MODULI SPACES AND INVARIANT THEORY

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ABSTRACT. A moduli space is a space that parametrizes geometric objects. For example, elliptic curves are classified by the so-called $J$-invariant, so the moduli space of elliptic curves is a line (with coordinate $J$). More generally, there exists a moduli space, called $M_g$, which parametrizes all projective algebraic curves of genus $g$ (equivalently, all compact Riemann surfaces of genus $g$). The Jacobian of a Riemann surface is a moduli space that classifies line bundles on a fixed Riemann surface.

The study of moduli spaces is an old branch of algebraic geometry with an abundance of technical tools: classical invariant theory, geometric invariant theory, period domains and variation of Hodge structures, stacks, derived categories, birational geometry, intersection theory, tropical geometry, etc. But we believe that a lot can be learned by studying examples using minimal machinery, as a motivation to learn more sophisticated tools.

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§1. Geometry of lines

Let’s start with a familiar example. Recall that the Grassmannian $G(r, n)$ parametrizes $r$-dimensional linear subspaces of $\mathbb{C}^n$. For example,

$$G(1, n) = \mathbb{P}^{n-1}$$

is the projective space\(^1\). Let’s try to understand $G(2, n)$. The projectivization of a 2-dimensional subspace $U \subset \mathbb{C}^n$ is a line $l \subset \mathbb{P}^{n-1}$, so in essence $G(2, n)$ parametrizes lines in the projective space. Most of our discussion remains valid for any $r$, but the case of $G(2, n)$ is notationally easier.

§1.1. Grassmannian as a complex manifold. Thinking about $G(2, n)$ as just a set is boring: we need to introduce some geometry on it. We care about two flavors of geometry, Analytic Geometry and Algebraic Geometry.

A basic object of analytic geometry is a complex manifold, i.e. a second countable Hausdorff topological space $X$ covered by charts $X_i$ homeomorphic to open subsets of $\mathbb{C}^n$:

$$\phi_i : U_i \hookrightarrow \mathbb{C}^n.$$ 

Coordinate functions on $\mathbb{C}^n$ are called local coordinates in the chart. On the overlaps $X_i \cap X_j$ we thus have two competing systems of coordinates, and the main requirement is that transition functions

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

between these coordinate systems are holomorphic. Complex manifolds form a category: maps between complex manifolds are holomorphic maps, i.e. maps that are holomorphic in charts.

For example, the projective space $\mathbb{P}^{n-1}$ can be covered by $n$ charts

$$X_i = \{[x_1 : \ldots : x_n], \quad x_i \neq 0\}.$$ 

What are the local coordinates and transition functions?

Let’s see how to generalize this to $G(2, n)$. Any 2-dimensional subspace $U \subset \mathbb{C}^n$ is a row space of a $2 \times n$ matrix $A$ of rank 2. Let $A_{ij}$ denote the $2 \times 2$ submatrix of $A$ with columns $i$ and $j$ and let $p_{ij} = \det A_{ij}$ be the corresponding minor. Since rank $A = 2$, we can find some $i < j$ such that $p_{ij} \neq 0$ (why?). Then $(A_{ij})^{-1}A$ has a canonical form

$$
\begin{bmatrix}
\ldots & * & 1 & * & \ldots & * & 0 & * & \ldots \\
\ldots & 0 & * & \ldots & * & 1 & * & \ldots
\end{bmatrix} \quad (1.1.1)
$$

where the $i$-th column is $\begin{bmatrix}1 \\ 0 \end{bmatrix}$, the $j$-th column is $\begin{bmatrix}0 \\ 1 \end{bmatrix}$, and the remaining $n - 2$ columns are arbitrary. Notice that multiplying $A$ by an invertible $2 \times 2$ matrix on the left does not change the row space.

Thus we can cover $G(2, n)$ by $\binom{n}{2}$ charts

$$X_{ij} = \{U \in G(2, n) \mid U \text{ is represented by a matrix } A \text{ like } (1.1.1)\}.$$

---

\(^1\)Whenever you see something italicized, ask yourself: do I know the definition of that? I hope that this will help you to learn the vocabulary faster.
Each subspace from $X_{ij}$ is a row space of a unique matrix (1.1.1), in particular $X_{ij}$ can be identified with $\mathbb{C}^{2(n-2)}$. Local coordinates on $X_{ij}$ are just entries of the matrix (1.1.1).

To show that $G(2, n)$ is a complex manifold we have to check that the transition functions between charts $X_{ij}$ and $X_{i'j'}$ are holomorphic. If a matrix $A$ represents a subspace $U \in X_{ij} \cap X_{i'j'}$ then $p_{ij} \neq 0$ and $p_{i'j'} \neq 0$. In the chart $X_{ij}$ we can choose $A$ as in (1.1.1). In the chart $X_{i'j'}$, the matrix will be $(A_{i'j'})^{-1}A$. Its matrix entries depend holomorphically (in fact rationally) on the matrix entries of $A$.

In this example (and in general) the structure of a topological space on $X$ is introduced simultaneously with constructing charts: a subset is declared open if its intersection with each chart is open. It is easy to check that $G(2, n)$ is indeed a Hausdorff space (why?).

§1.2. Moduli space or a parameter space? All lines in $\mathbb{P}^{n-1}$ are isomorphic. So maybe it’s better to call $G(2, n)$ not a parameter space: it doesn’t classify geometric objects up to isomorphism but rather it classifies subobjects (lines) in a fixed ambient space (projective space). This distinction is of course purely philosophical. A natural generalization of the Grassmannian, which we will study later, is given by Chow varieties and Hilbert schemes. They parametrize all algebraic subvarieties (and subschemes) of $\mathbb{P}^n$ with a fixed numerical data.

As a rule, parameter spaces are easier to construct than moduli spaces. To construct an honest moduli space $M$ of geometric objects $X$, one can

- embed these objects in some ambient space;
- construct a “parameter space” $H$ for embedded objects;
- Then $M$ will be the set of equivalence classes for the following equivalence relation on $H$: two embedded objects are equivalent if they are abstractly isomorphic.

In many cases there will be a group $G$ acting on $H$ and equivalence classes will be just orbits of the group. So we will have to learn how to construct an orbit space $M = H / G$ and a quotient map $H \to M$ that sends each point to its orbit. These techniques are provided by the invariant theory – the second component from the title of this course.

For example, suppose we want to construct $M_{3}$, the moduli space of curves of genus 3. These curves come in two flavors, they are either hyperelliptic or not. The following is known:

1.2.1. Theorem.

- A hyperelliptic curve of genus 3 is a double cover of $\mathbb{P}^1$ ramified at 8 points. These points are determined uniquely up to the action of $\text{PGL}_2$.
- A non-hyperelliptic curve of genus 3 is isomorphic to a smooth quartic curve in $\mathbb{P}^2$. Two smooth quartic curves are isomorphic as algebraic varieties if and only if they belong to the same $\text{PGL}_3$.

Let $H_{3} \subset M_{3}$ be the set of isomorphism classes of hyperelliptic curves and let $M_{3} \setminus H_{3}$ be the complement. Thus we can construct

$$H_{3} = \left[ \mathbb{P}(\text{Sym}^8 \mathbb{C}^2) \setminus D \right] / \text{PGL}_2$$
and
\[ \mathcal{M}_3 \setminus \mathcal{H}_3 = \left[ \mathbb{P}(\text{Sym}^4 \mathbb{C}^3) \setminus \mathcal{D} \right] / \text{PGL}_3. \]
Here \( \mathcal{D} \) is the discriminant locus. In the first case it parametrizes degree 8 polynomials in 2 variables without a multiple root (and hence such that its zero locus is 8 distinct points in \( \mathbb{P}^1 \)) and in the second case degree 4 polynomials in 3 variables such that its zero locus is a smooth curve in \( \mathbb{P}^2 \).

An easy dimension count shows that
\[ \dim \mathcal{H}_3 = \dim \mathbb{P}(\text{Sym}^8 \mathbb{C}^2) - \dim \text{PGL}_2 = 7 - 3 = 5 \]
and
\[ \dim \mathcal{M}_3 \setminus \mathcal{H}_3 = \dim \mathbb{P}(\text{Sym}^4 \mathbb{C}^3) - \dim \text{PGL}_3 = 14 - 8 = 6. \]

It is quite remarkable that there exists an irreducible moduli space \( \mathcal{M}_3 \) which contains \( \mathcal{H}_3 \) as a hypersurface. We will return to this example later.

§1.3. Stiefel coordinates. Let’s construct the Grassmannian as a quotient by the group action. The motivating idea is that \( G(1, n) \) is a quotient:
\[ \mathbb{P}^{n-1} = \left[ \mathbb{C}^n \setminus \{0\} \right] / \mathbb{C}^*. \]
Coordinates in \( \mathbb{C}^n \) are known as homogeneous coordinates on \( \mathbb{P}^{n-1} \). They are defined uniquely up to a common scalar factor. There are two ways to generalize them to \( r > 1 \), Stiefel coordinates and Plücker coordinates. Let
\[ \text{Mat}^0_{2,n} \subset \text{Mat}_{2,n} \]
be an open subset (in Zariski or complex topology) of matrices of rank 2. We have a map
\[ \Psi : \text{Mat}^0_{2,n} \to G(2, n) \]
which sends a matrix to its row space. \( \Psi \) can be interpreted as the quotient map for the action of \( \text{GL}_2 \) on \( \text{Mat}^0_{2,n} \) by left multiplication. Indeed, two rank 2 matrices \( X \) and \( X' \) have the same row space if and only if \( X = gX' \) for some matrix \( g \in \text{GL}_2 \). Matrix coordinates on \( \text{Mat}^0_{2,n} \) are sometimes known as Stiefel coordinates on the Grassmannian. They are determined up to a left multiplication with a \( 2 \times 2 \) matrix.

§1.4. Plücker coordinates. Let me use (1.3.1) as an example to explain how to use invariants to construct quotient maps.

1.4.1. Definition. We start with a very general situation: let \( G \) be a group acting on a set \( X \). A function \( f : X \to \mathbb{C} \) is called an invariant function if it is constant along \( G \)-orbits, i.e.
\[ f(gx) = f(x) \quad \text{for any } x \in X, \ g \in G. \]
Invariant functions \( f_1, \ldots, f_r \) form a complete system of invariants if they separate orbits. This means that for any two orbits \( O_1 \) and \( O_2 \), there exists at least one function \( f_i, i = 1, \ldots, r \), such that \( f_i|_{O_1} \neq f_i|_{O_2} \).

1.4.2. Lemma. If \( f_1, \ldots, f_r \) is a complete system of invariants then the map
\[ F : X \to \mathbb{C}^r, \quad F(x) = (f_1(x), \ldots, f_r(x)) \]
is a quotient map (onto its image) in the sense that its fibers are exactly the orbits.
Thus we need the following generalization:

1.4.3. Definition. Fix a homomorphism 
\[ \chi : G \to \mathbb{C}^*. \]

A function \( f : X \to \mathbb{C} \) is called a semi-invariant of weight \( \chi \) if
\[ f(gx) = \chi(g)f(x) \quad \text{for any } x \in X, \ g \in G. \]
(An invariant function is a special case of a semi-invariant of weight \( \chi = 1 \)).
Suppose \( f_0, \ldots, f_r \) are semi-invariants of the same weight \( \chi \). We will call them a complete system of semi-invariants of weight \( \chi \) if
- for any \( x \in X \), there exists a function \( f_i \) such that \( f_i(x) \neq 0 \);
- for any two points \( x, x' \in X \) not in the same orbit, we have
\[ [f_0(x) : \ldots : f_r(x)] \neq [f_0(x') : \ldots : f_r(x')]. \]

The first condition means that we have a map
\[ F : X \to \mathbb{P}^r, \quad F(x) = [f_0(x) : \ldots : f_r(x)], \]
which is clearly constant along \( G \)-orbits:
\[ [f_0(gx) : \ldots : f_r(gx)] = [\chi(g)f_0(x) : \ldots : \chi(g)f_r(x)] = [f_0(x) : \ldots : f_r(x)]. \]
The second condition means that \( F \) is a quotient map onto its image in the sense that its fibers are precisely the orbits.

1.4.4. Example. Let \( G = \text{GL}_2 \) act by left multiplication on \( \text{Mat}_{2,n}^0 \). Consider the \( 2 \times 2 \) minors \( p_{ij} \) (for \( i < j \)) as functions on \( \text{Mat}_{2,n}^0 \).

1.4.5. Proposition. The minors \( p_{ij} \) form a complete system of semi-invariants on \( \text{Mat}_{2,n}^0 \) of weight
\[ \det : \text{GL}_2 \to \mathbb{C}^*. \]

Proof. We have
\[ p_{ij}(gA) = \det(g)p_{ij}(A) \quad \text{for any } \ g \in \text{GL}_2, \ A \in \text{Mat}_{2,n}^0. \]
It follows that \( p_{ij} \)'s are semi-invariants of weight \( \det \). For any \( A \in \text{Mat}_{2,n}^0 \), at least one of the \( p_{ij} \)'s does not vanish. So we have a map
\[ F : \text{Mat}_{2,n}^0 \to \mathbb{P}(2)^{-1} \]
given by the minors \( p_{ij} \).

Now take \( A, A' \in \text{Mat}_{2,n}^0 \) such that \( F(A) = F(A') \). We have to show that \( A \) and \( A' \) are in the same \( G \)-orbit. Suppose \( p_{ij}(A) \neq 0 \), then certainly \( p_{ij}(A') \neq 0 \). By acting on \( A \) and \( A' \) by some elements of \( \text{GL}_2 \), we can assume that both \( A \) and \( A' \) have canonical form (1.1.1). Without loss of generality, we can take \( \{ij\} = \{12\} \). In particular,
\[ p_{12}(A) = p_{12}(A') = 1. \]
Since \( F(A) = F(A') \), we now have
\[ p_{ij}(A) = p_{ij}(A') \quad \text{for every } i < j. \]
Now it’s really easy to see that \( A = A' \): we can recover matrix entries of a matrix in the canonical form (1.1.1) as minors \( p_{11} \) and \( p_{2i} \) for \( i > 3 \). \( \square \)
This gives an inclusion

\[ i : G(2, n) = \text{Mat}_{2,n}^0 / \text{GL}_2 \hookrightarrow \mathbb{P}^{(2)}_{-1} \]

called the Plücker embedding. The minors \( p_{ij} \) are called Plücker coordinates.

1.4.6. Proposition. \( i \) is an immersion of complex manifolds.

Proof. For now we have only proved that \( i \) is an inclusion of sets. Let \( x_{ij} \) (for \( i < j \)) be homogeneous coordinates on \( \mathbb{P}^{(2)}_{-1} \). Each chart \( p_{ij} \neq 0 \) of the Grassmannian is mapped to the corresponding affine chart \( x_{ij} \neq 0 \) of the projective space. The map \( i \) in this chart is given by the remaining \( 2 \times 2 \) minors of a matrix (1.1.1). These minors are of course holomorphic functions. Therefore \( i \) is a holomorphic map of manifolds. To show that it is an embedding of complex manifolds, we have to verify more, namely that the complex differential of this map has maximal rank at every point. Working in the chart \( p_{12} \) as above, we observe again that matrix entries of a matrix in the canonical form (1.1.1) are minors \( p_{1i} \) and \( p_{2i} \) for \( i > 3 \). Therefore local coordinates of this chart are some of the components of the map \( i \), and therefore its differential has maximal rank. \( \square \)

Why minors \( p_{ij} \) are a natural choice for a complete system of semi-invariants? Let’s consider all possible semi-invariants on \( \text{Mat}_{2,n}^0 \) which are polynomials in matrix entries. By continuity, this is the same thing as polynomial semi-invariants on \( \text{Mat}_{2,n} \). Let

\[ \mathcal{O}(\text{Mat}_{2,n}) = \mathbb{C}[a_{11}, a_{2i}]_{1 \leq i \leq n} \]

be the algebra of polynomial functions on \( \text{Mat}_{2,n} \). It is easy to see (why?) that the only holomorphic homomorphisms \( \text{GL}_2(\mathbb{C}) \to \mathbb{C}^* \) are powers of the determinant. Let

\[ R_i = \mathcal{O}(\text{Mat}_{2,n})|_{\text{det}^i} \]

be a subset of polynomial semi-invariants of weight \( \det^i \). Notice that the scalar matrix \( \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \) acts on \( R_i \) by multiplying it on \( \det^i \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = t^{2i} \). It follows that all polynomials in \( R_i \) have degree \( 2i \), in particular

\[ R_i = 0 \quad \text{for } i < 0, \quad R_0 = \mathbb{C}. \]

We assemble all semi-invariants in one package (algebra of semi-invariants):

\[ R = \bigoplus_{i \geq 0} R_i \subset \mathcal{O}(\text{Mat}_{2,n}). \]

Since the product of semi-invariants of weights \( \chi \) and \( \chi' \) is a semi-invariant of weight \( \chi \cdot \chi' \), \( R \) is a graded subalgebra of \( \mathcal{O}(\text{Mat}_{2,n}) \).

1.4.7. Theorem (First Fundamental Theorem of invariant theory). The algebra \( R \) is generated by the minors \( p_{ij} \).

Thus considering only \( p_{ij} \)’s makes sense: all semi-invariants can’t separate orbits any more effectively than the generators. We will skip the proof of this theorem. But it raises some general questions:

- is the algebra of polynomial invariants (or semi-invariants) always finitely generated?
do these basic invariants separate orbits?
how to compute these basic invariants?

We will see that the answer to the first question is positive under very
general assumptions (Hilbert’s finite generation theorem). The answer to the
second question is “not quite” but a detailed analysis of what’s going on
is available (Hilbert–Mumford’s stability and the numerical criterion for it).
As far as the last question is concerned, the situation is worse: the genera-
tors can be computed explicitly only in a handful of cases.

§1.5. Grassmannian as a projective variety. What is the image of the Plücker
embedding? Let’s show that it is a projective algebraic variety.

1.5.1. DEFINITION. Let \( f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n] \) be homogeneous polyno-
mials. The vanishing set
\[
X = V(f_1, \ldots, f_r) = \{ x \in \mathbb{P}^n | f_1(x) = \ldots = f_r(x) = 0 \}
\]
is called a projective algebraic variety. For each chart \( U_i = \{ x_i \neq 0 \} \subset \mathbb{P}^n \),
\[
X \cap U_i \subset U_i \simeq \mathbb{A}^n
\]
is an affine algebraic variety given by vanishing of \( \tilde{f}_1, \ldots, \tilde{f}_r \), where \( \tilde{f}_j = f_j(x_0, \ldots, x_i-1, 1, x_i+1, \ldots, x_n) \) is a dehomogenization of \( f_j \).

Let \( U \) be a row space of a matrix
\[
A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2n}
\end{bmatrix}
\]
and consider a bivector
\[
b = (a_{11}e_1 + \ldots + a_{1n}e_n) \wedge (a_{21}e_1 + \ldots + a_{2n}e_n) = \sum_{i<j} p_{ij} e_i \wedge e_j.
\]

Thus if we identify \( \mathbb{P}^{n(n-1)/2} \) with the projectivization of \( \Lambda^2 \mathbb{C}^n \), the Plücker
embedding \( i \) simply sends a subspace \( U \) (generated by vectors \( u, u' \in \mathbb{C}^n \))
to the line \( \Lambda^2 U \subset \Lambda^2 \mathbb{C}^n \) (spanned by \( u \wedge u' \)). In particular, the image of \( i \) is
the projectivization of the subset of decomposable bivectors.

