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}(-D) & \xlongequal{\quad} & \mathcal{F}(-D) & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The sheaf \mathcal{F}' is called an elementary transformation of \mathcal{F} . It is a locally free sheaf of rank 2. Geometrically, consider \mathbb{P}^1 -bundles $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{F}')$, where say $\mathbb{P}(\mathcal{F}) = \mathbf{Proj}_{\mathcal{O}_X} \mathit{Sym}(\mathcal{F})$. Quotient maps π and π' give sections $s : D \rightarrow \mathbb{P}(\mathcal{F}|_D)$ and $s' : D \rightarrow \mathbb{P}(\mathcal{F}'|_D)$. Let $Z = s(D)$ and $Z' = s'(D)$ be their images. Note that they are local complete intersections of codimension 2. We have a canonical isomorphism

$$\mathrm{Bl}_Z \mathbb{P}(\mathcal{F}) \simeq \mathrm{Bl}_{Z'} \mathbb{P}(\mathcal{F}').$$

More concretely, $\mathbb{P}(\mathcal{F}')$ is obtained from $\mathrm{Bl}_Z \mathbb{P}(\mathcal{F})$ by blowing down a proper transform of the Cartier divisor $\mathbb{P}(\mathcal{F}|_D)$. Note that elementary transformations are functorial, i.e., for a map $g : Y \rightarrow X$, $\mathbb{P}(g^*\mathcal{F}')$ is the elementary transformation of $\mathbb{P}(g^*\mathcal{F})$ along the data $(g^{-1}(D), g^*s)$.

Lemma 5.1. *Let $p : Y \rightarrow X$ be a \mathbb{P}^1 -bundle and let $p' : Y' \rightarrow X$ be an elementary transformation given by data (D, Z) . Let $t : X \rightarrow Y$ be a global section and let T' denote the proper transform of $T = t(X)$ in Y' . If T and Z agree, or if they are disjoint, then T' is a section of p' .*

Let now t_1, t_2 be two global sections and let T'_1, T'_2 denote the proper transforms of $T_1 = t_1(X), T_2 = t_2(X)$. Assume T_1, Z agree over D .

- (a) *If T_2, Z are disjoint, then T'_1, T'_2 are disjoint over D .*
- (b) *Assume T_1, T_2, Z agree over D and for some point $x \in D$ (with X, D non-singular at x), we have at $p = s(x)$ that*

$$T_{p,T_1} \cap T_{p,T_2} = T_{p,Z} \subseteq T_{p,Y}.$$

Then T'_1, T'_2 are disjoint over x .

The condition on tangent spaces in (b) is equivalent to the differentials $dt_{1|x}, dt_{2|x}$ not having the same image. Alternatively, there exists a curve C in X smooth at x , such that in the ruled surface $S := p^{-1}(C) \rightarrow C$, the sections $T_1 \cap S$ and $T_2 \cap S$ are not tangent at p .

Proof of Lemma 5.1. If T and Z agree along D , the proper transform \tilde{T} in the blow-up \tilde{Y} of Y along Z is isomorphic to T , as it is the blow-up of T along Z (a Cartier divisor in T). As Y' is the blow-down of \tilde{Y} along the proper transform of $p^{-1}(D)$, which is disjoint from \tilde{T} , it follows that T' is isomorphic to \tilde{T} , hence T' is a section of p' .

Assume that T and Z are disjoint. Set $Y = \mathbb{P}(\mathcal{F}), Y' = \mathbb{P}(\mathcal{F}')$, for \mathcal{F}' the elementary transformation of \mathcal{F} along $\mathcal{F}|_D \rightarrow \mathcal{L}$ (corresponding to Z). The global section T corresponds to a quotient $\mathcal{F} \rightarrow \mathcal{M}$. Since T and Z are disjoint, the induced map $\mathcal{F}|_D \rightarrow \mathcal{M}|_D \oplus \mathcal{L}$ is an isomorphism (hence, the first exact sequence in the commutative diagram relating \mathcal{F} and \mathcal{F}' is split). The induced map $\mathcal{F}' \rightarrow i_*\mathcal{M}|_D$ factors through $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{M}$. It follows that $\mathcal{F}' \rightarrow \mathcal{M}$ is surjective (it is enough to check this on D) and $T' = \mathbb{P}(\mathcal{M})$, i.e., T' is a section of p' .