Notice that we have
\[
0 = b \wedge b = 2 \sum_{i<j<k<l} (p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}) e_i \wedge e_j \wedge e_k \wedge e_l.
\]
Thus quadratic polynomials
\[
x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}
\]
vanish along the image of \( i \). These polynomials are called Plücker quadrics.

1.5.2. PROPOSITION. \( i(G(2, n)) \) is a projective algebraic variety given by vanish-
ing of Plücker quadrics. It is also a compact complex submanifold of \( \mathbb{P}^{n(n-1)/2} \).

Proof. We already know that Plücker quadrics vanish along the image of \( i \),
so we just have to show that the vanishing set
\[
X = V(x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}) \subset \mathbb{P}^{n(n-1)/2}
\]
does not contain any other points. We can do this in charts. For simplicity, let’s only consider the chart \( U_{12} \), where we have \( x_{12} = 1 \). What are the equations of \( X \cap U_{12} \)? Taking equations with \( ij = 12 \) gives equations

\[
x_{kl} = x_{1k}x_{2l} - x_{1l}x_{2k}, \quad 2 < k < l \leq n.
\]

\[(1.5.3)\]

It follows that homogeneous coordinates of any point \( x \in X \) are minors of the matrix

\[
A = \begin{bmatrix}
1 & 0 & -x_{23} & -x_{24} & \ldots & -x_{2n} \\
0 & 1 & x_{13} & x_{14} & \ldots & x_{1n}
\end{bmatrix}.
\]

It follows that \( x \) corresponds to the row space of \( A \).

Any projective algebraic variety is compact in Euclidean topology, and since we already know that \( i \) is an immersion, \( \Gamma(G(2, n)) \) is a complex submanifold of \( \mathbb{P}^{\binom{n}{2}-1} \).

\[\square\]

Of course not every complex algebraic variety \( X \) is a complex manifold because it can be singular. If \( X \) is non-singular then it admits a structure of a complex manifold of the same dimension denoted by \( X^{an} \).

\section{Second fundamental theorem – relations.}

We proved that \( G(2, n) \) is cut out by Plücker quadrics. Is it possible to describe all polynomials in \( \binom{n}{2} \) variables \( x_{ij} \) that vanish along \( G(2, n) \)? In other words, what is the homogeneous ideal \( I \) of \( G(2, n) \)? Recall that \( G(2, n) \) is the image of the map \( \Psi \), see (1.3.1). Thus \( I \) is the kernel of the homomorphism of polynomial algebras

\[
\psi : \mathbb{C}[x_{ij}]_{1 \leq i < j \leq n} \to \mathbb{C}[a_{11}, a_{21}]_{1 \leq i \leq n}, \quad x_{ij} \mapsto p_{ij}(A).
\]

The image of \( \psi \) is equal to the algebra of GL\(_2\)-semi-invariants by the First Fundamental Theorem but we are not going to use this. One can think about elements of \( I \) as relations between \( 2 \times 2 \) minors of a \( 2 \times n \) matrix.

\begin{theorem} \textbf{(Second Fundamental Theorem of invariant theory).} The ideal \( I \) is generated (as an ideal of a polynomial ring) by Plücker quadrics

\[
x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}, \quad i < j < k < l.\]
\end{theorem}

\textbf{Proof.} The proof consists of several steps.

\textbf{Step 1.} Plücker quadrics are in \( I \). This was proved in Proposition 1.5.2. Let \( I' \subset I \) be the ideal generated by the Plücker relations. The goal is to show that \( I = I' \).

\textbf{Step 2,} called the straightening law. It was introduced by Alfred Young (who has Young diagrams named after him). We encode each monomial \( x_{i_1j_1} \ldots x_{i_kj_k} \) in a Young tableau

\[
\begin{bmatrix}
i_1 & i_2 & \ldots & i_k \\
\hline
j_1 & j_2 & \ldots & j_k
\end{bmatrix}
\]

\[(1.6.2)\]

Note that the columns of this matrix are increasing. The tableau is called \textit{standard} if its rows are non-decreasing:

\[i_1 \leq i_2 \leq \ldots \leq i_k \quad \text{and} \quad j_1 \leq j_2 \leq \ldots \leq j_k.\]

In this case we also call the corresponding monomial a \textit{standard monomial}. We claim that every monomial is equivalent modulo \( I' \) (i.e. modulo Plücker quadrics) to a linear combination of standard monomials.
Suppose \( x = x_{i_1 j_1} \ldots x_{i_k j_k} \) is a non-standard monomial. Then it contains a pair of variables \( x_{ab}, x_{cd} \) such that \( a < c \) but \( b > d \). Then we have
\[
a < c < d < b.
\]
We have
\[
x_{ab}x_{cd} = x_{ac}x_{db} - x_{ad}x_{cb} \mod I'.
\]
(1.6.3)
We can plug-in the right-hand-side into \( x \) and rewrite it as a sum of two monomials. However, we need to come up with some way to compare monomials to finish the argument by applying induction.

To every monomial \( x = x_{i_1 j_1} \ldots x_{i_k j_k} \), associate a sequence of positive integers
\[
j_1 - i_1, \ldots, j_k - i_k,
\]
reordered in non-increasing order. We order all monomials by ordering these sequences of integers lexicographically (and if monomials have the same sequences, for example monomials \( x_{12} \) and \( x_{34} \), order them randomly).

We argue by induction on lexicographical order that every non-standard monomial is equivalent modulo Plücker quadrics to a linear combination of standard monomials. Suppose this is verified for all monomials smaller than \( x = x_{i_1 j_1} \ldots x_{i_k j_k} \). Do the substitution (1.6.3) and notice that \( c - a, b - d, d - a \), and \( b - c \) are all strictly less than \( b - a \). Therefore both of these monomials are smaller than \( x \) in lexicographic order and so can be re-written as linear combinations of standard monomials modulo \( I' \) by the inductive assumption.

**Step 3.** Finally, we claim that standard monomials are linearly independent modulo \( I \), and in particular \( I = I' \). Concretely, we claim that
\[
\{ \psi(x) \mid x \text{ is a standard monomial}\}
\]
is a linearly independent subset of \( \mathbb{C}[a_{1i}, a_{2i}]_{1 \leq i \leq n} \). A cool idea is to order the variables as follows:
\[
a_{11} < a_{12} < \ldots < a_{1n} < a_{21} < a_{22} < \ldots < a_{2n}
\]
and to consider the corresponding lexicographic ordering of monomials in \( \mathbb{C}[a_{1i}, a_{2i}]_{1 \leq i \leq n} \). For any polynomial \( f \), let \( \text{in}(f) \) be the initial monomial of \( f \) (i.e. the smallest monomial for lexicographic ordering). Notice that \( \text{in}(f) \) is multiplicative:
\[
\text{in}(fg) = \text{in}(f) \text{in}(g)
\]
(1.6.4)
for any (non-zero) polynomials. We have
\[
\text{in} p_{ij} = a_{1i}a_{2j},
\]
and therefore
\[
\text{in}(\psi(x)) = \text{in}(p_{i_1 j_1} \ldots p_{i_k j_k}) = a_{1i_1}a_{1i_2} \ldots a_{1i_k}a_{2j_1}a_{2j_2} \ldots a_{2j_k}.
\]
Notice that a standard monomial \( x \) is completely determined by \( \text{in}(\psi(x)) \).

Now we argue by contradiction that the set of polynomials \( \{ \psi(x) \} \), for all standard monomials \( x \), is linearly independent. Indeed, consider a trivial linear combination
\[
\lambda_1 \psi(x_1) + \ldots + \lambda_r \psi(x_r) = 0
\]
where all $\lambda_i \neq 0$. Then the minimum of initial terms in $(\psi(x_i))$, $i = 1, \ldots, r$ should be attained at least twice, a contradiction with the fact they are all different. □

Many calculations in Algebraic Geometry can be reduced to similar manipulations with polynomials.

§1.7. Hilbert polynomial. Equations of the moduli space are rarely known so explicitly as in the case of the Grassmannian. But some numerical information about these equations is often available.

We start with a general situation: let $X \subset \mathbb{P}^n$ be a projective variety and let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal of polynomials that vanish along $X$. The algebra $R = \mathbb{C}[x_0, \ldots, x_n]/I$ is known as a homogeneous coordinate algebra of $X$. Note that $R$ is graded by degree

$$R = \bigoplus_{j \geq 0} R_j, \quad R_0 = \mathbb{C},$$

and $R$ is generated by $R_1$ as an algebra. The function

$$h(k) = \dim R_k$$

is called the the Hilbert function of $X$. Notice that knowing $h(k)$ is equivalent to knowing $\dim I_k$ for every $k$:

$$h(k) + \dim I_k = \binom{n+k}{k} \quad (why?)$$

We have the following fundamental theorem:

1.7.1. Theorem. There exists a unique polynomial $H(t)$, the Hilbert polynomial, such that

$$h(k) = H(k) \quad \text{for} \quad k \gg 0.$$

The Hilbert polynomial has degree $r = \dim X$ and has a form

$$\frac{d}{r!} t^r + \text{(lower terms)},$$

where $d$ is the degree of $X$, i.e. the number of points in the intersection of $X$ with a general projective subspace of codimension $r$.

The word “general” here is used in the standard sense of algebraic geometry: there exists a dense Zariski open subset $S \subset G(n+1-r, n+1)$ (the Grassmannian of all projective subspaces of codimension $r$) such that $X \cap L$ is a finite set of $d$ points for every projective subspace $L \in S$.

1.7.2. Proposition. Let $n \geq 3$. The Hilbert function of $G(2, n)$ in the Plücker embedding is

$$h(k) = \binom{n+k-1}{k}^2 - \binom{n+k}{k+1} \binom{n+k-2}{k-1}. \quad (1.7.3)$$

The degree of $G(2, n)$ in the Plücker embedding is the Catalan number

$$\frac{1}{n-1} \binom{2n-4}{n-2} = 1, 2, 5, 14, 42, 132, \ldots$$
Proof. While proving the Second Fundamental Theorem 1.6.1 we have already established that \( h(k) \) is equal to the number of standard monomials of degree \( k \), i.e. to the number of standard tableaux with \( k \) columns. Let \( N_l \) be the number of non-decreasing sequences \( 1 \leq i_1 \leq \ldots \leq i_l \leq n \). Then

\[
N_l = \binom{n + l - 1}{l}.
\]

Indeed, this is just the number of ways to choose \( l \) objects from \{1, \ldots, n\} with repetitions (so it is for example equal to the dimension of the space of polynomials in \( n \) variables of degree \( l \)). The number of all tableaux

\[
\begin{bmatrix}
i_1 & i_2 & \ldots & i_k \\
j_1 & j_2 & \ldots & j_k
\end{bmatrix}, \quad 1 \leq i_1 \leq \ldots \leq i_k \leq n, \ 1 \leq j_1 \leq \ldots \leq j_k \leq n
\]

is then

\[
N_k^2 = \binom{n + k - 1}{k}^2.
\]

To prove (1.7.3), we claim that the number of non-standard tableaux is

\[
\binom{n + k}{k + 1} \binom{n + k - 2}{k - 1}.
\]

More precisely, we claim that there is a bijection between the set of non-standard tableaux and the set of pairs \((A, B)\), where \( A \) is a non-decreasing sequence of length \( k + 1 \) and \( B \) is a non-decreasing sequence of length \( k - 1 \).

Suppose that \( l \) is the number of the first column where \( i_l \geq j_l \). Then we can produce two sequences:

\[
\begin{bmatrix}
i_1 & i_2 & \ldots & i_k \\
j_1 & j_2 & \ldots & j_k
\end{bmatrix}, \quad 1 \leq i_1 \leq \ldots \leq i_k \leq n, \ 1 \leq j_1 \leq \ldots \leq j_k \leq n
\]

of length \( k + 1 \) and

\[
\begin{bmatrix}
i_1 & \ldots & i_{l-1} & i_{l+1} & i_{l+2} & \ldots & i_k \\
i_1 & \ldots & i_{l-1} & j_l & \ldots & j_{k-1}
\end{bmatrix}
\]

of length \( k - 1 \). In an opposite direction, suppose we are given sequences

\[
i_1 \leq \ldots \leq i_{k+1} \quad \text{and} \quad j_1 \leq \ldots \leq j_{k-1}.
\]

Let \( l \) be the minimal index such that \( i_l \leq j_l \) and take a tableau

\[
\begin{bmatrix}
j_1 & \ldots & j_{l-1} & i_{l+1} & i_{l+2} & \ldots & i_k \\
i_1 & \ldots & i_{l-1} & j_l & \ldots & j_{k-1}
\end{bmatrix}
\]

If \( i_l > j_l \) for any \( l \leq k - 1 \), then take the tableau

\[
\begin{bmatrix}
j_1 & \ldots & j_{k-1} & i_k \\
i_1 & \ldots & i_{k-1} & i_{k+1}
\end{bmatrix}
\]

After some manipulations with binomial coefficients, (1.7.3) can be rewritten as

\[
\binom{n + k - 1}{n - 1}^2 - \binom{n + k}{n - 1} \binom{n + k - 2}{n - 1} =
\]

\[
\frac{(n + k - 1)^2 \ldots (k + 1)^2}{(n - 1)!^2} - \frac{(n + k - 1)(n + k - 2) \ldots k}{(n - 1)!} \frac{(n + k - 1)(n + k - 2) \ldots k}{(n - 1)!} =
\]

\[
\frac{(k + n - 1)(k + n - 2)^2 \ldots (k + 2)^2(k + 1)}{(n - 1)!^2} \frac{(n + k - 1)(k + 1) - (n + k)k}{(n - 1)!}.
\]
This is a polynomial in $k$ of degree $2n-4$ with a leading coefficient $rac{1}{(n-1)! (n-2)!}$. Since the degree of $G(2, n)$ is equal to $(2n-4)!$ multiplied by the leading coefficient of $h(k)$, we see that this degree is indeed the Catalan number. □

§1.8. Enumerative geometry. Why do we want to study moduli spaces? Their geometry reflects delicate properties of parametrized objects. To get a flavor of this, let’s relate the degree of the Grassmannian (i.e. the Catalan number) to geometry of lines.

1.8.1. Theorem. The number of lines in $\mathbb{P}^{n-1}$ that intersect $2n-4$ general codimension 2 subspaces is equal to the Catalan number

$$\frac{1}{n-1} \binom{2n-4}{n-2}.$$

The precise meaning of the word “general”, as before, is the following. There exists a dense Zariski open subset

$$U \subset G(n-2, n) \times \ldots \times G(n-2, n) \quad (2n-4 \text{ copies})$$

such that for every $(2n-4)$-tuple of codimension two subspaces

$$(W_1, \ldots, W_{2n-4}) \in U,$$

the number of lines that intersect $W_1, \ldots, W_{2n-4}$ is the Catalan number.

For example, there is only one line in $\mathbb{P}^2$ passing through 2 general points, 2 lines in $\mathbb{P}^3$ intersecting 4 general lines, 5 lines in $\mathbb{P}^4$ intersecting 6 general planes, and so on. This is a typical problem from the classical branch of Algebraic Geometry called enumerative geometry, which was described by H. Schubert (around 1870s) as a field concerned with questions like: How many geometric figures of some type satisfy certain given conditions? If these figures are lines (or projective subspaces), the enumerative geometry is nowadays known as Schubert calculus, and is more or less understood. Recently enumerative geometry experienced renaissance due to advances in moduli theory and physics (Gromov–Witten invariants, etc.).

Proof. For every codimension 2 subspace $W \subset \mathbb{P}^{n-1}$, we consider the locus

$$D_W \subset G(2, n)$$

of all lines that intersect $W$. This is an example of a Schubert variety. We claim that $D_W$ can be described as the intersection with a hyperplane:

$$D_W = G(2, n) \cap H_W, \quad H_W \subset \mathbb{P}^{n-1}.$$

Indeed, without loss of generality we can take

$$W = \{ x_1 = x_2 = 0 \} = \mathbb{P}(e_3, \ldots, e_n).$$

Let $\pi : \mathbb{C}^n \to \langle e_1, e_2 \rangle$ be a linear projection with kernel $\langle e_3, \ldots, e_n \rangle$. Lines intersecting $W$ are projectivizations of 2-dimensional subspaces $U$ such that $\text{Ker} \pi|_U \neq 0$, or equivalently $\text{dim} \pi|_U < 2$. It follows that $U \in D_W$ if and only if the Plücker minor $p_{12}(U) = 0$. So $D_W$ is precisely the intersection with hyperplane

$$H_W = \{ x_{12} = 0 \}.$$
We will show that if 
\[ W_1, \ldots, W_{2n-4} \subset \mathbb{P}^{n-1} \]
are sufficiently general codimension 2 subspaces then the intersection
\[ D_{W_1} \cap \ldots \cap D_{W_{2n-4}} = G(2, n) \cap \left( H_{W_1} \cap \ldots \cap H_{W_{2n-4}} \right) \]
is a finite union of the Catalan number of points. We are going to see that 
\[ H_{W_1} \cap \ldots \cap H_{W_{2n-4}} \]
is a codimension \(2n - 4\) projective subspace and would like to invoke the definition of the degree. The problem, however, is that this subspace is not going to be a “general” subspace. So we need a refined notion of the degree that can be verified for a specific subspace. For this we need to introduce the notion of transversality.

§1.9. Transversality. Recall that any algebraic variety \(X\) carries a structure sheaf \(\mathcal{O}_X\). If \(x \in X\), then we also have
- \(\mathcal{O}_{X,x}\) is the local ring of \(x\).
- \(m_{X,x} \subset \mathcal{O}_{X,x}\) is the maximal ideal of \(x\).
- \(T^*_X, x = m_{X,x}/m_{X,x}^2\) is the Zariski cotangent space of \(x\). For every function \(f \in m_{X,x}\), its coset \(f + m_{X,x}^2 \in T^*_X, x\) is called the differential \(df\).
- This is a usual differential if \(X = \mathbb{A}^n\).
- \(T_X, x = (m_{X,x}/m_{X,x}^2)^*\) is the Zariski tangent space of \(x\).

We say that \(X\) is smooth (or non-singular) at \(x\) if
\[ \dim_C T_{X, x} = \dim X \]
(and otherwise \(\dim_C T_{X, x} > \dim X\)). The locus of all smooth points is called the smooth locus of \(X\). It is Zariski open and dense. Its complement is called the singular locus.

Let \(Y \subset X\) be an algebraic subvariety defined by a sheaf of radical ideals \(\mathcal{I}_Y \subset \mathcal{O}_X\).

For every \(y \in Y\), we have the following.
- \(\mathcal{O}_{Y,y} = \mathcal{O}_{X,y}/\mathcal{I}_{Y,y}\).
- \(T_{Y,y} \subset T_{X,y}\) is a vector subspace cut out by differentials \(df \in T^*_{X,y}\) of functions \(f \in \mathcal{I}_{Y,y}\).
- If \(X\) is smooth at \(y\) and \(\text{codim}_X Y = r\) then \(Y\) is smooth at \(y\) if and only if one can find \(r\) functions \(f_1, \ldots, f_r \in \mathcal{I}_{Y,y}\) with linearly independent differentials \(df_1, \ldots, df_r \in T^*_{X,y}\) (the Jacobian criterion).

1.9.1. Definition. Let \(X\) be an algebraic variety with subvarieties \(Y_1, \ldots, Y_r\). We say that these subvarieties intersect transversally at \(y \in Y_1 \cap \ldots \cap Y_r\) if \(X\) and \(Y_1, \ldots, Y_r\) are smooth at \(y\) and
\[ \text{codim}_{T_{X,y}} [T_{Y_1,y} \cap \ldots \cap T_{Y_r,y}] = \text{codim}_X Y_1 + \ldots + \text{codim}_X Y_r, \]
the largest possible. One can prove that in this case \(Y_1 \cap \ldots \cap Y_r\) is smooth at \(y\), of codimension \(\text{codim}_X Y_1 + \ldots + \text{codim}_X Y_r\).

We are going to invoke a powerful
1.9.2. **Theorem (Kleiman–Bertini).** Let $G$ be a complex algebraic group acting regularly and transitively on an algebraic variety $X$. Let $Y, Z \subset X$ be irreducible subvarieties. Then, for a sufficiently general $g \in G$, subvarieties $Y$ and $gZ$ intersect properly, i.e. their intersection is either empty or

$$\text{codim}_X(Y \cap gZ) = \text{codim}_X Y + \text{codim} gZ.$$ 

Moreover, if $x \in Y \cap gZ$ is a smooth point of both $Y$ and $gZ$ then these subvarieties intersect transversally at $x$.

**Proof.** I will only sketch the “intersect properly” part, to show that it makes sense. Consider the algebraic “incidence” subset

$$I \subset Y \times Z \times G, \quad Z = \{(y, z, g) \mid y = gz\}.$$ 

Let $\pi : I \to Y \times Z$ be the projection. Then

$$\pi^{-1}(y, z) = \{g \mid y = gz\} \simeq G_y$$

because the action is transitive. These fibers (which don’t have to be irreducible) all have the same dimension equal to

$$\dim G_y = \dim G - \dim G_y = \dim G - \dim X.$$ 

By a theorem on dimension of fibers,

$$\dim I = (\dim Y + \dim Z) + (\dim G - \dim X)$$

(here one has to be careful because $I$ may have several irreducible components). Now let’s consider the projection $p : I \to G$. Notice that

$$p^{-1}(g) \simeq Y \cap gZ.$$ 

There are two possibilities. If $p$ is not dominant then $Y$ does not intersect $gZ$ for a general $g \in G$. If $p$ is dominant then we can apply the theorem on dimension of fibers (and again using the same care) to conclude that

$$\dim Y \cap gZ = \dim I - \dim G = \dim Y + \dim Z - \dim X$$

and so

$$\text{codim}_X(Y \cap gZ) = \text{codim}_X Y + \text{codim} gZ.$$ 

This proves the “intersect properly” part.

1.9.3. **Corollary (Bertini’s Theorem).** An irreducible subvariety $Y \subset \mathbb{P}^n$ intersects a general hyperplane $H \subset \mathbb{P}^n$ (or a general projective subspace of fixed dimension) properly and transversally at all smooth points of $Y$.

**Proof.** Apply Kleiman–Bertini to $X = \mathbb{P}^n, G = \text{GL}_{n+1}$, and $Z = H$. 

1.9.4. **Remark.** In the set-up of Bertini Theorem one can prove more: $Y \cap H$ is non-empty (unless $Y$ is a point) and irreducible (unless $Y$ is a curve).

The following refined definition of degree is a special case of a general principle called “conservation of number”.

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2We are going to discuss algebraic groups and their actions later. The only relevant case for now is the transitive action of $G = \text{GL}_n(\mathbb{C})$ on $X = G(k, n)$, for example on $\mathbb{P}^{n-1}$.

3There exists a dense Zariski open subset $U \subset G$ such that this is true for every $g \in U$. 
1.9.5. Proposition. Let $X \subset \mathbb{P}^n$ be a projective variety of dimension $r$. A general projective subspace of codimension $r$ intersects $X$ transversally in all intersection points. For subspaces with this property, $X \cap L$ is a finite union of the same number of points (the degree of $X$).

Now let’s go back to the Schubert variety $D_W$ with $W = \{x_1 = x_2 = 0\}$. Recall that $D_W = G(2, n) \cap H_W$, where $H_W = \{x_{12} = 0\}$. Let’s describe this intersection in charts of the Grassmannian. $D_W$ is exactly the complement of the chart $U_{12} \subset G(2, n)$. In charts $U_{1i}$ and $U_{2i}$ for $i \geq 3$, the minor $p_{12}$ reduces to a linear equation. In the remaining charts, $D_W$ is given by a quadric $a_{11}a_{22} - a_{12}a_{21} = 0$. Its singular locus is
\[ \{a_{11} = a_{22} = a_{12} = a_{21} = 0\} \]

To summarize,


It remains to show that for sufficiently general codimension 2 subspaces $W_1, \ldots, W_{2n-4} \subset \mathbb{P}^{n-1}$, the intersection
\[ G(2, n) \cap H_{W_1} \cap \ldots \cap H_{W_{2n-4}} \subset \mathbb{P}^{(2)} \]
is transversal at all intersection points. Equivalently, we have to show that $D_{W_1}, \ldots, D_{W_{2n-4}} \subset G(2, n)$ intersect transversally at all intersection points. We will apply the Kleiman–Bertini Theorem inductively $2n - 4$ times. Take $G = GL_n$, acting transitively on $X = G(2, n)$. Let $Y = Z = D_W$, $W_1 = W$.

By the Kleiman–Bertini Theorem, $D_{W_1}$ and $D_{W_2}$ intersect properly if $W_2$ is a general codimension 2 subspace. Continuing inductively, we see that $D_{W_1}, \ldots, D_{W_{2n-4}}$ intersect properly, i.e. in finitely many points.

Let $S_W \subset D_W$ be the singular locus. By the Kleiman–Bertini Theorem, $D_{W_1}, \ldots, S_{W_i}, \ldots, D_{W_{2n-4}}$ intersect properly, i.e. this intersection is empty, for every $i = 1, \ldots, 2n - 4$. It follows that $D_{W_1}, \ldots, D_{W_{2n-4}}$ are smooth at every point of their intersection and intersect transversally there. \[\square\]

§1.10. Grassmannian as a fine moduli space. Finally, I would like to use the well-understood example of the Grassmannian to illustrate central notions of the fine moduli space and the universal family. We will also introduce a functorial (or categorical) approach to moduli problems.

Every point of $G(2, n)$ corresponds to a 2-dimensional subspace of $\mathbb{C}^n$. We would like to organize these subspaces into a “family”. Lem me remind the definition of a vector bundle.

1.10.1. Definition. A trivial vector bundle over an algebraic variety $X$ with the fiber $\mathbb{C}^r$ is the product $X \times \mathbb{C}^r$ along with the projection
\[ \pi : X \times \mathbb{C}^r \to X. \]

More generally, an $r$-dimensional vector bundle over an algebraic variety $X$ is an algebraic variety $E$ and a surjective morphism
\[ \pi : E \to X \]
such that the following holds.
• There exists a covering $X = \bigcup U_\alpha$ and

isomorphisms (called trivializations)

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r, \quad p_2 \circ \psi_\alpha = \pi$$

• given by linear maps over overlaps $U_\alpha \cap U_\beta$, i.e. the induced map

$$(U_\alpha \cap U_\beta) \times \mathbb{C}^r \xrightarrow{\psi_\beta \circ \psi_\alpha^{-1}} (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

takes

$$(x, v) \mapsto (x, \phi_{\alpha\beta}(x)v),$$

where $\phi_{\alpha\beta}(x)$ is an invertible $r \times r$ matrix with entries in $O(U_\alpha \cap U_\beta)$.

A map of vector bundles $(E, p_E) \to (F, p_F)$ is map of underlying varieties $L : E \to F$ which commutes with projections

$$\pi_F \circ L = \pi_E$$

and such that $L$ is given by linear transformations in an atlas $\{U_\alpha\}$ which trivializes both $E$ and $F$. Concretely, if $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ and $\phi_\alpha : \pi^{-1}_F(U_\alpha) \to U_\alpha \times \mathbb{C}^s$ are trivializations then the map

$$\phi_\alpha \circ L \circ \psi_\alpha^{-1} : U_\alpha \times \mathbb{C}^r \to U_\alpha \times \mathbb{C}^s$$

takes

$$(x, v) \mapsto (x, L_\alpha(x)v),$$

where $L_\alpha(x)$ is an $s \times r$ matrix with entries in $O(U_\alpha)$. If $E \subset F$ and the inclusion map is the map of vector bundles then $E$ is a sub-bundle of $F$.

Going back to the Grassmannian $G(k, n)$, consider the universal bundle (also known as tautological)

$$U = \{([U], v) \mid v \in U\} \subset G(k, n) \times \mathbb{C}^n.$$ 

Its fiber at a point that corresponds to a subspace $U \subset \mathbb{C}^n$ is $U$ itself. We claim that $U$ is a vector bundle, in fact a subbundle of the trivial bundle with fiber $\mathbb{C}^n$. It is trivialized in standard affine charts of the Grassmannian, for example let’s consider the chart $U = U_{12}$ defined by $p_{12} \neq 0$. For any point of this chart, rows $(v_1, v_2)$ of the matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} & \ldots & a_{1n} \\ 0 & 1 & a_{23} & a_{24} & \ldots & a_{2n} \end{bmatrix}$$

give a basis of the corresponding subspace in $\mathbb{C}^n$. The trivialization of $U$ is defined as follows:

$$\phi^{-1} : U \times \mathbb{C}^2 \to \pi^{-1}(U), \quad (u, z_1, z_2) \mapsto z_1v_1 + z_2v_2.$$

In what sense is the bundle $U$ universal? Suppose we have an algebraic variety $X$ which parametrizes a family of $2$-dimensional subspaces of $\mathbb{C}^n$. In other words, we are given a 2-dimensional sub-bundle $(E, \pi)$ of a trivial bundle $X \times \mathbb{C}^n$. Then we have a map

$$f_E : X \to G(2, n), \quad x \mapsto \pi^{-1}(x).$$
What are the properties of this map? Let \( U \subset X \) be a trivializing chart with trivialization \( \psi : \pi^{-1}_E(U) \to U \times \mathbb{C}^2 \). Composing \( \psi^{-1} \) with the embedding of \( E \) into \( X \times \mathbb{C}^n \) gives a map of trivial vector bundles

\[
U \times \mathbb{C}^2 \to U \times \mathbb{C}^n,
\]

which is given by an \( 2 \times n \) matrix \( A \) with coefficients in \( \mathcal{O}_X(U) \). In other words, the restriction of the map \( f_E \) to \( U \) factors as the composition of the map \( U \to \text{Mat}^0(2, n) \) given by \( A \) and the map \( \text{Mat}^0(2, n) \to G(2, n) \) which sends a matrix to its row space. One corollary of this is that \( f_E|U \) is a morphism of algebraic varieties, and therefore the same is true for \( f_E \).

It turns out that the morphism \( f_E \) completely determines \( E \) by pulling back the universal bundle. Recall that if \( \pi : E \to Y \) is a vector bundle and \( f : X \to Y \) is a morphism then we can define a pull-back vector bundle

\[
f^*E = \{(x, v) \mid f(x) = \pi(v)\} \subset X \times E.
\]

The map \( f^*\pi : f^*E \to X \) is induced by the projection \( X \times E \to X \). To see that \( f^*E \) is indeed a vector bundle, we can assume without loss of generality that \( E \) is trivial (by trivializing it), in which case \( f^*E \) is also a trivial vector bundle. Thus a trivializing atlas for \( f^*E \) can be obtained by just taking preimages of open sets in a trivializing atlas of \( E \). If \( E \subset F \) is a subbundle then we can define a subbundle \( f^*E \subset f^*E \) in an obvious way. After this preparation it is clear that

\[
E = f^*_E U \subset f^*_E [G(2, n) \times \mathbb{C}^n] = X \times \mathbb{C}^n.
\]

To summarize,

1.10.2. **Proposition.** Given a 2-dimensional subbundle \( (E, \pi) \) of a trivial bundle \( X \times \mathbb{C}^n \), there exists a unique map \( f_E : X \to G(2, n) \) such that \( f_E^*U = E \) as a subbundle of the trivial bundle \( X \times \mathbb{C}^n = f_E^*[G(2, n) \times \mathbb{C}^n] \).

Now let’s repackage this in a categorical language. Recall that if \( C \) is a category and \( G \) is an object of \( C \) then there is a contravariant functor

\[
h_G : C \to \text{Sets}
\]

which sends every object \( X \) of \( C \) to the set of morphisms \( \text{Mor}_C(X, G) \) and every morphism \( X \to Y \) to the function \( \text{Mor}_C(Y, G) \to \text{Mor}_C(X, G) \) given by precomposing with \( f \). A contravariant functor \( G : C \to \text{Sets} \) is called representable by an object \( G \) if functors \( h_G \) and \( G \) are isomorphic (or naturally equivalent). Recall that this means the following: we have to associate to every object \( X \) of \( C \) a bijection

\[
\eta_X : h_G(X) = \text{Mor}(X, G) \to G(X).
\]

This bijection should be such that, for every morphism \( g : X \to Y \) in \( C \), we have

\[
\eta_Y \circ h_G(g) = G(g) \circ \eta_X.
\]

**Yoneda’s Lemma** says that the functors \( h_G \) and \( h_{G'} \) are naturally equivalent if and only if \( G \) and \( G' \) are isomorphic. In other words, a functor can be represented by only one object, up to an isomorphism.

We can define a contravariant **Grassmannian functor**

\[
G(2, n) : \text{Algebraic Varieties} \to \text{Sets}
\]
which sends every algebraic variety \( X \) to the set \( G(2, n)(X) \) of all rank 2 subbundles \( E \) of the trivial vector bundle \( X \times \mathbb{C}^n \). A morphism \( f : X \rightarrow Y \) gives a function \( G(2, n)(Y) \rightarrow G(2, n)(X) \) given by the pull-back \( E \mapsto f^* E \) (inside the trivial vector bundle \( X \times \mathbb{C}^n \)). Notice that as sets

\[
G(2, n) = G(2, n)(\text{point})
\]

because both sides parametrize 2-dimensional subspaces of \( \mathbb{C}^n \).

1.10.3. PROPOSITION. \( G(2, n) \) is represented by \( G(2, n) \).

Proof. By definition of an isomorphism of functors, we have to associate to every algebraic variety \( X \) a bijection

\[
\eta_X : \text{Mor}(X, G(2, n)) \rightarrow G(2, n)(X).
\]

This is exactly what we have done above: \( \eta_X(f) := f^* U \). This bijection should be such that, for every morphism \( g : X \rightarrow Y \), we have

\[
\eta_Y \circ h_{G(2, n)}(g) = G(2, n)(g) \circ \eta_X,
\]

which translates into \( (f \circ g)^* U = g^* f^* U \).

In moduli theory one is interested in a class of contravariant functors

\[
\mathcal{M} : \text{Algebraic Varieties} \rightarrow \text{Sets},
\]

called moduli functors, such that \( \mathcal{M}(X) \) is a set of isomorphism classes of families parametrized by \( X \) of geometric objects, i.e. a set of morphisms \( \pi : E \rightarrow X \) such that fibers of \( \pi \) are geometric objects of interest. Typically there are various additional assumptions on \( \pi \) and \( E \), for example in the case of \( G(2, n) \) \( E \) has to be a subbundle of a trivial bundle with a fixed fiber \( \mathbb{C}^m \). For a morphism \( f : X \rightarrow Y \), the corresponding function

\[
\mathcal{M}(f) : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)
\]

is usually called the pull-back and denoted by \( f^* \). Specifically, in most cases,

\[
f^* E = X \times_Y E = \{(x, e) \mid f(x) = \pi(e)\} \subset X \times E
\]

is the fiber product (we will discuss them in detail later). We can recover the set of isomorphism classes of our objects as \( \mathcal{M}(\text{point}) \).

1.10.4. DEFINITION. Let \( \mathcal{M} \) be a moduli functor. An algebraic variety \( M \) together with a family \( U \in \mathcal{M}(M) \) is called a fine moduli space with a universal family \( U \) if the functor \( h_M \) is isomorphic to \( \mathcal{M} \) and in fact

\[
\eta_X(f) = f^* U \quad \text{for every morphism } f : X \rightarrow M.
\]

Thus the Grassmannian (with its universal bundle) is a fine moduli space.

The following lemma is not very useful but it gives a neat statement.

1.10.5. LEMMA. Let \( \mathcal{M} \) be a moduli functor. An algebraic variety \( M \) is a fine moduli space if and only if the functor \( h_M \) is isomorphic to \( \mathcal{M} \).

Proof. We have to find a universal family. Applying \( \eta_M \) to the identity map \( [M \rightarrow M] \in h_M \) gives a family \( U \in \mathcal{M}(M) \). We claim that it is a universal family. Let \( f : X \rightarrow M \) be a morphism. Then

\[
\eta_X(f) = \eta_X(h_M(f) \text{Id}) = f^* (\eta_M(\text{Id})) = f^* U.
\]

\[\square\]
For a simple example of a moduli functor without a fine moduli space, consider the functor
\[ P : \text{Algebraic Varieties} \to \text{Sets} \]
which sends every algebraic variety \( X \) to its Picard group \( \text{Pic}(X) \) of all line bundles on \( X \) modulo isomorphism. To a morphism \( f : X \to Y \) we associate a pull-back of line bundles \( f^* : \text{Pic}(Y) \to \text{Pic}(X) \), which turns \( P \) into a functor. We claim that it is not representable. Indeed, suppose it is representable by an algebraic variety \( P \). The set of points of \( P \) will be identified with \( \text{Pic}(\text{point}) \), which is just a one-element set, the trivial line bundle. So \( P = \text{point} \). But then every line bundle on every \( X \) will be a pull-back of the trivial line bundle on the point, i.e. will be trivial, which is of course not the case.

1.10.6. REMARK. The same argument will work for every moduli functor of families with isomorphic fibers, for example for the functor of isomorphism classes of vector bundles, \( \mathbb{P}^1 \)-bundles, etc.

1.10.7. REMARK. I am not assuming familiarity with schemes, in fact I will use moduli problems as an excuse to introduce schemes. But you know what they are, notice the argument above will be a bit harder if one works with algebraic schemes instead of algebraic varieties. The problem is that there is only one algebraic variety with one point, but there are plenty of schemes with one point, for example \( \text{Spec} \mathbb{C}[t]/(t^k) \) for every \( k > 0 \).

Let’s re-examine the projective space \( \mathbb{P}^n = G(1, n+1) \) from this point of view. The universal line bundle is
\[ \mathcal{O}_{\mathbb{P}^n}(-1) = \{(L,v) \mid v \in L \} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}. \]
So \( \mathbb{P}^n \) represents a functor
\[ \text{Algebraic Varieties} \to \text{Sets} \]
which sends every algebraic variety \( X \) to the set of all line sub-bundles \( L \) of the trivial vector bundle \( X \times \mathbb{C}^{n+1} \). Given this subbundle, we have an obvious map \( X \to \mathbb{P}^n \) which sends \( x \in X \) to the fiber of \( L \) over \( x \) (viewed as a line in \( \mathbb{C}^{n+1} \)). And \( L \) is then a pull-back of \( \mathcal{O}_{\mathbb{P}^n}(-1) \) inside the trivial line bundle \( X \times \mathbb{C}^{n+1} \).

In algebraic geometry it is more common to use an isomorphic functor. To define it, we need the following standard definitions.