We now prove the second part of the lemma. As proved above, T'_1 and T'_2 are sections of p' . Assume we are in situation (a). We prove that T'_1, T'_2 are disjoint above any point $x \in D$. Consider a general curve C in X through x . By functoriality, the ruled surface $S = p^{-1}(C) \rightarrow C$ undergoes an elementary transformation given by data (x, z) , where $z = s(x)$. As the section T_1 passes through z , while T_2 does not, it follows immediately that T'_1, T'_2 are disjoint over x . Assume now that we are in situation (b). As before, we reduce to the ruled surface case. We may choose C a curve through x that is transverse to D at x and let $S = p^{-1}(C)$. It follows that $\dim(T_{p,Z} \cap T_{p,S}) = 0$ and sections $T_1 \cap S, T_2 \cap S$ are transverse at p ; hence, T'_1, T'_2 are disjoint above x . \square

Definition 5.2. Let X be a non-singular variety and let D_1, \dots, D_N be irreducible divisors in X with simple normal crossings. For simplicity let's assume that intersections $D_{ij} := D_i \cap D_j$ and $D_{ijk} := D_i \cap D_j \cap D_k$ are irreducible (or empty). We denote the interiors of these intersections by D_{ij}^0 and D_{ijk}^0 , respectively. Let $1 \leq M \leq N$. Let $p : Y \rightarrow X$ be a \mathbb{P}^1 -bundle and let $s_i : D_i \rightarrow p^{-1}(D_i)$ be sections with images Z_i for $i = M, \dots, N$. We say that these sections form a *compatible system of sections* if the following conditions are satisfied:

- (1) Let $M \leq i < j$. If $D_{ij} \neq \emptyset$ then either
 - (a) Z_i and Z_j agree over D_{ij} , or
 - (b) Z_i and Z_j are disjoint over D_{ij}^0 , in which case let

$$\mathcal{K} := \{k \in \{1, \dots, N\} \mid Z_i \text{ and } Z_j \text{ agree over } D_{ijk}\}.$$

Then, for any $k \in \mathcal{K}$, we have $M \leq k < i$, Z_k agrees with Z_i over D_{ik} and with Z_j over D_{jk} , and, for any $p \in s_k(D_{ijk}^0)$,

$$T_{p, s_i(D_{ij})} \cap T_{p, s_j(D_{ij})} = T_{p, s_k(D_{ijk})}. \quad (5.1)$$

- (2) If $D_{ijk} \neq \emptyset$, $M \leq i, j, k$, then there exists a subset $\{a, b\}$ of $\{i, j, k\}$ such that Z_a and Z_b agree over D_{ab} .

Proposition 5.3. *Given a compatible system of sections Z_1, \dots, Z_N , let $p' : Y' \rightarrow X$ be an elementary modification given by data (D_1, Z_1) . Let $Z'_2, \dots, Z'_N \subset Y'$ be the proper transforms of Z_2, \dots, Z_N . Then Z'_2, \dots, Z'_N are sections of p' which form a compatible system of sections. In particular, we can iterate elementary transformations, which gives a sequence of \mathbb{P}^1 -bundles $Y_0 = Y, Y_1 = Y', \dots, Y_N$ over X .*

Proof. We first show that each Z'_i is a section for each $i > 1$. By Lemma 5.1, it suffices to show that Z_1 and Z_i are either disjoint or agree over D_{1i} . Suppose they do not agree over D_{1i} and are not disjoint. Then we are in the situation (b) of condition (1) in Def. 5.2. If G denotes the locus in D_{1i} where Z_i and Z_1 agree, we have that $G \subset D_{1i} \setminus D_{1i}^0 = \cup_k D_{1ik}$. Since there are no indices less than 1, G does not contain any of the D_{1ik} . It follows that all components of G have codimension ≥ 4 in X . But this is a contradiction, as Z_1 and Z_i are distinct sections of the \mathbb{P}^1 -bundle $p^{-1}(D_{1i}) \rightarrow D_{1i}$, hence their intersection has pure codimension 4 in Y (hence, G has pure codimension 3 in X).

Next we show that Z'_2, \dots, Z'_N form a compatible system of sections. Notice that condition (2) is obvious because the elementary transformation is an isomorphism outside of D_1 (if Z_a and Z_b agree over D_{ab} , then Z'_a and Z'_b agree over D_{ab} as well). So we only need to check condition (1). Take $1 < i < j$ such that $D_{ij} \neq \emptyset$. As before, if Z_i and Z_j agree over D_{ij} , then Z'_i and Z'_j agree over D_{ij} as well. If Z_i and

Z_j do not agree, then let \mathcal{K} be the subset from condition (1) of the compatible system Z_1, \dots, Z_N and let \mathcal{K}' be the corresponding subset for Z'_1, \dots, Z'_N . It is clear that $\mathcal{K}' \setminus \{1\} = \mathcal{K} \setminus \{1\}$ and (5.1) is satisfied for these indices k (because the elementary transformation is an isomorphism over D_{ijk}^0). So we only need to check that $1 \notin \mathcal{K}'$, i.e., that Z'_i and Z'_j do not agree over D_{1ij} . We can assume that $D_{1ij} \neq \emptyset$, as otherwise there is nothing to prove. Consider two cases. Firstly, suppose $1 \notin \mathcal{K}$. By condition (2) of Def. 5.2, we may assume without loss of generality that Z_1 and Z_i agree over D_{1i} . Then Z_1 and Z_j do not agree over D_{1j} and therefore, must be disjoint as proved above. It follows by Lemma 5.1(a) (applied to Z_i and Z_j over D_{ij}) that Z'_i and Z'_j are disjoint over D_{1ij} . Secondly, suppose $1 \in \mathcal{K}$. Then by Lemma 5.1(b) applied to Z_i and Z_j over D_{ij} , we have that Z'_i, Z'_j are disjoint over D_{1ij}^0 and hence, $1 \notin \mathcal{K}'$. \square