1.10.8. DEFINITION. Let \( \pi : E \to X \) be a vector bundle on an algebraic variety. A morphism \( s : X \to E \) is called a global section if \( \pi \circ s = \text{Id}_X \). All global sections form a \( \mathbb{C} \)-vector space denoted by \( H^0(X,E) \). Any linear map of vector bundles \( L : E \to F \) induces a linear map
\[ H^0(X,E) \to H^0(X,F), \quad s \mapsto L \circ s. \]

1.10.9. DEFINITION. A linear bundle \( \pi : L \to X \) is called globally generated if for every \( x \in X \) there exists a global section \( s \in H^0(X,L) \) such that \( s(x) \neq 0 \).
1.10.10. THEOREM. \( \mathbb{P}^n \) represents a functor

\[ \text{Algebraic Varieties} \to \text{Sets} \]

which sends every algebraic variety \( X \) to the set of isomorphism classes of data

\[ \{ L^*; s_0, \ldots, s_n \}, \]

where \( L^* \) is a globally generated line bundle on \( X \) and \( s_0, \ldots, s_n \in H^0(X, L^*) \) have the property that for every \( x \in X \), there exists an \( s_i \) such that \( s_i(x) \neq 0 \). The universal family on \( \mathbb{P}^n \) is given by \( \{ \mathcal{O}_{\mathbb{P}^n}(1); z_0, \ldots, z_n \} \).

Given a datum \( \{ L^*; s_0, \ldots, s_n \} \) on \( X \), the corresponding morphism to \( \mathbb{P}^n \) sends \( x \in X \) to a point with homogeneous coordinates \([s_0(x) : \ldots : s_n(x)]\), where we identify the fiber of \( L^* \) over \( x \) with \( \mathbb{C} \) linearly.

Proof. We just have to construct an isomorphism of the new functor with our old functor of all possible inclusions \( i : L \hookrightarrow X \times \mathbb{C}^{n+1} \). First we dualize the inclusion to obtain the surjection \( \alpha : X \times (\mathbb{C}^{n+1})^* \to L^* \) (surjection on each fiber). Given \( \alpha \), we can define \( n + 1 \) global sections \( s_0, \ldots, s_n \) of \( L^* \) by taking images of constant global sections of \( X \times (\mathbb{C}^{n+1})^* \) which send every point \( x \in X \) to \( z_0, \ldots, z_n \in (\mathbb{C}^{n+1})^* \), standard coordinate functions on \( \mathbb{C}^{n+1} \). And vice versa, suppose we have global sections

\[ s_0, \ldots, s_n \in H^0(X, L^*) \]

such that for every \( x \in X \) we have \( s_i(x) \neq 0 \) for some \( i \). Then we can define \( \alpha \) by sending \( (x, z_i) \) to \( (x, s_i(x)) \) and extending linearly. \( \square \)
§1.11. Homework 1. Due February 17.

Name:
Attach this problem sheet to your written solutions and leave this space empty for me to record problems presented during office hours.

**Problem 1.** (2 points) Let $L_1, L_2, L_3 \subset \mathbb{P}^3$ be three general lines. (a) Show that there exists a unique smooth quadric surface $S \subset \mathbb{P}^3$ which contains $L_1, L_2, L_3$. (b) Use the previous part to give an alternative proof of the fact that 4 general lines in $\mathbb{P}^3$ intersect exactly two lines.

**Problem 2.** (2 points) Identify $\mathbb{C}^2 \times \mathbb{C}^2$ with the space of skew-symmetric $n \times n$ matrices in such a way that $G(2, n)$ becomes the projectivization of the set of skew-symmetric matrices of rank 2 and Plücker quadrics become $4 \times 4$ sub-Pfaffians.

**Problem 3.** (1 point) For any line $L \subset \mathbb{P}^3$, let $[L] \in \mathbb{C}^6$ be its Plücker vector. The Grassmannian $G(2, 4) \subset \mathbb{P}^6$ is a quadric, and therefore can be described as the vanishing set of a quadratic form $Q$, which has an associated inner product such that $Q(v) = v \cdot v$. Describe this inner product in coordinates and show that $[L_1] \cdot [L_2] = 0$ if and only if $L_1$ and $L_2$ intersect.

**Problem 4.** (4 points) In the notation of the previous problem, show that five given lines $L_1, \ldots, L_5$ all intersect a line if and only if

$$
\begin{vmatrix}
\end{vmatrix} = 0
$$

**Problem 5.** (1 point) Prove multiplicativity of initial terms

$$\text{in}(fg) = \text{in}(f) \text{in}(g)$$

of polynomials in lexicographic ordering.

**Problem 6.** (1 point) Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d$. Compute its Hilbert function and Hilbert polynomial.

**Problem 7.** (1 point) Let $p_1, p_2, p_3 \in \mathbb{P}^2$ be points which (a) don’t lie on a line or (b) lie on line. Compute the Hilbert function and the Hilbert polynomial of the union $X = \{p_1, p_2, p_3\}$.

**Problem 8.** (2 points) Let $L_1, L_2 \subset \mathbb{P}^3$ be skew lines. Compute the Hilbert function and the Hilbert polynomial of the union $X = L_1 \cup L_2$.

**Problem 9.** (3 points) Let $X \subset \mathbb{P}^n$ be a hypersurface and let $F_X \subset G(2, n)$ be the subset of all lines contained in $X$. Show that $F_X$ is a projective algebraic variety.

**Problem 10.** (3 points) For any point $p \in \mathbb{P}^3$ (resp. any plane $H \subset \mathbb{P}^3$) let $L_p \subset G(2, 4)$ (resp. $L_H \subset G(2, 4)$) be a subset of lines containing $p$ (resp. contained in $H$). (a) Show that every $L_p$ and $L_H$ is isomorphic to $\mathbb{P}^2$ in the Plücker embedding of $G(2, 4)$. (b) Show that any $\mathbb{P}^2$ contained in $G(2, 4) \subset \mathbb{P}^5$ has a form $L_p$ or $L_H$ for some $p$ or $H$. 


Problem 11. (4 points) Consider the Segre map
\[\psi : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n^2-1} = \mathbb{P}(\text{Mat}_{n,n}),\]
\[\psi([x_1 : \ldots : x_n], [y_1 : \ldots : y_n]) = [x_1 y_1 : \ldots : x_i y_j : \ldots : x_n y_n] \]
Its image is called the Segre variety. (a) Show that this map is an embedding of complex manifolds. (b) Show that the homogeneous ideal of the Segre variety in \(\mathbb{P}(\text{Mat}_{n,n})\) is generated by \(2 \times 2\) minors \(a_{ij}a_{kl} - a_{il}a_{kj}\). (c) Compute the Hilbert polynomial and the degree of the Segre variety.

Problem 12. (1 point) Let the symmetric group \(S_n\) act on \(\mathbb{C}^n\) by permuting coordinates. Show that the elementary symmetric functions \(\sigma_1 = x_1 + \ldots + x_n, \ldots, \sigma_n = x_1 \ldots x_n\) form a complete set of invariants for this action.

Problem 13. (2 points) An alternative way of thinking about a matrix \(X = \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ x_{21} & \ldots & x_{2n} \end{bmatrix}\) is that it gives \(n\) points \(p_1, \ldots, p_n\) in \(\mathbb{P}^1\) (with homogeneous coordinates \([x_{11} : x_{21}], \ldots, [x_{1n} : x_{2n}]\) as long as \(X\) has no zero columns. Suppose \(n = 4\) and consider the rational normal curve \(f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\).

(a) Show that points \(f(p_1), \ldots, f(p_4)\) lie on a plane if and only if
\[F(X) = \det \begin{bmatrix} x_1^3 & x_1 x_2 x_3 & x_1 x_2^2 & x_1 x_3^2 & x_2^3 \\ x_2^3 & x_2 x_1 x_3 & x_2 x_1^2 & x_2 x_3^2 & x_3^3 \\ x_3^3 & x_3 x_1 x_2 & x_3 x_1^2 & x_3 x_2^2 & x_1^3 \\ x_4^3 & x_4 x_1 x_2 & x_4 x_1^2 & x_4 x_2^2 & x_2^3 \end{bmatrix} = 0.\]

(b) Express \(F(X)\) as a polynomial in \(2 \times 2\) minors of the matrix \(X\).

Problem 14. (3 points) Let \(V_i = \langle e_1, \ldots, e_i \rangle \subset \mathbb{C}^n\) for \(i = 0, \ldots, n\),
\[0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n.\]
Fix integers \(n - 2 \geq a \geq b \geq 0\) and define the Schubert cell \(W_{a,b} \subset G(2,n)\) consists of all subspaces \(U\) such that
\[\dim(U \cap V_k) = \begin{cases} 0 & \text{if } k < n - 1 - a \\
1 & \text{if } n - 1 - a \leq k < n - b \\
2 & \text{if } n - b \leq k.\end{cases}\]
(a) Show that \(W_{1,0}\) is the special Schubert variety \(D_W\).
(b) Show that \(W_{a,b}\) is isomorphic to \(\mathbb{C}^{2(n-2)-a-b}\).
(c) Use part (b) to compute the Euler characteristic of \(G(2,n)\).
§2. Algebraic curves and Riemann surfaces

After the projective line $\mathbb{P}^1$, the easiest algebraic curve to understand is an elliptic curve (Riemann surface of genus 1). Let

$$M_1 = \{ \text{isom. classes of elliptic curves} \}.$$ 

We are going to assign to each elliptic curve a number, called its $j$-invariant and prove that as a set

$$M_1 = \mathbb{A}^1.$$ 

We will then define the moduli functor of families of elliptic curves and show that it has no fine moduli space! We will discuss various ways to fix this. For example, we will show that $\mathbb{A}^1$ is a coarse moduli space.

2.0.1. Definition. We say that the algebraic variety $M$ is a coarse moduli space of the moduli functor $\mathcal{M} : \text{Algebraic Varieties} \to \text{Sets}$ if

1. We have a natural transformation of functors $\mathcal{M} \to h_M$ (but not necessarily an equivalence), which induces a bijection of sets $\mathcal{M}(\text{point}) = \text{Mor}(\text{point}, M) = M$.

This means that Points of $M$ correspond to isomorphism classes of objects and every family $\pi E \to X$ induces a morphism $X \to M$.

2. If $M'$ is another algebraic variety satisfying (1), an obvious map $M \to M'$ is a morphism.

More generally, we introduce

$$M_g = \{ \text{isom. classes of smooth projective curves of genus } g \}$$

and

$$M_{g,n} = \left\{ \text{isom. classes of smooth projective curves } C \text{ of genus } g \right\}.$$ 

In order to understand these spaces, we will need to study GIT in detail.

First we recall some basic facts about projective algebraic curves = compact Riemann surfaces. We refer to [G] and [Mi] for a detailed exposition.

§2.1. Elliptic and Abelian integrals. The theory of algebraic curves has its roots in analysis. In 1655 Wallis began to study the arc length of an ellipse

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1.$$ 

The equation can be solved for $Y$

$$Y = \frac{b}{a} \sqrt{(a^2 - X^2)},$$

differentiated

$$Y' = \frac{-bX}{a\sqrt{a^2 - X^2}},$$

squared and put into the integral $\int \sqrt{1 + (Y')^2} \, dX$ for the arc length. Now the substitution $x = X/a$ results in

$$s = a \int \sqrt{\frac{1 - e^2 x^2}{1 - x^2}} \, dx,$$
between the limits $0$ and $X/a$, where $e = \sqrt{1 - (b/a)^2}$ is the eccentricity. This is the result for the arc length from $X = 0$ to $X/a$ in the first quadrant, beginning at the point $(0, b)$ on the $Y$-axis. Notice that we can rewrite this integral as an elliptic integral

$$\int \frac{a - ae^2 x^2}{\sqrt{(1 - e^2 x^2)(1 - x^2)}} \, dx = \int P(x, y) \, dx,$$

where $P(x, y)$ is a rational function and $y$ is a solution of the equation

$$y^2 = (1 - e^2 x^2)(1 - x^2),$$

which defines an elliptic curve in $\mathbb{A}^2$. More generally,

2.1.1. Definition. An algebraic function $y = y(x)$ is a solution of

$$y^n + a_1(x)y^{n-1} + \ldots + a_n(x) = 0, \quad (2.1.2)$$

where $a_i(x) \in \mathbb{C}(x)$ are rational functions. Without loss of generality, we can assume that this equation is irreducible over $\mathbb{C}$.\footnote{Let us point out for clarity that after Abel and Galois we know that for $n \geq 5$ not every algebraic function is a nested radical function like $y(x) = \sqrt[3]{x^3 - 7x} \sqrt[4]{x}$.}

An Abelian integral is the integral of the form

$$\int P(x, y) \, dx$$

where $y = y(x)$ is an algebraic function and $P(x, y)$ is a rational function.

All functions of the form $P(x, y)$, where $y = y(x)$ is a solution of (2.1.2), form a field $K$, which is finitely generated and of transcendence degree 1 over $\mathbb{C}$ (because $x$ and $y$ are algebraically dependent).

2.1.3. Lemma. Every finitely generated field $K$ such that $\text{tr.deg.}_\mathbb{C} K = 1$ can be obtained in this way.

Proof. Let $x \in K$ be transcendental over $\mathbb{C}$. Then $K/\mathbb{C}(x)$ is a finitely generated, algebraic (hence finite), and separable (because we are in characteristic 0) field extension. By a theorem on the primitive element, we have $K = \mathbb{C}(x, y)$, where $y$ is a root of an irreducible polynomial of the form (2.1.2).

Notice that of course there are many choices for $x$ and $y$ in $K$, thus the equation (2.1.2) is not determined by the field extension $K/\mathbb{C}$. But it turns out that this choice is not important from the perspective of computing integrals because we can always do $u$-substitutions.

On a purely algebraic level we are should study the moduli problem

$$\{\text{isomorphism classes of f.g. field extensions } K/\mathbb{C} \text{ with } \text{tr.deg.}_\mathbb{C} K = 1\}$$

Clearing denominators in (2.1.2) gives an irreducible affine plane curve

$$C = \{f(x, y) = 0\} \subset \mathbb{A}^2$$

and its projective completion, an irreducible plane curve in $\mathbb{P}^2$. Recall that the field of rational functions $\mathbb{C}(X)$ on an irreducible affine variety $X$ is the quotient field of its ring of regular functions $\mathcal{O}(X)$. The field of rational
functions on an arbitrary algebraic variety \( X \) is the field of rational functions of any of its affine charts. In our case
\[
\mathcal{O}(C) = \mathbb{C}[x, y]/(f) \quad \text{and so} \quad \mathbb{C}(C) = K.
\]
Recall by the way that the word \( \textit{curve} \) means “of dimension 1”, and \( \textit{dimension} \) of an irreducible affine or projective variety is by definition the transcendence degree of its field of rational functions. So we can restate our moduli problem as understanding
\[
\{\text{birational equivalence classes of irreducible plane curves.}\}
\]
Here we use the following definition

2.1.4. Definition. Irreducible algebraic varieties \( X \) and \( Y \) are called \textit{birational} if their fields of rational functions \( \mathbb{C}(X) \) and \( \mathbb{C}(Y) \) are isomorphic. Equivalently, there exist dense Zariski open subsets \( U \subset X \) and \( V \subset Y \) such that \( U \) and \( V \) are isomorphic.

More generally, we can consider arbitrary irreducible affine or projective curves because every curve \( C \) is birational to a curve in \( \mathbb{A}^2 \) by Lemma 2.1.3.\(^5\) Thus our moduli problem is equivalent to the study of
\[
\{\text{birational equivalence classes of irreducible algebraic curves.}\}
\]

2.1.5. Theorem. For any algebraic curve \( C \), there exists a smooth projective curve \( C' \) birational to \( C \).

Sketch. One can assume that \( C \) is projective by taking the projective closure. There are two ways to proceed. One effective argument involves finding a projective plane curve \( \tilde{C} \) birational to \( C \) (using projections as above) and then resolving singularities of \( \tilde{C} \) by consecutively blowing-up \( \mathbb{P}^2 \) in singular points of the proper transform of \( \tilde{C} \). The difficulty is to show that the process terminates. Nevertheless, this is the only approach known to work in higher dimension. Hironaka’s \textit{resolution of singularities theorem} states that, for every projective algebraic variety \( X \subset \mathbb{P}^n \) in characteristic 0, there exists a sequence of blow-ups of \( \mathbb{P}^n \) (but not only in points of course, one has to blow-up loci of higher dimension) such that eventually the proper transform of \( X \) is non-singular (and birational to \( X \)). This theorem is extremely hard and its analogue in characteristic \( p \) is still an open question.

Another approach is to construct the \textit{normalization} of \( C \). Recall that if \( X \) is an irreducible affine variety then its normalization \( \tilde{X} \) has coordinate algebra given the \textit{integral closure} of \( \mathcal{O}(X) \) in its field of fractions \( \mathbb{C}(X) \) (the fact the integral closure is finitely generated is non-trivial). If \( X \) is arbitrary then one can cover \( X \) by affine charts, normalize the charts, and then glue them back together using the fact that the integral closure commutes with localization. One can show that if \( X \) is a projective variety then \( \tilde{X} \) is also projective (despite apriori being defined abstractly, by gluing charts). It turns out that the \textit{singular locus of a normal variety has codimension at least 2}. In particular, an algebraic curve \( C \) is normal iff it is non-singular. \( \square \)

\(^5\)A constructive approach to find a birational model of \( C \) in \( \mathbb{P}^2 \) is to take the image of \( C \) after a general \textit{linear projection} \( \mathbb{P}^n \to \mathbb{P}^2 \). How is this related to the standard proof of the primitive element theorem?
Thus our moduli problem can be restated as the study of
\{ birational equivalence classes of smooth projective algebraic curves. \}

So far everything we said could have been done in any dimension. But
the last step is specific for curves. It is known that

2.1.6. Theorem ([III, 2.3.3]). If \( X \) is a smooth algebraic variety and \( f : X \to \mathbb{P}^n \)
is a rational map then the indeterminancy locus of \( f \) has codimension 2.

Thus if \( C \) is a smooth curve and \( f : C \to \mathbb{P}^n \) is a rational map then \( f \) is
regular. Suppose now that \( C \) and \( C' \) are birational projective curves. Then
we have a birational map \( C \to C' \), which has to be regular by the above
theorem. So birational curves are in fact isomorphic.

To summarize, we have the following

2.1.7. Theorem. There is a natural bijection between
\[ \bigcup M_g = \{ \text{isom. classes of smooth projective algebraic curves} \} \]
and
\[ \{ \text{isomorphism classes of finitely generated} \}
\{ \text{field extensions } K/\mathbb{C} \text{ with tr.deg.} C K = 1 \} \].

The bijection sends a curve \( C \) to its field of fractions \( \mathbb{C}(C) \).

The analytic approach is to consider Riemann surfaces (one-dimensional
complex manifolds) instead of algebraic curves. It turns out that this gives
the same moduli problem:

\{ biholomorphic isom. classes of compact Riemann surfaces \}.

Moreover, categories of smooth projective curves and of compact Riemann
surfaces are equivalent. Indeed, if \( X \) is a smooth projective algebraic curve
then \( X^{an} \) is a compact Riemann surface and any morphism \( X \to Y \) of
smooth algebraic curves gives a holomorphic map \( X^{an} \to Y^{an} \). Thus we
have a functor, the analytification, from one category to another.

2.1.8. Lemma. Analytification is fully faithful, that is every holomorphic map
\( f : X \to Y \) between two smooth projective algebraic curves is a regular morphism.

Sketch. Let \( G \subset X \times Y \) be the graph of \( f \). We embed \( X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m \), and
\[ G \subset X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{nm+n+m} \]
by the Segre embedding. Thus \( G \) is a complex submanifold of \( \mathbb{P}^{nm+n+m} \).
But very general Chow embedding theorem [GH] asserts that every complex
submanifold of \( \mathbb{P}^N \) is algebraic.\(^6\) Thus \( G \subset X \times Y \) is algebraic and therefore
the map \( f : X \to G \to Y \) is regular. \( \square \)

2.1.9. Example. Every meromorphic function \( f \) on a smooth projective algebraic curve \( C \) can be viewed as a holomorphic map from \( C \) to \( \mathbb{P}^1 \). Thus \( f \) is
in fact a rational function.

\(^6\)In fact the theorem says that every analytic (i.e. locally given as a vanishing set of
holomorphic functions) subset of \( \mathbb{P}^N \) is algebraic. It doesn’t have to be a submanifold.
A difficult part is to show that analytification is essentially surjective: any compact Riemann surface is biholomorphic to a smooth projective curve. It is hard to construct a single meromorphic function, but once this is done the rest is straightforward. It is enough to find a harmonic function (why?). Klein (following Riemann) “covers the surface with tin foil... Suppose the poles of a galvanic battery are placed at the points $A_1$ and $A_2$. A current arises whose potential $u$ is single-valued, continuous, and satisfies the equation $\Delta u = 0$ across the entire surface, except for the points $A_1$ and $A_2$, which are discontinuity points of the function.” A modern treatment can be found in [GH].

§2.2. Genus and meromorphic forms.

2.2.1. LEMMA. Every meromorphic form $\omega$ on a smooth projective algebraic curve is rational, i.e. can be written as $\omega = f \, dg$, where $f$ and $g$ are rational functions.

Proof. This follows from Example 2.1.9. Indeed, let $g$ be a non-constant rational function on $C$. Then $dg$ is a non-zero meromorphic form. So we can write $\omega = f \, dg$, where $f$ is a meromorphic function. But all meromorphic functions are rational. $\square$

In particular, the vector space of holomorphic differential forms, i.e. meromorphic differential forms without poles is the same as the vector space of regular differential forms, i.e. rational differential forms without poles. The fundamental result is that this space is finite-dimensional and its dimension is equal to the genus $g$, the number of handles on a Riemann surface.

Another way to compute the genus is to use the genus formula:

$$2g - 2 = (\text{number of zeros of } \omega) - (\text{number of poles of } \omega)$$

(2.2.2)

of any meromorphic (rational) differential form $\omega = f \, dg$.

2.2.3. EXAMPLE. Consider a form $\omega = dx$ on $\mathbb{P}^1$. It has no zeros or poles in the $x$-chart of 0. In the $y$-chart at infinity we have

$$dx = d(1/y) = -(1/y^2)dy.$$ 

So it has a pole of order 2 at infinity, which shows that $g(\mathbb{P}^1) = 0$ by (2.2.2).

2.2.4. EXAMPLE. A smooth plane curve $C \subset \mathbb{P}^2$ of degree $d$ has genus

$$g = \frac{(d - 1)(d - 2)}{2}.$$ 

(2.2.5)

In algebraic geometry, it is more natural to write this as

$$2g - 2 = d(d - 3)$$

(2.2.6)

because this can be generalized to an adjunction formula which computes the genus of a curve on any algebraic surface $S$, not just $\mathbb{P}^2$.

There is a nice choice of a holomorphic form on $C$: suppose $C \cap \mathbb{A}^2_{x,y}$ is given by the equation $f(x, y) = 0$. Differentiating this equation shows that

$$\omega = \frac{dx}{f_y} = -\frac{dy}{f_x}$$

along $C$, where the first (resp. second) expression is valid where $f_y \neq 0$ (resp. $f_x \neq 0$), i.e. where $x$ (resp. $y$) is a holomorphic coordinate. Thus $\omega$ has no zeros or poles in $C \cap \mathbb{A}^2$. We will show that $\omega$ has zeros at each
of the \(d\) intersection points of \(C\) with the line at infinity and each zero has multiplicity \(d - 3\). Combined with (2.2.2), this will give (2.2.6).

Indeed, switching from the chart \((x, y) = [x : y : 1]\) of \(\mathbb{P}^2\) to the chart \((x, z) = [x : 1 : z]\) gives

\[
-\frac{dy}{f_x(x, y)} = -\frac{d\frac{1}{z}}{f_x\left(\frac{z}{x}, \frac{1}{z}\right)} = \frac{dz}{z^2 - 1} g(x, z) = z^{d-3} \frac{dz}{g(x, z)},
\]

where we can arrange homogeneous coordinates from the start so that at every point at infinity \(g\) doesn’t vanish and \(z\) is a holomorphic coordinate.

Notice that in this language an Abelian integral is the integral of a meromorphic form on a Riemann surface.

\[\text{§2.3. Divisors on curves.}\]

A (Weil) divisor \(D\) on a smooth projective curve \(C\) is an integral linear combination \(\sum a_i P_i\) of points \(P_i \in C\). Divisors form an Abelian group, denoted by \(\text{Div} C\), freely generated by classes of points. There is a homomorphism \(\deg : \text{Div} C \rightarrow \mathbb{Z}\), called the degree, namely

\[\deg D = \sum a_i.\]

2.3.1. \text{Definition.} A principal divisor of a rational function \(f\) on \(C\) is

\[(f) = \sum_{P \in C} \text{ord}_P(f)P,\]

where \(\text{ord}_P(f)\) is the order of zeros or poles of \(f\) at \(P\). Recall that it can be defined analytically: if \(z\) is a holomorphic coordinate on \(C\) centered at \(P\) then near \(P\)

\[f(z) = z^n g(z),\]

where \(g(z)\) is holomorphic and does not vanish at \(p\). Then \(\text{ord}_P(f) = n.\)

2.3.2. \text{Definition.} A canonical divisor of a meromorphic (=rational) form \(\omega\) is defined as follows:

\[(\omega) = \sum_{P \in C} \text{ord}_P(\omega)P,\]

where if \(z\) is a holomorphic coordinate (or a local parameter) at \(P\) then we can write \(\omega = f\ dz\) and \(\text{ord}_P(\omega) = \text{ord}_P(f).\)

\[\text{If we want to work algebraically then, instead of choosing a holomorphic coordinate, we can choose a local parameter, i.e. a rational function } z \text{ regular at } P, z(P) = 0, \text{ and such that any rational function } f \text{ on } C \text{ can be written (uniquely) as}\]

\[f = z^n g,\]

where \(g\) is regular at \(P\) and does not vanish there (see [III, 1.1.5]). This is an instance of a general strategy in Algebraic Geometry: if there is some useful analytic concept (e.g. a holomorphic coordinate) that does not exist algebraically, one should look for its dsitable properties (e.g. a factorization \(f = z^n g\) as above). Often it is possible to find a purely algebraic object (e.g. a local parameter) satisfying the same properties.

A local parameter at \(x\) can also be described as follows:

- a uniformizer of the DVR \(O_{C, P}\) (with valuation \(\text{ord}\));
- any element in \(m_{C, P} \setminus m_{C, P}^2\);
- a coordinate of an affine chart such that the tangent space \(T_P C\) projects onto the corresponding coordinate axis;
- a rational function \(z\) such that \(\text{ord}_P(z) = 1.\)
We can rewrite (2.2.2) as

\[ \deg(\omega) = 2g - 2. \]

Two divisors \( D \) and \( D' \) are called \emph{linearly equivalent} if \( D - D' \) is a principal divisor. Notation: \( D \sim D' \). For example, any two canonical divisors are linearly equivalent. Indeed, if \( \omega \) and \( \omega' \) are two rational forms then \( \omega = g\omega' \) for some rational function \( g \) and it is easy to see that \( (\omega) = (g) + (\omega') \).

A linear equivalence class of a canonical divisor is usually denoted by \( K_C \).

\section*{2.4. Riemann–Hurwitz formula.}

Suppose \( f : C \to C' \) is a non-constant map of smooth projective algebraic curves. It is often called a \emph{branched cover}. Its degree \( \deg f \) can be computed in two ways:

- topologically: number of points in the preimage of a general point.
- algebraically: degree of the induced field extension \( \mathbb{C}(C)/\mathbb{C}(C') \), where \( \mathbb{C}(C') \) is embedded in \( \mathbb{C}(C) \) by pull-back of functions \( f^* \).

Suppose \( f(Q) = P \) and suppose \( e_Q = \text{ord}_Q f^*(z) > 1 \), where \( z \) is a local parameter at \( P \). Then \( P \) is called a \emph{branch point}, \( Q \) is called a \emph{ramification point} and \( e_Q \) is called a \emph{ramification index} (or multiplicity).

Let \( f^{-1}(P) = \{Q_1, \ldots, Q_r\} \). It turns out that

\[ \deg f = \sum_{i=1}^r e_{Q_i}. \]

In particular, viewing a rational function \( f \) on \( C \) as a map \( f : C \to \mathbb{P}^1 \), its degree is equal to both the number of zeros and the number of poles of \( f \) (counted with multiplicities), and so

\[ \deg(f) = (\deg f) - (\deg f) = 0 \quad \text{for every } f \in k(C). \]

In particular, linearly equivalent divisors have the same degree.

We extend the map \( [P] \mapsto \sum_{i=1}^r e_{Q_i} [Q_i] \) by linearity to a homomorphism

\[ f^* : \text{Div} C' \to \text{Div} C. \]

Then \( \deg f^* D = (\deg f) \deg D \) for every divisor \( D \) on \( C' \).

\begin{enumerate}
\item \textbf{2.4.1. Theorem (Riemann–Hurwitz).} For every branched cover \( f : C \to C' \),

\[ K_C \sim f^* K_{C'} + \sum_{Q \in C} (e_Q - 1)[Q]. \]

and comparing the degrees and using (2.2.2),

\[ 2g(C) - 2 = \deg f [2g(C') - 2] + \sum_{Q \in C} (e_Q - 1). \]

\textbf{Proof.} Choose a meromorphic form \( \omega \) on \( C' \) without zeros or poles at branch points. Then \( K_{C'} = (\omega) \) and \( K_C = (f^* \omega) \). Every zero (resp. pole) of \( \omega \) contributes to \( \deg f \) zeros (resp. poles) of \( f^* \omega \). In addition, if \( t \) is a local parameter at a ramification point \( Q \) and \( z \) is a local parameter at \( P = f(Q) \) then \( f^*(z) = t^{e_Q} g \), where \( g \) is regular at \( Q \). Then

\[ f^*(dz) = d(t^{e_Q} g) = e_Q t^{e_Q - 1} g dt + t^{e_Q} dg, \]
which shows that each ramification point is a zero of $f^*\omega$ of order $e_Q - 1$. So

$$(f^*\omega) = f^*(\omega) + \sum_{Q \in C} (e_Q - 1)[Q],$$

which proves the Riemann–Hurwitz formula. \qed

§2.5. **Riemann–Roch formula and linear systems.**

2.5.1. **Theorem (Riemann–Roch).** For every divisor $D$ on $C$, we have

$$l(D) - i(D) = 1 - g + \deg D,$$

where

$$l(D) = \dim \mathcal{L}(D), \quad \text{where} \quad \mathcal{L}(D) = \{ f \in \mathcal{C}(C) \mid (f) + D \geq 0 \}$$

and

$$i(D) = \dim \mathcal{K}(D), \quad \text{where} \quad \mathcal{K}(D) = \{ \text{meromorphic forms } \omega \mid (\omega) \geq D \}.$$  

2.5.2. **Example.** If $D = 0$ then $l(D) = 1$ and therefore $i(D) = g$. Notice that $\mathcal{K}(0)$ is the space of holomorphic differentials.

Here $l(D) = 1$ because the only rational functions which are regular everywhere are constants. Analytically, this is Liouville’s Theorem for Riemann surfaces (see also the maximum principle for harmonic functions). Algebraically, this is

2.5.3. **Theorem.** If $X$ is an irreducible projective variety then the only functions regular on $X$ are constants.

**Proof.** A regular function is a regular morphism $X \to \mathbb{A}^1$. Composing it with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ gives a regular morphism $f : X \to \mathbb{P}^1$ such that $f(X) \subset \mathbb{A}^1$. But the image of a projective variety under any morphism is closed, thus $f(X)$ must be closed in $\mathbb{P}^1$ and so $f(X)$ must be a point. \qed

2.5.4. **Example.** If $D = K$ then $L(D)$ is the space of homomorphic forms and $\mathcal{K}(D)$ is the space of holomorphic functions. So in this case Riemann–Roch gives (2.2.2)!

2.5.5. **Example.** Suppose $g(C) = 0$. Let $D = P$ be a point. Then RR gives

$$l(P) = i(P) + 2 \geq 2.$$  

It follows that $\mathcal{L}(D)$ contains a non-constant rational function $f$, which then has a unique simple pole at $P$. This function gives an branch cover $C \to \mathbb{P}^1$ of degree 1, therefore an isomorphism. So $\mathcal{M}_0 = \{ pt \}$.

2.5.6. **Definition.** A divisor $D'$ is called effective if $D' \geq 0$ (i.e. all its coefficients are positive). We define a linear system of divisors

$$|D| = \{ (f) + D \mid f \in \mathcal{L}(D) \} = \{ D' \mid D' \sim D, \; D' \geq 0 \},$$

which consists of all effective divisors linearly equivalent to $D$.

More generally, if $L \subset \mathcal{L}(D)$ is a vector subspace then an incomplete linear system $|L| \subset |D|$ consists of all divisors of the form $\{ (f) + D \mid f \in L \}$. 

Choosing a basis \( f_0, \ldots, f_r \) of \( \mathcal{L}(D) \) gives a map
\[
\phi_D : C \to \mathbb{P}^r, \quad \phi_D(x) = [f_0, \ldots, f_r].
\]
Since \( C \) is a smooth curve, this map is regular. More generally, we define a similar map \( \phi_{|L|} \) for every incomplete linear system \( L \).

2.5.7. DEFINITION. Let \( D \) be an effective divisor. Its base locus is the intersection of all divisors in the linear system \( |D| \). Its fixed part is a maximal effective divisor \( E \) such that \( D' - E \geq 0 \) for every \( D' \in |D| \). Notice that \( E \) is a sum of points in the base locus with positive multiplicities. More generally, we can define the base-locus and the fixed part of an incomplete linear system, in an obvious way.

2.5.8. LEMMA. We have
\[
|D| = |D - E| + E,
\]
and
\[
\mathcal{L}(D) = \mathcal{L}(D - E)
\]
and
\[
\phi_D = \phi_{D - E}.
\]

2.5.9. LEMMA. \( D \) has no base points iff \( l(D - P) = l(D) - 1 \) for every \( P \in C \).

Proof. If \( P \) is in the base locus then \( l(D - P) = l(D) \). If the base locus is empty then \( |D - P| + P \) is strictly contained in \( |D| \) and so \( l(D - P) < l(D) \). Thus we can assume that \( D \) is effective and doesn’t contain \( P \). In this case, we can identify \( \mathcal{L}(D - P) \) with a hyperplane in \( \mathcal{L}(D) \) of all rational functions that vanish at \( P \).

If we are interested in maps \( \phi_D \) then we can always assume that \( D \) is effective and base-point-free, i.e. its base locus is empty. In addition, \( D \) is called very ample if \( \phi_D \) is an embedding \( C \subset \mathbb{P}^r \). One has the following very useful criterion generalizing the previous lemma.

2.5.10. THEOREM. A divisor \( D \) is very ample if and only if
\begin{itemize}
  \item \( \phi_D \) separates any points \( P, Q \in C \), i.e. \( l(D - P - Q) = l(D) - 2 \);
  \item \( \phi_D \) separates tangents, i.e. \( l(D - 2P) = l(D) - 2 \) for any point \( P \in C \).
\end{itemize}

2.5.11. PROPOSITION. Every morphism \( \phi : C \to \mathbb{P}^r \) is given by a base-point-free linear system (possibly incomplete) as long as \( \phi(C) \) is not contained in a projective subspace of \( \mathbb{P}^r \) (in which case we can just switch from \( \mathbb{P}^r \) to \( \mathbb{P}^s \) for \( s < r \)).

Proof. Indeed, \( \phi \) is obtained by choosing rational functions
\[
f_0, \ldots, f_r \in k(C).
\]
Consider their divisors \((f_0), \ldots, (f_r)\) and let \( D \) be the smallest effective divisor such that \((f_i) + D \) is effective for every \( i \). Then of course every \( f_i \in \mathcal{L}(D) \) and \( D \) is base-point-free (otherwise it’s not the smallest).  

\[8\text{This is a special feature of algebraic curves as in higher dimension the base locus is not necessarily a divisor.} \]
Divisors \((f_0) + D, \ldots, (f_r) + D\) have very simple meaning: they are just “pull-backs” of coordinate hyperplanes in \(\mathbb{P}^r\). More precisely, suppose \(h\) is a local parameter at a point \(P \in C\), which contributes \(nP\) to \(D\). Then \(\phi\) (in the neighborhood of \(P\)) can be written as

\[
[f_0 h^n : \ldots : f_r h^n],
\]

where at least one of the functions does not vanish. So pull-backs of coordinate hyperplanes near \(P\) are given by divisors \((f_0) + nP, \ldots, (f_r) + nP\).
§2.6. **Homework 2. Due March 3.**

Name:
Don’t write anything on the problem sheet and attach it to your written solutions. I will use it to record problems presented during office hours.

**Problem 1.** (1 point) Let $F : \text{Sets} \to \text{Sets}$ be a contravariant functor that sends a set $S$ to the set of subsets of $S$ and any function $f : S \to S'$ to a function that sends $U \subset S'$ to $f^{-1}(U) \subset S$. Show that $F$ is representable.

**Problem 2.** (2 points) Let $F$ be a contravariant functor from the category of topological spaces to $\text{Sets}$ which sends a topological space $X$ to the set of open sets of $X$. Is it representable?

**Problem 3.** (3 points) Let $C$ be a category and let $D$ be the category of contravariant functors $C \to \text{Sets}$ with natural transformations as morphisms. Show how to extend the assignment $X \mapsto h_X$ to a fully-faithful functor $C \to D$.

**Problem 4.** (1 point) Let $F : \text{AlgebraicVarieties} \to \text{Sets}$ be a functor which assigns to each $X$ the set $\mathcal{O}(X)$ of all regular functions on $X$ and to each morphism $f : X \to Y$ the pullback of functions $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Show that $F$ is representable.

**Problem 5.** (2 points) Let $F : \text{AlgebraicVarieties} \to \text{Sets}$ be a functor which assigns to each $X$ the subset $S \subset \mathcal{O}(X)$ of all regular functions $f$ which can be written as a square $f = g^2$ of a regular function. Is $F$ representable?

**Problem 6.** (2 points) Let $M$ be the set of isomorphism (=conjugacy) classes of invertible complex $3 \times 3$ matrices. (a) Describe $M$ as a set. (b) Let’s define the following moduli problem: a family over a variety $X$ is a $3 \times 3$ matrix $A(x)$ with coefficients in $\mathcal{O}(X)$ such that $\det A(x) \in \mathcal{O}^*(X)$, i.e. $A(x)$ is invertible for any $x \in X$. Describe the corresponding moduli functor. (c) Show that this moduli functor has no coarse moduli space.

**Problem 7.** (2 points) The formula $j = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda(\lambda - 1)^2}$ gives a 6 : 1 cover $\mathbb{P}^1 \to \mathbb{P}^1$. Thinking about $\mathbb{P}^1$ as a Riemann sphere, let’s color $\mathbb{P}^1$ in two colors: color the upper half-plane $\mathcal{H}$ white and the lower half-plane $-\mathcal{H}$ black. Draw the pull-back of this coloring to $\mathbb{P}^1$.

**Problem 8.** (2 points) Using a birational isomorphism between $\mathbb{P}^1$ and the circle $\{x^2 + y^2 = 1\} \subset \mathbb{A}^2$ given by stereographic projection from $(0,1)$, describe an algorithm for computing integrals of the form

$$\int P(x, \sqrt{1-x^2}) \, dx$$

where $P(x,y)$ is an arbitrary rational function.