Proof of Thm. 1.1. Choose general points $q_1, \dots, q_n \in \mathbb{P}^{n-2}$ and let $\pi : \text{Bl}_{q_n} \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-3}$ be a resolution of the linear projection away from q_n . Then π is a \mathbb{P}^1 -bundle. Let $p_i = \pi(q_i)$ for $i = 1, \dots, n-1$. For any subset I of $\{1, \dots, n-1\}$ such that $1 \leq |I| \leq n-4$, let $L_I \subset \mathbb{P}^{n-3}$ be a linear subspace spanned by p_i for $i \in I$. Notice that we have sections $t_I : L_I \rightarrow \pi^{-1}(L_I)$ that send L_i to a proper transform of a linear subspace in \mathbb{P}^{n-2} spanned by q_i , for $i \in I$. Let $\Psi : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ be the Kapranov map such that $\Psi(\delta_{I \cup \{n\}}) = L_I$ for any subset I as above [Kap93]. Let $p : Y \rightarrow \overline{M}_{0,n}$ be the pull-back of π and let $s_I : \delta_{I \cup \{n\}} \rightarrow \pi^{-1}(\delta_{I \cup \{n\}})$ be the pull-back of t_I for each subset I as above. We order boundary divisors $\delta_{I \cup \{n\}}$ according to $|I|$ (in the increasing order) and arbitrarily for fixed $|I|$. We claim that this is a compatible system of sections. If we can show the claim then, by Prop. 5.3, the last elementary transformation Y_N is a SQM of the blow-up of \mathbb{P}^{n-2} along points q_1, \dots, q_n and proper transforms of linear subspaces spanned by $\{q_i\}_{i \in I}$ for subsets $I \subset \{1, \dots, n-1\}$ with $\leq n-4$ elements. It follows that the required small modification \widetilde{LM}_{n+1} is the blow-up of Y_N in proper transforms of linear subspaces spanned by $\{q_i\}_{i \in I}$ for subsets I with $n-3$ elements.

Now we prove the claim. Set $D_I := \delta_{I \cup \{n\}}$. Suppose $I \neq J$, $|I| \leq |J|$, $D_{IJ} \neq \emptyset$. Then either $I \subset J$, in which case Z_I and Z_J agree on D_{IJ} , or there exists a partition $A \sqcup B \sqcup C = \{1, \dots, n-1\}$ such that $I = A \cup B$ and $J = A \cup C$. In this case, the set \mathcal{K} from condition (1) of the compatible system is the set of all non-empty subsets of A . This shows condition (2) and all of condition (1), except (5.1). If $A = \emptyset$ then there is nothing to check. Assume $A \neq \emptyset$. Let $\alpha \in D_{KIJ}^0$. It is enough to find a curve C in D_{IJ} passing through α , such that in the ruled surface

$S := p^{-1}(C)$, s_I and s_J are not tangent above α . As we have

$$\Psi(D_{IJ}) = L_I \cap L_J \cong \mathbb{P}^{|A|}, \quad \Psi(D_{IJK}) = L_K \subseteq L_A \cong \mathbb{P}^{|A|-1},$$

we may choose l to be any line in $L_I \cap L_J$ that passes through $\Psi(\alpha)$ and is not contained in L_A . Let C be any curve in D_{IJ} that maps to l and is smooth at α . We claim that C has the desired property, i.e., that $s_I(C)$ and $s_J(C)$ are not tangent above α . It suffices to check this after composing with the maps $\Psi' : Y \rightarrow \text{Bl}_{q_n} \mathbb{P}^{n-2}$, the pull-back of the Kapranov map, and the blow-up map $\text{Bl}_{q_n} \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-2}$. Let Λ be the plane in \mathbb{P}^{n-2} which is the image of $p^{-1}(l)$. If Z_I is the linear subspace in \mathbb{P}^{n-2} spanned by the points q_i for $i \in I$, then $Z_I \cap Z_J = Z_A$. Clearly, the linear subspaces $Z_I \cap \Lambda$ and $Z_J \cap \Lambda$ are intersecting only at a point (lying above $L_A \cap l = \Psi(\alpha)$). Equivalently, $Z_I \cap \Lambda$ and $Z_J \cap \Lambda$ are not tangent at their intersection point. This proves the claim. \square

The proof of Thm. 1.1 and Cor. 1.4 yield the following:

Corollary 5.4. *Let $p_1, \dots, p_{n-2} \in \mathbb{P}^{n-3}$ be points in linearly general position and let X_n be the toric variety which is the blow-up of \mathbb{P}^{n-3} along proper transforms of linear subspaces of codimension ≥ 3 spanned by the points p_i , in order of increasing dimension. Let e denote the identity of the open torus of X_n . Then $\text{Bl}_e X_{n+1}$ is a SQM of a \mathbb{P}^1 -bundle over $\overline{M}_{0,n}$. If $\text{char } k = 0$ and $n \geq 134$, then $\text{Bl}_e X_{n+1}$ is not a MDS.*

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