**Problem 9.** (2 points) Let $X$ and $Y$ be irreducible quasi-projective varieties with fields of rational functions $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. Show that these fields are isomorphic if and only if there exist non-empty affine open subsets $U \subset X$ and $V \subset Y$ such that $U$ is isomorphic to $V$. 
Problem 10. (2 points) Let $R = \sum_{k \geq 0} R_k$ be an integral finitely generated graded algebra. Show that $\text{MaxSpec } R$ is non-singular if and only if $R$ is isomorphic to a polynomial algebra.

Problem 11. (3 points) Suppose that $R = \sum_{k \geq 0} R_k$ is a graded algebra generated by finitely many generators all in degree 1. Show that there exists a Hilbert polynomial $H(t)$ such that $\dim_{\mathbb{C}} R_k = H(k)$ for $k \gg 0$. Show that this can fail if $R$ is finitely generated but not all generators are in degree 1.

Problem 12. (2 points) Let $C$ be an algebraic curve. Then one can describe morphisms $C \to \mathbb{P}^r$ either using Theorem 1.10.10 or using linear systems of divisors. Explain how these methods are related.

Problem 13. (2 points) Let $R = \sum_{k \geq 0} R_k$ be a finitely generated graded algebra with Hilbert function $h(k)$. Its generating function $P(t) = \sum h(k)t^k$ is called the Poincare series. (a) Show that $P(t) = \frac{1}{(1-t)^n}$ for $R = \mathbb{C}[x_1, \ldots, x_n]$. (b) Suppose that $R$ is generated by homogeneous generators of degrees $k_1, \ldots, k_s$. Show that the Poincare series $P(t)$ is a rational function of the form

$$\frac{f(t)}{(1-t^{k_1}) \cdots (1-t^{k_s})},$$

where $f(t)$ is a polynomial with integer coefficients.

Problem 14. (1 point) Solve a cross-word puzzle.
Across

1 A linear system that corresponds to a proper subset of a basis in \( L(D) \) is called ...
4 Theorem of zeros
8 A friendly giant hiding inside the j-function
9 All effective divisors linearly equivalent to a given one
11 An algebraic curve over \( \mathbb{C} \) is also known as a ...
13 He coined a term “coarse moduli space”
14 A cross-ratio is also known as an ...
15 Word used by Riemann for a number of parameters
16 An attribute of a divisor contributing to the Riemann-Roch formula
19 Example of a fine moduli space
22 Dimension of the space of holomorphic differentials on the algebraic curve
23 Something you can do with families
24 An algebraic variety is called … if it is not a union of two proper closed subsets
25 A normal form of a cubic curve is named after him

Down

2 Type of an algebraic curve most revered by number theorists
3 Divisor of a meromorphic form
5 An algebraic geometry’s analogue of a holomorphic coordinate
6 He made important contributions to the study of integrals of algebraic functions
7 A fancy word used by algebraic geometers instead of “compact”
10 Holomorphic map with surjective differential
12 The best kind of a moduli space
17 Image of a ramification point
18 The second best kind of a moduli space
20 A divisor with non-negative multiplicities is called ...
21 An everywhere defined rational map is called …
§3. Moduli of elliptic curves

§3.1. Curves of genus 1. Let us recall the following basic result.

3.1.1. Theorem. Let $C$ be a smooth projective curve. TFAE:

1. $C$ has a plane model in $\mathbb{P}^2$ given by the Weierstrass equation
   $$y^2 = 4x^3 - g_2x - g_3, \quad \Delta = g_2^3 - 27g_3^2 \neq 0.$$

2. $C$ is isomorphic to a cubic curve in $\mathbb{P}^2$.

3. $C$ admits a $2 : 1$ cover of $\mathbb{P}^1$ ramified at 4 points.

4. $C$ has genus 1.

In addition, every cubic curve in $\mathbb{P}^2$ has a flex point and admits a unique (up to scalar) regular form $\omega$. Moreover, this form has no zeros.

Proof. Easy steps.

1. $\Rightarrow$ 2. It is easy to check that the projective closure is smooth at all points including $[0 : 1 : 0]$, which in fact is a flex point (and the line at infinity $z = 0$ is the flex line). This curve has genus 1 by the formula for the genus of a plane curve.

2. $\Rightarrow$ 3. Use a linear projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from any point $p \in C$ (projecting from points away from $C$ gives a triple cover). Use Riemann–Hurwitz to find the number of branch points.

3. $\Rightarrow$ 4. Use Riemann–Hurwitz to compute the genus. Then $L(K)$ is one-dimensional. Let $\omega$ be a generator. Since $\deg K = 0$, $\omega$ has no zeros.

The Riemann–Roch analysis. Assume 4. Let $D$ be a divisor of positive degree. Since $\deg D > \deg K$, we have $i(D) = 0$ and

$$l(D) = \deg D.$$

It follows that $D$ is base-point-free if $\deg D \geq 2$ and very ample if $\deg D \geq 3$.

Fix a point $P \in C$. Notice that $L(kP) \subset L(lP)$ for $k \leq l$ and that $L(kP) \cdot L(lP) \subset L((k+l)P)$. Thus we have a graded algebra

$$R(C, P) = \bigoplus_{k \geq 0} L(kP) \subset \mathbb{C}(C).$$

$L(0) = L(P)$ is spanned by 1. $L(2P)$ is spanned by 1 and by some non-constant function, which we will call $x$. Since $2P$ has no base-points, we have a $2 : 1$ map

$$\psi_{2P} : C \to \mathbb{P}^1$$
given by $[x : 1]$. This shows $4 \Rightarrow 3$. $P$ is one of the ramification points.

$L(3P)$ is spanned by 1, $x$, and a new function, which we will call $y$. Since $3P$ is very ample, we have an embedding

$$\psi_{3P} : C \to \mathbb{P}^2,$$
given by $[x : y : 1]$. The image is a curve of degree 3. Moreover, $P$ is a flex point. This shows that $4 \Rightarrow 2$.

Notice that $L(6P)$ has dimension 6 but contains seven functions

$$1, x, y, x^2, xy, x^3, y^2.$$  

Thus, they are linearly dependent. Moreover, $x^3$ and $y^2$ are the only functions on the list that have a pole of order 6 at $P$. Therefore, they must both
contribute to the linear combination. After rescaling them by constants, we can assume that the equation has form

$$y^2 + axy = 4x^3 - g_1x^2 - g_2x - g_3.$$  

After making the changes of variables $y \mapsto y - \frac{a}{2}x$ and $x \mapsto x + \frac{g_1}{12}$, we get the Weierstrass form. This shows that (4) $\Rightarrow$ (1).

**Logically unnecessary but fun implications:**  
(2) $\Rightarrow$ (1). Move a flex point to $[0 : 1 : 0]$, make the line at infinity $z = 0$ the flex line, etc. (this is analogous in spirit to the Riemann-Roch analysis above but more tedious). One can also prove existence of a flex point directly, by intersecting with the Hessian cubic.

(1) $\Rightarrow$ (3). Project $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_z$. Three ramification points are at the roots of $4x^3 - g_2x - g_3 = 0$, the last ramification point is at $\infty$. □

From the complex-analytic perspective, we have the following

**3.1.2. Theorem.** Let $C$ be a compact Riemann surface. TFAE:

1. $C$ is biholomorphic to a projective algebraic curve of genus 1. It admits a unique (up to scalar) holomorphic form $\omega$. This form has no zeros.
2. $C$ has genus 1.
3. $C$ is biholomorphic to a complex torus $\mathbb{C}/\Lambda$, where $\Lambda \simeq \mathbb{Z} \oplus \mathbb{Z}\tau$, $\text{Im} \tau > 0$.

**Proof.**  
(1) $\Rightarrow$ (2). Apply analytification.

(3) $\Rightarrow$ (2). $C/\Lambda$ is topologically a torus and has a structure of a Riemann surface induced from a translation-invariant complex structure on $\mathbb{C}$. Also notice that $dz$ descends to a non-vanishing holomorphic form.

(3) $\Rightarrow$ (1). One can invoke a general theorem about the equivalence of categories here, but it’s more fun to show directly that every complex torus is biholomorphic to a projective cubic curve. Let $P \in C/\Lambda$ be the image of $0 \in \mathbb{C}$. From the Riemann–Roch analysis, we should expect to find a meromorphic function in with pole of order 2 at $P$ and holomorphic elsewhere. Its pull-back to $\mathbb{C}$ will be a doubly-periodic (i.e. $\Lambda$-invariant) meromorphic function on $\mathbb{C}$ with poles only of order 2 and only at lattice points. Luckily, this function was constructed explicitly by Weierstrass:

$$\wp(z) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda, \gamma \neq 0} \left( \frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2} \right).$$

Notice that $\wp'(z)$ has poles of order 3 at lattice points, and therefore

$$\{1, \wp(z), \wp'(z)\}$$

should be a basis of $\mathcal{L}(3P)$. Indeed, one checks directly that the map

$$\mathbb{C} \to \mathbb{C}^2, \quad z \mapsto [\wp(z) : \wp'(z) : 1]$$

gives an embedding $C \subset \mathbb{P}^2$ as a cubic curve. In fact it is easy to see that $\wp'$ and $\wp$ satisfy the Weierstrass equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$
Periods. (2) ⇒ (3). Let’s assume that $C$ is compact Riemann surface which has a nowhere vanishing holomorphic form $\omega$ and topological genus 1. We fix a point $P \in C$ and consider a multi-valued holomorphic map

$$\pi : C \to \mathbb{C}, \quad z \mapsto \int_p^z \omega.$$ 

It is multi-valued because it depends on the choice of a path of integration. Notice however that near every point $z \in C$ we can choose a branch of $\pi$ by specifying paths of integration (say connecting $P$ to $z$ and then $z$ to a nearby point by a segment in a holomorphic chart) and this branch of $\pi$ is conformal (because $\omega$ has no zeros and $\int dz = z$).

Topologically, we can obtain $C$ by gluing opposite sides of the rectangle, in other words we have a homeomorphism $\mathbb{C}/\mathbb{Z}^2 \to C$ which sends segments $\alpha, \beta \subset \mathbb{C}$ connecting the origin to $(1,0)$ and $(0,1)$ to generators of the first homology group $H_1(C) = \mathbb{Z} \alpha + \mathbb{Z} \beta$. We can then define periods

$$A = \int_\alpha \omega \quad \text{and} \quad B = \int_\beta \omega.$$ 

They generate a subgroup $\Lambda \subset \mathbb{C}$. Integrals along paths in $C$ are uniquely defined modulo $\Lambda$.

3.1.3. Lemma. $A$ and $B$ are linearly independent over $\mathbb{R}$.

Proof. If not then we can assume that $\Lambda \subset \mathbb{R}$ (by multiplying $\omega$ by a constant). Then $\text{Im} \pi$ is a single-valued harmonic function, which must be constant by the maximum principle because $C$ is a compact Riemann surface. This is a contradiction: a branch of $\pi$ is a local isomorphism near $P$. \(\square\)

So $\Lambda$ is a lattice and $\pi$ induces a holomorphic map

$$f : C \to \mathbb{C}/\Lambda.$$ 

Notice that its composition with a homeomorphism $\mathbb{C}/\mathbb{Z}^2 \to C$ is a homeomorphism $\mathbb{C}/\mathbb{Z}^2 \to \mathbb{C}/\Lambda$. Indeed, it is induced by the integration map $\mathbb{C} \to \mathbb{C}$ which sends $\alpha \mapsto A$ and $\beta \mapsto B$. Thus $f$ is bijective. \(\square\)

3.1.4. Remark. An important generalization is a beautiful Klein–Poincare Uniformization Theorem: a universal cover of a compact Riemann surface is

- $\mathbb{P}^1$ if $g = 0$;
- $\mathbb{C}$ if $g = 1$;
- $\mathbb{H}$ (the upper half-plane) if $g \geq 2$.

In other words, every compact Riemann surface of genus $\geq 2$ is isomorphic to a quotient of $\mathbb{H}$ by a discrete subgroup

$$\Gamma \subset \text{Aut}(\mathbb{H}) = \text{PGL}_2(\mathbb{R}),$$

which acts freely on $\mathbb{H}$.

\(\text{If } C \text{ is a cubic curve in the Weierstrass normal form then } \omega = \frac{dx}{y} \text{ (see Example 2.2.4) and so these integrals are elliptic integrals}

$$\int \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}.$$
§3.2. \textit{J-invariant.} Now we would like to classify elliptic curves up to isomorphism, i.e. to describe \( M_1 \) as a set. As we will see many times in this course, automorphisms of geometric objects can cause problems. A curve of genus 1 has a lot of automorphisms: a complex torus \( \mathbb{C}/\Lambda \) admits translations by vectors in \( \mathbb{C} \). These translations are biholomorphic, and therefore regular, automorphisms. In fact \( \mathbb{C}/\Lambda \) is an algebraic group:

3.2.1. \textbf{Definition.} An algebraic variety \( X \) with a group structure is called an \textit{algebraic group} if the multiplication \( X \times X \to X \) and the inverse \( X \to X \) maps are morphisms of algebraic varieties.

In a cubic plane curve realization, this group structure is a famous “three points on a line” group. We can eliminate translations by fixing a point.

3.2.2. \textbf{Definition.} An \textit{elliptic curve} is a pair \((C, P)\), where \( C \) is a smooth projective curve of genus 1 and \( P \in C \). It is convenient to choose \( P \) to be the unity of the group structure on \( C \) if one cares about it.

Of course as a set we have

\[ M_1 = M_{1,1}. \]

Every pointed curve \((C, P)\) still has at least one automorphism, namely the involution given by permuting the two branches of the double cover

\[ \phi_{2P} : C \to \mathbb{P}^1. \]

In the complex torus model this involution is given by the formula \( z \mapsto -z \), which reflects the fact that the Weierstrass \( \wp \)-function is even.

Let’s work out when two elliptic curves are isomorphic and when the automorphism group \( \text{Aut}(C, P) \) is larger than \( \mathbb{Z}/2\mathbb{Z} \).

3.2.3. \textbf{Theorem.} \begin{enumerate}
\item Curves with Weierstrass equations \( y^2 = 4x^3 - g_2x - g_3 \) and \( y^2 = 4x^3 - g'_2x - g'_3 \) are isomorphic if and only if there exists \( t \in \mathbb{C}^* \) such that \( g'_2 = t^4g_2 \) and \( g'_3 = t^3g_3 \).
\item Two smooth cubic curves \( C \) and \( C' \) are isomorphic if and only if they are projectively equivalent: \( C \simeq A(C') \) for some \( A \in \text{PGL}_3(\mathbb{C}) \).
\item Let \( C \) (resp. \( C' \)) be a double cover of \( \mathbb{P}^1 \) with a branch locus \( p_1, \ldots, p_4 \) (resp. \( p'_1, \ldots, p'_4 \)). Then \( C \simeq C' \) if and only if there exists \( g \in \text{PGL}_2(\mathbb{C}) \) such that

\[ \{p_1', p_2', p_3', p_4'\} = g\{p_1, p_2, p_3, p_4\}. \]

In particular, we can always assume that branch points are 0, 1, \( \lambda, \infty \).
\item \( \mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda' \) if and only if \( \Lambda = \alpha \Lambda' \) for some \( \alpha \in \mathbb{C}^* \). If \( \Lambda = \mathbb{Z} + \mathbb{Z}\tau \) and \( \Lambda' = \mathbb{Z} + \mathbb{Z}\tau' \) with \( \text{Im}\, \tau, \text{Im}\, \tau' > 0 \) then this is equivalent to

\[ \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for some} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}_2(\mathbb{Z}) \quad (3.2.4) \]
\end{enumerate}

There are only two curves with special automorphisms:

\[ \text{Aut}(y^2 = x^3 + 1) = \mathbb{Z}/6\mathbb{Z} \quad \text{and} \quad \text{Aut}(y^2 = x^3 + x) = \mathbb{Z}/4\mathbb{Z}. \]

Their lattices in \( \mathbb{C} \) are the hexagonal and the square lattices. These curves are double covers of \( \mathbb{P}^1 \) branched at 0, 1, \( \lambda, \infty \), with \( \lambda = e^{\pi i/3} \) and \(-1\), respectively.
Proof. Let $C$ be and $C'$ be two plane smooth cubic curves, which are abstractly isomorphic. Let $P \in C$ and $P' \in C'$ be flex points. Then embeddings $C \hookrightarrow \mathbb{P}^2$ and $C' \hookrightarrow \mathbb{P}^2$ are given by linear systems $\mathcal{L}(3P)$ and $\mathcal{L}(3P')$, respectively. After translation by an element $C'$, we can assume that an isomorphism $\phi : C \rightarrow C'$ takes $P$ to $P'$. Then $\mathcal{L}(3P) = \phi^* \mathcal{L}(3P')$. Applying projective transformations to $C$ and $C'$ is equivalent to choosing bases in the linear systems. If we choose a basis in $\mathcal{L}(3P')$ and pull it back to the basis of $\mathcal{L}(3P)$, we will have

$$\phi_{3P} = \phi_{3P'} \circ \phi,$$

i.e. $C$ and $C'$ are equal cubic curves. This proves (2). In the Weierstrass form, the only possible linear transformations are $x \mapsto tx$ and $y \mapsto \pm t^{3/2}y$, which proves (1).

A similar argument proves (3). Notice that in this case $\text{Aut}(C, P)$ modulo the hyperelliptic involution acts on $\mathbb{P}^1$ by permuting branch points. In fact, $\lambda$ is simply the cross-ratio:

$$\lambda = \frac{(p_4 - p_1)(p_2 - p_3)}{(p_2 - p_1)(p_4 - p_3)},$$

but branch points are not ordered, so we have an action of $S_4$ on possible cross-ratios. However, it is easy to see that the Klein’s four-group $V$ does not change the cross-ratio. The quotient $S_4/V \simeq S_3$ acts non-trivially:

$$\lambda \mapsto \{\lambda, 1/\lambda, (\lambda - 1)/\lambda, \lambda/(\lambda - 1), 1/(1 - \lambda)\} \quad (3.2.5)$$

Special values of $\lambda$ correspond to cases when some of the numbers in this list are equal. For example, $\lambda = 1/\lambda$ implies $\lambda = -1$ and the list of possible cross-ratios boils down to $-1, 2, 1/2$ and $\lambda = 1/(1 - \lambda)$ implies $\lambda = e^{\pi i}$, in which case the only possible cross-ratios are $e^{\pi i}$ and $e^{-\pi i}$. To work out the actual automorphism group, we look at the Weierstrass models. For example, if $\lambda = -1$ then the branch points are $0, 1, -1, \infty$ and the equation is $y^2 = x^3 + x$. One can make a change of variables $x \mapsto ix, y \mapsto e^{3\pi i/4}$, then the equation becomes

$$y^2 = x^3 + x$$

as required and the branch points now are $0, i, -i, \infty$. An automorphism of $\mathbb{P}^1$ permuting these branch points is $x \mapsto -x$. To keep the curve in the Weierstrass form, we also have to adjust $y \mapsto iy$. This gives an automorphism of $C$ and its square is an automorphism $x \mapsto x, y \mapsto -y$, i.e. an involution permuting branches of the double cover.

(4) Consider an isomorphism $f : \mathbb{C}/\Lambda' \rightarrow \mathbb{C}/\Lambda$. Composing it with translations on the source and on the target, we can assume that $f(0 + \Lambda') = 0 + \Lambda$. Then $f$ induces a holomorphic map $\mathbb{C} \rightarrow \mathbb{C}/\Lambda$ with kernel $\Lambda'$, and its lift to the universal cover gives an isomorphism $F : \mathbb{C} \rightarrow \mathbb{C}$ such that $F(\Lambda') = \Lambda$. But it is proved in complex analysis that all automorphisms of $\mathbb{C}$ preserving the origin are maps $z \mapsto az$ for $a \in \mathbb{C}^*$. So we have

$$\mathbb{Z} + \mathbb{Z}r = a(\mathbb{Z} + \mathbb{Z}r'),$$

which gives

$$a r' = a + b r, \quad a = c + d r,$$

which gives (3.2.4). \qed
3.2.6. THEOREM. We can define the $j$-invariant by any of the two formulas:

\[ j = 1728g_3^2/\Delta = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \] (3.2.7)

The $j$-invariant uniquely determines an isomorphism class of an elliptic curve. The special values of the $j$-invariant are $j = 0$ (for $\mathbb{Z}/6\mathbb{Z}$) and $j = 1728$ (for $\mathbb{Z}/4\mathbb{Z}$).

Proof. It is easy to see that the expression $256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$ does not change under the transformations (3.2.5). Thus, for fixed $j$, the polynomial

\[ 256(\lambda^2 - \lambda + 1)^3 - j\lambda^2(\lambda - 1)^2 \]

has 6 roots related by transformations (3.2.5). So the $j$-invariant uniquely determines an isomorphism class of an elliptic curve. The rest is left to the homework exercises. \[ \square \]

§3.3. Monstrous Moonshine. Trying to compute the $j$-invariant in terms of the lattice parameter $\tau$ produces some amazing mathematics. Notice that $j(\tau)$ is invariant under the action of $\text{PSL}_2(\mathbb{Z})$ on $\mathcal{H}$. This group is called the modular group. It is generated by two transformations,

\[ S : z \mapsto -1/z \quad \text{and} \quad T : z \mapsto z + 1 \]

It has a fundamental domain (see the figure). The $j$-invariant maps the fundamental domain to the plane $\mathbb{A}^1$. 

FIGURE 1. Modular group.
Since the \( j \)-invariant is invariant under \( z \mapsto z + 1 \), it can be expanded in a variable \( q = e^{2\pi i \tau} \):

\[
j = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots
\]

According to the classification of finite simple groups, there are a few infinite families (like alternating groups \( A_n \)) and several sporadic groups. The largest sporadic group \( F_1 \) is called the Monster. It has about \( 10^{54} \) elements. Its existence was predicted by Robert Griess and Bernd Fischer in 1973 and it was eventually constructed by Griess in 1980 as the automorphism group of a certain (commutative, non-associative) algebra of dimension 196884. In other words, the Monster has a natural \( 196884 \)-dimensional representation, just like \( S_n \) has a natural \( n \)-dimensional representation. This dimension appears as one of the coefficients of \( j(q) \) and in fact all coefficients in this \( q \)-expansion are related to representations of the Monster group. This is a *Monstrous Moonshine Conjecture* of McKay, Conway, and Norton proved in 1992 by Borcherds (who won a Fields medal for this work).

§3.4. *Families of elliptic curves.* We want to upgrade \( \mathcal{M}_g \) and \( \mathcal{M}_{g,n} \) to moduli functors. What is a family of smooth projective curves? It should be a morphism \( f : X \to B \) such that every fiber of \( f \) is a smooth projective curve. We have to impose a technical condition: the morphism \( f \) must be *smooth* and *proper*. We postpone definitions until later and focus on the following good properties, which will allow us to define the moduli functor.

3.4.1. **Theorem.**

- If \( f : X \to B \) is a smooth morphism of algebraic varieties and \( B \) is non-singular then \( X \) is also non-singular. Moreover, the induced map of analytifications \( X^{an} \to B^{an} \) is a submersion of complex manifolds, i.e. its differential is surjective at every point.
- Let \( f : X \to B \) be a morphism of non-singular algebraic varieties such that the corresponding map of analytifications is a submersion of complex manifolds. Then \( f \) is a smooth morphism.
- Let \( f : X \to B \) be a smooth morphism and let \( g : B' \to B \) be any morphism of algebraic varieties. Consider the fiber product

\[
X \times_B B' = \{(x, b') \mid f(x) = g(b')\} \subset X \times B'.
\]

Then \( X \times_B B' \to B' \) is a smooth morphism of algebraic varieties. In particular, every fiber of \( f \) is non-singular.

3.4.2. **Theorem.** Let \( f : X \to B \) be a morphism of non-singular algebraic varieties. It is proper iff the corresponding map of analytifications \( X^{an} \to B^{an} \) is a proper holomorphic map, i.e. the preimage of every compact set is compact.

3.4.3. **Definition.** A family of smooth projective curves of genus \( g \) with \( n \) marked points is a smooth proper morphism of algebraic varieties \( f : X \to B \) such that all fibers are smooth projective curves of genus \( g \). In addition, we need \( n \) disjoint sections, i.e. morphisms \( s_1, \ldots, s_n : B \to X \) such that \( f \circ s_i = \text{Id}_B \) for every \( i \) and such that \( s_i(b) \neq s_j(b) \) for every \( b \in B \) and \( i \neq j \).

The moduli functor

\[
\mathcal{M}_{g,n} : \text{AlgebraicVarieties} \to \text{Sets}
\]
sends every algebraic variety $B$ to the set of isomorphism classes of families $f : X \to B$ of smooth projective curves of genus $g$ with $n$ marked points and every morphism $B' \to B$ a pull-back function $\mathcal{M}_{g,n}(B) \to \mathcal{M}_{g,n}(B')$, $X \mapsto X \times B'$. (Try to define the pullback of sections $s_1, \ldots, s_n$ yourself.)

3.4.4. REMARK. A family of smooth projective curves of genus 1 with a marked point is also called an elliptic fibration.

It would be nice to have a better structure theory of elliptic fibrations. We will later show the following:

3.4.5. THEOREM. A morphism $\pi : X \to B$ with a section $\sigma : B \to X$ is an elliptic fibration if and only if every point $b \in B$ has an affine neighborhood $U = \text{Spec} \ R$ such that $\pi^{-1}(U)$ is isomorphic to a subvariety of $U \times \mathbb{P}^2_{[x:y:z]}$ given by the Weierstrass equation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3,$$

where $g_2, g_3 \in R = \mathcal{O}(U)$ are regular functions such that $\Delta = g_2^3 - 27 g_3^2 \in R^* \text{ is invertible. Moreover, } g_2 \text{ and } g_3 \text{ are defined uniquely up to transformations}
\begin{align*}
g_2 &\mapsto t^4 g_2, \\
g_3 &\mapsto t^6 g_3
\end{align*}$

for some invertible $t \in R^*$.  

3.4.7. REMARK. Dependence on $t$ comes from the following basic observation: multiplying $y$ by $t^3$ and $x$ by $t^2$ will induce transformation (3.4.6). Notice that if $t \in \mathcal{O}^*(pt) = \mathbb{C}^*$ then we can take a square root and multiply by it instead. This gives Theorem 3.2.3 (1). But if $t$ is a non-constant regular function then the square root of course doesn’t have to exist. For example, take $t$ to be a coordinate in $\mathbb{A}^1 \setminus \{0\}$.

§3.5. The $j$-line is a coarse moduli space.

3.5.1. THEOREM. The $j$-line $\mathbb{A}^1$ is a coarse moduli space for $\mathcal{M}_{1,1}$.

Proof. Suppose we have an elliptic fibration $\pi : X \to B$. Then we have a function $j_B : B \to \mathbb{A}^1$ which sends every $b \in B$ to the $j$-invariant of $\pi^{-1}(b)$.

We have to show that this function is a morphism of algebraic varieties. This can be checked in affine charts on $B$, and thus by Theorem 3.4.5 we can assume that the fiber is in the Weierstrass normal form. But then $j_B$ can be computed by the usual formula (3.2.7). Since $g_2$ and $g_3$ are regular functions in the chart, $j_B$ is a regular function as well.

A tricky part is to check the last condition in the definition of a coarse moduli space. Suppose we have another variety $Z$ and a natural transformation $\mathcal{M}_{1,1} \to i_Z$ such that points of $Z$ are in 1-1 correspondence with isomorphism classes of elliptic curves. Every elliptic fibration $\pi : X \to B$ then gives a regular morphism $j_Z : B \to Z$. We want to show that it factors through a morphism $\mathbb{A}^1 \to Z$. Let $I \subset Z \times \mathbb{A}^1$ be the locus of pairs corresponding to curves with the same $j$-invariant. It is a bijective correspondence between $Z$ and $\mathbb{A}^1$. Suppose we know that $I$ is closed. Then both projections $I \to Z$ and $I \to \mathbb{A}^1$ are bijective maps of algebraic varieties, and therefore normalization maps. But $\mathbb{A}^1$ is a normal variety, and therefore the second projection is an isomorphism. Thus $I$ is a graph of a morphism $\mathbb{A}^1 \to Z$. 
It remains to show that $I$ is closed. It is easy to construct an elliptic fibration $X \to B$ over a smooth algebraic curve such that every elliptic curve appears as one of the fibers. Just take $B = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and define $X \subset B \times \mathbb{P}^2_{[x:y:z]}$ by the Weierstrass equation
\[ y^2z = x(x-z)(x-\lambda z). \]

The $j$-invariant in this case is the map
\[ \mathbb{P}^1 \setminus \{0, 1, \infty\} = \text{Spec} \mathbb{C} \left[ \lambda, \frac{1}{\lambda}, \frac{1}{\lambda - 1} \right] \to \mathbb{A}^1_j \]
given by (3.2.7). Notice that this morphism is finite: $\lambda$ is a root of a monic polynomial with coefficients in $\mathbb{C}[j]$. Therefore $j_Z \times j : B \to Z \times \mathbb{A}^1_j$ is also finite (see the exercises) and therefore its image is closed.

\[ \square \]

§3.6. The $j$-line is not a fine moduli space.

3.6.1. PROPOSITION. $\mathcal{M}_{1,1}$ has no fine moduli space.

Proof. Indeed, if it has one then then it should be $\mathbb{A}^1_j$, because a fine moduli space is also automatically a coarse one. Thus the natural transformation $\mathcal{M}_{1,1} \to h_{\mathbb{A}^1}$ should be an isomorphism, i.e.
\[ \eta_B : \mathcal{M}_{1,1}(B) \to \text{Mor}(B, \mathbb{A}^1) \]
should be a bijection for every $B$. This fails in multiple ways.

Let $B = \text{MaxSpec } R$, where $R$ contains an element $s$ without a square root (for example $R = \mathbb{C}[s]$). Consider any elliptic fibration $f : E \to B$ with Weierstrass equation
\[ y^2 = 4x^3 - g_2x - g_3 \]
and let $E' \to B$ be a “twisted” fibration with Weierstrass equation
\[ y^2 = 4x^3 - s^2g_2x - s^3g_3. \]

This fibrations have isomorphic fibers (over every $b \in B$) hence give the same maps $B \to \mathbb{A}^1_j$. However, they are not isomorphic. If they were, we would have
\[ s^2 = t^4, \quad s^3 = t^6 \]
for some $t \in R$ by Theorem 3.4.5. Thus $s = t^2$, a contradiction. It follows that $\eta_B$ is not injective. For example, $\eta_{\mathbb{A}^1}$ is not injective.

In fact, $\eta_{\mathbb{A}^1}$ is also not surjective. Indeed, the preimage of the identity map would be a universal elliptic fibration, i.e. a fibration over $\mathbb{A}^1_j$ such that the $j$-invariant of the fiber over $j$ is $j$. Theorem 3.4.5 will be applicable this fibration and so locally at every point $P \in \mathbb{A}^1$ we would have
\[ j = 1728 \frac{g_2^3(j)}{g_2^2(j) - 27g_3^2(j)} \]
for some rational functions $g_2$ and $g_3$ defined near $P$. Taking $P = 0$ gives a contradiction because $j$ has zero of order 1, whereas the order of zeros of the RHS at 0 is divisible by 3!
Taking $P = 1728$ gives another contradiction because

$$j - 1728 = 1728 \frac{27g_3^2(j)}{g_2^3(j) - 27g_3^3(j)}.$$ 

The LHS has simple zero at 1728 but the RHS has even multiplicity. □
§3.7. Homework 2. Due March 24.

Name:
Don’t write anything on the problem sheet and attach it to your written solutions. I will use it to record problems presented during office hours.

**Problem 1.** (1 point) Show that a coarse moduli space (of any moduli functor) is unique (if exists) up to an isomorphism. Show that a fine moduli space is always also a coarse moduli space.

**Problem 2.** (2 points) Compute $j$-invariants of elliptic curves

- (a) $y^2 + y = x^3 + x$
- (b) $y^2 = x^4 + bx^3 + cx$

**Problem 3.** (1 point) Show that every elliptic curve is isomorphic to a curve of the form $y^2 = (1 - x^2)(1 - e^2x^2)$.

**Problem 4.** (2 points) Show that the two formulas in (3.2.7) agree.

**Problem 5.** (2 points)

(a) Compute the $j$-invariant of an elliptic curve $y^2 + xy = x^3 - 36q - 1728x - 1q - 1728$, where $q$ is some parameter.

(b) Show that $\mathbb{A}^1 \setminus \{0, 1728\}$ carries a family of elliptic curves with $j$-invariant $j$.

**Problem 6.** (1 point) In Definition 3.4.3, explain how to pullback sections.

**Problem 7.** (2 points) Let $f$ be a rational function on an algebraic curve $C$ such that all zeros of $f$ have multiplicity divisible by 3 and all zeros of $f - 1728$ have multiplicities divisible by 2. Show that $C \setminus \{f = \infty\}$ carries an elliptic fibration with $j$-invariant $f$.

**Problem 8.** (3 points) Let $(C, P)$ be an elliptic curve. (a) By considering a linear system $\phi|_{4P}$, show that $C$ embeds in $\mathbb{P}^3$ as a curve of degree 4.

(b) Show that quadrics in $\mathbb{P}^3$ containing $C$ form a pencil $\mathbb{P}^1$ with 4 singular fibers. (c) These four singular fibers define 4 points in $\mathbb{P}^1$. Relate their cross-ratio to the $j$-invariant of $C$.

**Problem 9.** (2 points) Let $E$ be an elliptic curve and consider the trivial family $\mathbb{P}^1 \times E$ over $\mathbb{P}^1$. Now take two copies of this algebraic surface and glue them along $\{0\} \times E$ by identifying $E$ with $E$ by identity and along $\{\infty\} \times E$ by identifying $E$ with $E$ via a non-trivial involution. This gives an elliptic fibration over a reducible curve obtained by gluing two copies of $\mathbb{P}^1$ along 0 and $\infty$. Show that all elliptic curves in this family are isomorphic but the family is not trivial.

**Problem 10.** (3 points) Let $(C, P)$ be an elliptic curve. Let $\Gamma \subset C$ be the ramification locus of $\phi|_{2P}$. (a) Show that $\Gamma \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is precisely the 2-torsion subgroup in the group structure on $C$. (b) A level 2 structure on $(C, P)$ is a choice of an ordered basis $\{Q_1, Q_2\} \in \Gamma$ (considered as a $\mathbb{Z}_2$-vector space). Based on Theorem 3.4.5, describe families of elliptic curves with level 2 structure. Define a moduli functor of elliptic curves with a level 2 structure. Show that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ carries a family of elliptic curves with a level 2 structure such that every curve with a level 2 structure appears (uniquely) as one of the fibers. Is this family universal?
Problem 11. (2 points) Consider the family of cubic curves
\[ C_a = \{ x^3 + y^3 + z^3 + axyz = 0 \} \subset \mathbb{P}^2 \]
parametrized by \( a \in \mathbb{A}^1 \). (a) Find all \( a \) such that \( C_a \) is smooth and find its inflection points. (b) Compute \( j \) as a function on \( a \) and find all \( a \) such that \( C_a \) has a special automorphism group.

Problem 12. (3 points) Let \((C, P)\) be an elliptic curve equipped with a map \( C \to C \) of degree 2. By analyzing the branch locus \( \phi_2P \), show that the \( j \)-invariant of \( C \) has only 3 possible values and find these values.

Problem 13. (3 points) Let \( \pi : X \to Y \) be a morphism of algebraic varieties. Recall that \( \pi \) is called finite if \( Y \) can be covered by affine open sets \( Y = \bigcup_i U_i \) such that every \( V_i := \pi^{-1}(U_i) \) is affine and \( \mathcal{O}(V_i) \) is a finitely generated \( \mathcal{O}(U_i) \) module. Equivalently, \( \mathcal{O}(V_i) \) is generated (as a \( \mathbb{C} \)-algebra) by finitely many elements which are roots of monic polynomials with coefficients in \( \mathcal{O}(U_i) \). (a) Prove the “equivalently” part. (b) Prove that every finite morphism has finite fibers. (c) Prove that every finite morphism is surjective. (d) Not every surjective morphism with finite fibers is finite.

Problem 14. (2 points) Let \( X \) be an algebraic variety. Recall that it is called normal if it can be covered by affine open sets \( X = \bigcup_i U_i \) such that every \( \mathcal{O}(U_i) \) is integrally closed in its field of fractions. Show that \( X \) is normal if and only if every birational finite map \( Y \to X \) is an isomorphism.

Problem 15. (2 points) Let \( f : X \to Y \) be a finite morphism of algebraic varieties and let \( g : X \to Z \) be an arbitrary morphism. Show that \( f \times g : X \to Y \times Z \) is a finite morphism of algebraic varieties.

Problem 16. (2 points) Let \( f : X \to Y \) be a non-constant morphism of smooth projective curves. Show that \( f \) is finite.

Problem 17. (2 points) Let \( f : X \to Y \) be a finite map of algebraic varieties and suppose that \( Y \) is normal. Then every fiber \( f^{-1}(y) \) has at most \( \deg f \) points, where \( \deg f \) is defined as the degree of the field extension \( [\mathbb{C}(X) : \mathbb{C}(Y)] \). Moreover, let \( U \subset Y \) be a subset of points \( y \in Y \) such that \( f^{-1}(y) \) has exactly \( \deg f \) points. Then \( U \) is open and non-empty.
§4. Families of algebraic varieties

Our next goal is to sketch the proof of Theorem 3.4.5 following [MS]. In order to do that, we will have to introduce a more advanced viewpoint on families of algebraic varieties. Recall that our goal is to describe explicitly any smooth proper morphism \( \pi : X \to B \) with a section \( \sigma \) such that all of the fibers are elliptic curves. For starters, we can shrink \( B \) and assume that it is an affine variety \( \text{MaxSpec } R \). We can think about \( X \) as “an elliptic curve over the ring \( R \)” generalizing “an elliptic curve \( E \) over the field \( \mathbb{C} \).” Eventually we will write a Weierstrass equation for it with coefficients in \( R \).

We adopt the same strategy as in the proof of Theorem 3.1.1, namely the Riemann–Roch analysis of linear systems \( L(kP) \) on \( E \). But we will have to upgrade our technology so that we can work with a family over the base \( B \) and with a section \( A = \sigma(B) \subset X \) instead of a single point \( P \in E \).

We will go back and forth between introducing general techniques and filling the gaps in the proof of Theorem 3.4.5.

§4.1. Short exact sequence associated with a subvariety. Let \( X \) be an algebraic variety and let \( Z \subset X \) be a subvariety. Algebraically, it is given by a sheaf of (radical) ideals \( I_Z \subset \mathcal{O}_X \). Namely,

\[
I_Z(U) = \{ f \in \mathcal{O}_X(U) \mid f|_{Z \cap U} = 0 \} \subset \mathcal{O}_X(U)
\]

for every open subset \( U \subset X \). These sheaves sit in the following very useful short exact sequence of sheaves

\[
0 \to I_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0, \tag{1}
\]

where \( i : Z \hookrightarrow X \) is the inclusion map and \( i_* \) is the push-forward of sheaves\(^{10}\). If \( U = \text{MaxSpec } R \) is an affine open then \( I = I_Z(U) \) is an ideal of \( R \) and \( R/I \) is a coordinate ring of \( Z \cap U \). Taking sections of sheaves in (1) over \( U \) gives a short exact sequence

\[
0 \to I \to R \to R/I \to 0.
\]

For example, if \( P \in C \) is a point of a smooth curve then (1) becomes

\[
0 \to \mathcal{I}_P \to \mathcal{O}_C \to \mathbb{C}_P \to 0, \tag{2}
\]

where \( \mathbb{C}_P \) is the skyscraper sheaf of the point \( P \).

§4.2. Cartier divisors and invertible sheaves.

4.2.1. Definition. A Cartier divisor \( D \) on an algebraic variety \( X \) is an (equivalence class of) data

\[
(U_\alpha, f_\alpha), \quad \alpha \in I,
\]

where \( X = \bigcup U_\alpha \) is an open covering and \( f_\alpha \in \mathbb{C}(X) \) are non-zero rational functions such that \( f_\alpha/f_\beta \) is an invertible function on each overlap \( U_\alpha \cap U_\beta \). A Cartier divisor is called effective if \( f_\alpha \in \mathcal{O}_X(U_\alpha) \) for every \( \alpha \).

We can think about a Cartier divisor \( D \) as a divisor given by equation \( f_\alpha = 0 \) in each chart \( U_\alpha \). The fact that \( f_\alpha/f_\beta \) is invertible on every overlap makes this consistent. More precisely, we have the following definition:

\[^{10}\text{Recall that if } X \to Y \text{ is a continuous map of topological spaces and } \mathcal{F} \text{ is a sheaf on } X \text{ then the pushforward } f_*\mathcal{F} \text{ has sections } f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \text{ for every open set } U \subset Y.\]
4.2.2. Definition. Suppose the singular locus of $X$ has codimension at least 2$^{11}$. Then an associated divisor is
\[
\sum \text{ord}_H(f_\alpha)[H],
\]
the summation over all prime divisors (=irreducible hypersurfaces) $H \subset X$ and $\text{ord}_H(f_\alpha)$ is the order of zeros–poles along $H$. It is defined using any $f_\alpha$ such that $H \cap U_\alpha \neq \emptyset$. The fact that $f_\alpha/f_\beta$ is invertible on $U_\alpha \cap U_\beta$ makes this definition independent of $\alpha$. The sum of course turns out to be finite.

In general, not every prime divisor is Cartier: locally the former correspond to prime ideals of height 1 and the latter to locally principal ideals.

4.2.3. Example. Suppose $n_1P_1 + \ldots + n_rP_r$ is a divisor on a curve $C$. Choose a covering $C = U_0 \cup U_1 \cup \ldots \cup U_r$, where $U_0 = C \setminus \{P_1, \ldots, P_r\}$ and $U_i$ is defined as follows: choose a local parameter $g_i$ for $P_i$ and define $U_i$ to be $C$ with removed points $P_j, j \neq i$ and removed zeros and poles of $g_i$ except for its simple zero at $P_i$.$^{13}$ Finally, define $f_i = g_i^{n_i}$.

A Cartier divisor $D$ has an associated line bundle $X = L_D$, which has trivializing atlas $\cup U_\alpha$ and transition functions $f_\alpha/f_\beta$. In algebraic geometry we often bypass this line bundle and work with its sheaf of sections, denoted by $\mathcal{O}_X(D)$. One can define it directly: for every open subset $U \subset X$,
\[
\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X) \mid ff_\alpha \in \mathcal{O}_X(U \cap U_\alpha)\}.
\]
The sheaf $\mathcal{O}_X(D)$ is invertible, i.e. every point of $X$ has a neighborhood $U$ such that $\mathcal{O}_X(D)|_U \cong \mathcal{O}_U$. Namely, for every $\alpha$,
\[
\mathcal{O}_X(D)|_{U_\alpha} = \frac{1}{f_\alpha} \mathcal{O}_X|_{U_\alpha}.
\]

4.2.4. Example. One very useful application is a special case of the exact sequence (1) when $Z = D$ is a prime divisor, which happens to be Cartier:
\[
0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.
\]
For example, the exact sequence (2) can be rewritten as
\[
0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathbb{C}_P \to 0.
\]

4.2.5. Example. The linear system $\mathcal{L}(D)$ in this language is the space of global sections:
\[
\mathcal{L}(D) = \Gamma(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D)).
\]

---

$^{11}$Recall that this is always the case if $X$ is normal.

$^{12}$A difficult step of the proof is that non-singular varieties are locally factorial, i.e. every local ring is a UFD. It is easy to show that on locally factorial varieties all divisors are Cartier.

$^{13}$In the analytic category one can take $U_i$ to be a small neighborhood of $P_i$ and $g_i$ a corresponding local coordinate.
§4.3. **Morphisms with a section.** A section is always a subvariety:

4.3.1. **Lemma.** Suppose a morphism of algebraic varieties $\pi : X \to B$ has a section $\sigma : B \to X$. Then $A = \sigma(B)$ is a closed subvariety isomorphic to $B$.

**Proof.** It suffices to show that $A$ is closed because then $\pi|_{\sigma(B)}$ and $\sigma$ give a required isomorphism. Arguing by contradiction, take $x \in \bar{A} \setminus A$. Let $b = \pi(x)$. Then $y = \sigma(b) \neq x$. Since $X$ is quasi-projective, there exists an open subset $U$ of $X$, which contains both $x$ and $y$, and a function $f \in \mathcal{O}(U)$ such that $f(x) \neq f(y)$. But this gives a contradiction: $\sigma^*f$ must be regular in a neighborhood of $b$, but then $(\sigma^*f)(b)$ is equal to both $f(x)$ and $f(y)$.

§4.4. **Morphisms with reduced fibers.** Let $\pi : X \to B$ be a morphism of algebraic varieties, $b \in B$. What are the equations of the fiber $X_b = \pi^{-1}(b)$? Notice that $x \in X_b$ if and only if $\pi^*(f(x)) = 0$ for every function $f$ on $B$ regular and vanishing at $b$. Therefore the ideal $\mathcal{I}_{X_b,x} \subset \mathcal{O}_{X,x}$ of the fiber is equal to the radical of the ideal $I \subset \mathcal{O}_{X,x}$ generated by $\pi^*(\mathfrak{m}_{B,b})$.

4.4.1. **Definition.** The fiber $X_b$ is reduced at $x \in X_b$ if $I$ is a radical ideal. We say that $X_b$ is a reduced fiber if it is reduced at every point.

4.4.2. **Example.** Let $f : C \to D$ be a non-constant morphism of algebraic curves. For every point of a curve, its maximal ideal in the local ring is generated by a local parameter at that point. Thus the fiber $f^{-1}(y)$ is reduced at $x$ if and only if $f$ is unramified at $x$.

4.4.3. **Lemma.** Under assumptions of Lemma 4.3.1, suppose in addition that every fiber $X_b$ is reduced. Let $x \in A$, $b = \pi(x)$. Then the maximal ideal $\mathfrak{m}_{X_b,x}$ of $x$ in the fiber is generated by the image of the ideal $\mathcal{I}_{A,x}$ of the section.

**Proof.** It is clear that the ideal $\mathcal{I}_{A,x}$ restricts to the ideal $J \subset \mathcal{O}_{X_b,x}$ which is contained in $\mathfrak{m}_{X,b,x}$. So it suffices to prove that $\mathcal{O}_{X_b,x}/J = \mathbb{C}$. Since the fiber is reduced, we have $\mathcal{O}_{X_b,x} = \mathcal{O}_{X,x}/\pi^*(\mathfrak{m}_{B,b})$. Thus it suffices to show that $\mathcal{O}_{X,x}/(\pi^*(\mathfrak{m}_{B,b}) + \mathcal{I}_{A,x}) = \mathbb{C}$.

And indeed, doing factorization in a different order gives

$$\mathcal{O}_{X,x}/(\mathcal{I}_{A,x} + \pi^*(\mathfrak{m}_{B,b})) = \mathcal{O}_{A,x}/\pi^*(\mathfrak{m}_{B,b}) \simeq \mathcal{O}_{B,b}/\mathfrak{m}_{B,b} = \mathbb{C}.$$  

This proves the Lemma.

If all fibers of $\pi$ are curves then

$$\dim A = \dim B = \dim X - 1,$$

i.e. the section $A$ is a divisor. We claim that it is a Cartier divisor if $\pi$ has reduced fibers and all of them are smooth curves.

---

14The proof reflects the fact that every quasi-projective algebraic variety is separated. Moreover, an algebraic variety is separated whenever any two points $x, y \in X$ are contained in an open subset $U$ which admits a regular function $f \in \mathcal{O}(U)$ separating $x$ and $y$, i.e. such that $f(x) \neq f(y)$. It is easy to construct an unseparated variety by gluing affine open varieties. A simple example is an affine line with two origins $X$ obtained by gluing two copies of $\mathbb{A}^1$ along an open subset $\{x \neq 0\}$ using the identity map. The two origins cannot be separated by a function. Notice by the way that a projection map $X \to \mathbb{A}^1$ has a section with the non-closed image! In the category of manifolds, the “separation of points by a continuous function” property is equivalent to Hausdorffness.
4.4.4. **Lemma.** Let $\pi : X \to B$ be a morphism with a section $\sigma$, reduced fibers, and such that all fibers are smooth curves. Then $A = \sigma(B)$ is a Cartier divisor.

**Proof.** We have to show that every point $x = \sigma(b) \in A$ has an affine neighborhood $U$ such that $\mathcal{I}_A(U)$ is a principal ideal. It suffices to prove that its localization, $I := \mathcal{I}_{A,x} \subset \mathcal{O}_{X,x}$ is a principal ideal. Let $J := \mathcal{I}_{B,x} \subset \mathcal{O}_{X,x}$ be the vanishing ideal of the fiber.

By Lemma 4.4.3, the maximal ideal $m_{B,x}$ of $x$ in the fiber is generated by the restriction of $I$. Since the fiber is a non-singular curve, $\mathcal{O}_{X,x}$ is a DVR and $m_{X,x}$ is generated by one element $z$ (a uniformizer). Choose $f \in I$ that restricts to $z \in m_{X,x}$.

We claim that $I = (f)$, or equivalently $I/(f) = 0$. Since this is a finitely generated module of a local ring $\mathcal{O}_{X,x}$, Nakayama’s lemma applies, and it suffices to show that

$$ I/(f) = m_{X,x}I/(f), $$

i.e. that

$$ I = (f) + m_{X,x}I. $$

Choose $\alpha \in I$ and choose a multiple $gf$ which has the same restriction to the fiber as $\alpha$. Then $\alpha - gf$ restricts trivially, i.e. belongs to $J$. Thus it suffices to prove that $I \cap J \subset m_{X,x}I$. In fact we claim that

$$ I \cap J = IJ. $$

The pull-back by $\pi^*$ and by $\sigma^*$ give an isomorphism of $\mathcal{O}_{B,b}$ modules

$$ \mathcal{O}_{X,x} = \pi^* \mathcal{O}_{B,b} + I. $$

Since the fiber $X_b$ is reduced, we have

$$ J = \mathcal{O}_{X,x} \pi^* m_{B,b} = \pi^* m_{B,b} + I \pi^* m_{B,b} = \pi^* m_{B,b} + IJ. $$

Thus if $\alpha \in I \cap J$ then, modulo $IJ$, we can assume that $\alpha = \pi^* \beta$. But then

$$ \beta = \pi^* \beta = \pi^* \alpha = 0 $$

because $\alpha$ vanishes along the section. Thus $\alpha = \pi^* \beta = 0$ as well. □

§4.5. **Flat and smooth morphisms.** Let’s finally define smooth morphisms.

4.5.1. **Definition.** A morphism $f : X \to Y$ of algebraic varieties is called **flat** if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module for every point $x \in X$ and $y = f(x)$.

4.5.2. **Definition.** A morphism $f : X \to Y$ of algebraic varieties is called **smooth** if it is flat, has reduced fibers, and every fiber is non-singular.

Let $X \to B$ be an elliptic fibration, i.e. a smooth proper morphism with a section $A = \sigma(B)$ such that all fibers are elliptic curves. We would like to write down Weierstrass equation of $X$, possibly after shrinking the base $B$. If $B = pt$, the argument will be identical to what we have seen so far, using linear systems

$$ \mathcal{L}(E, kP) = H^0(E, \mathcal{O}_E(kP)). $$

Recall that if $\mathcal{F}$ is a sheaf of Abelian groups on an algebraic variety $X$ then we can define higher cohomology groups $H^k(X, \mathcal{F})$ in addition to the group $H^0(X, \mathcal{F})$ of global sections. There are several important facts to know about these groups.
4.5.3. **Theorem.** For every short exact sequence of sheaves
\[ 0 \to F \to F' \to F'' \to 0, \]
we have a long exact sequence of cohomology groups
\[ \ldots \to H^k(X, F) \to H^k(X, F') \to H^k(X, F'') \to H^{k+1}(X, F) \to \ldots, \]
functorial with respect to commutative diagrams of short exact sequences
\[ \begin{array}{ccc}
0 & \longrightarrow & F \\
\downarrow & & \downarrow \\
0 & \longrightarrow & G
\end{array} \quad \begin{array}{ccc}
0 & \longrightarrow & F' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & G'
\end{array} \quad \begin{array}{ccc}
0 & \longrightarrow & F'' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & G''
\end{array} \]

4.5.4. **Definition.** A sheaf $F$ on an algebraic variety $X$ is called *locally free* of rank $r$ if every point has a neighborhood $U$ such that $F|_U \cong \mathcal{O}_U^\oplus r$. Equivalently, $F$ is a sheaf of sections of some vector bundle $\pi : F \to X$, i.e.
\[ F(U) = \{ s : X \to F \mid \pi \circ s = \text{Id}_X \}. \]

For example, a locally free sheaf of rank 1 is the same thing as an invertible sheaf, which is the same thing as a sheaf of sections of a line bundle.

4.5.5. **Theorem.** Let $X$ be a projective algebraic variety and let $F$ be a locally free sheaf on $X$. Then all cohomology groups $H^k(X, F)$ are finite-dimensional vector spaces which vanish for $k > \dim X$. Their dimensions are denoted by $h^k(X, F)$.

If $X$ is a smooth projective curve, then one also has

4.5.6. **Theorem (Serre duality).** Let $D$ be any divisor on $X$ and let $K$ be the canonical divisor. Then we have duality of vector spaces
\[ H^i(X, \mathcal{O}_X(D)) \cong H^{1-i}(X, \mathcal{O}_X(K-D))^*. \]
In particular,
\[ i(D) = \dim K(D) = h^0(K-D) = h^1(D) \]
and we can rewrite Riemann–Roch in a (less useful) form
\[ h^0(D) - h^1(D) = 1 - g + \deg D. \]

Recall that if $P$ is a point on an elliptic curve then $h^0(kP) = k$ and $h^1(kP) = 0$ for $k > 0$. This was the main observation for our analysis.

Now let $\pi : X \to B$ be an elliptic fibration with section $A = \sigma(B)$ and consider invertible sheaves $\mathcal{O}_X(kA)$ for $k \geq 0$. We would like to understand the space of global sections $H^0(X, \mathcal{O}_X(kA))$, but since we need a freedom to shrink $B$, it is better to study the push-forward $\pi_*\mathcal{O}_X(kA)$. Its sections over an open subset $U \subset B$ is the space of global sections
\[ H^0(\pi^{-1}(U), \mathcal{O}_X(kA)). \]

Recall that this is the space of rational functions on $X$ regular at any point $x \in \pi^{-1}(U) \setminus A$ and that can be written as $g/f^k$ at any point $x \in A$, where $g$ is regular and $f$ is a local defining equation of the Cartier divisor $A$.

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15More generally, one can reach the same conclusion if $F$ is a coherent sheaf.
16In fact much more generally but we won’t need that.
§4.6. Pushforwards and derived pushforwards. If \( \mathcal{F} \) is a sheaf of Abelian groups on an algebraic variety \( X \) and \( \pi : X \to B \) is a morphism then one can define derived pushforward sheaves \( R^k \pi_* \mathcal{F} \) on \( B \) in addition to the push-forward \( R^0 \pi_* \mathcal{F} := \pi_* \mathcal{F} \). When \( B \) is a point, \( \pi_* \mathcal{F} = H^0(X, \mathcal{F}) \) and \( R^k \pi_* \mathcal{F} = H^0(X, \mathcal{F}) \). One has an analogue of Theorem 4.5.3:

4.6.1. Theorem. For every short exact sequence of sheaves

\[
0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0,
\]

we have a long exact sequence of derived push-forwards

\[
\ldots \to R^k \pi_* \mathcal{F} \to R^k \pi_* \mathcal{F}' \to R^k \pi_* \mathcal{F}'' \to R^{k+1} \pi_* \mathcal{F} \to \ldots,
\]

functorial with respect to commutative diagrams of short exact sequences

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{G}
\end{array}
\quad \begin{array}{ccc}
0 & \longrightarrow & \mathcal{F}' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{G}'
\end{array}
\quad \begin{array}{ccc}
0 & \longrightarrow & \mathcal{F}'' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{G}''
\end{array}
\]

4.6.2. Remark. Suppose \( \mathcal{F} \) is a locally free sheaf on \( X \) and let \( \pi : X \to B \) be a morphism. Notice that \( \pi_* \mathcal{F} \) is not just a sheaf of Abelian groups but a sheaf of \( O_B \)-modules. Indeed, for every open subset \( U \subset B \),

\[
\pi_* \mathcal{F}(U) = \mathcal{F}(\pi^{-1}U)
\]

is an \( O_X(\pi^{-1}U) \)-module but we have a homomorphism of rings

\[
f^* : O_B(U) \to O_X(\pi^{-1}U)
\]

which makes its a \( O_B(U) \)-module as well. In particular, the stalk \( (\pi_* \mathcal{F})_b \) at \( b \in B \) is a \( O_{B,b} \)-module. Derived pushforwards are also sheaves of \( O_B \)-modules, generalizing the fact that cohomology groups are vector spaces.

Here’s the analogue of Theorem 4.5.5:

4.6.3. Theorem. Let \( \pi : X \to B \) be a proper morphism of algebraic varieties and let \( \mathcal{F} \) be a locally free\(^\text{17}\) sheaf on \( X \). Then all derived push-forwards \( R^k \pi_* \mathcal{F} \) are sheaves of finitely generated \( O_B \)-modules.

§4.7. Cohomology and base change.

4.7.1. Definition. Take any vector bundle \( E \to X \) and a subvariety \( Z \subset X \). The restriction \( E|_Z \) of \( E \) is a vector bundle on \( Z \). If \( \mathcal{F} \) is a (locally free) sheaf of sections of \( E \) then the sheaf of sections \( \mathcal{F}|_Z \) of \( E|_Z \) can be described as follows: if \( U \subset X \) is an affine open set then

\[
\mathcal{F}|_Z(U \cap Z) = \mathcal{F}(U)/\mathcal{I}_Z(A)\mathcal{F}(U).
\]

In other words, we have an exact sequence of sheaves on \( X \)

\[
0 \to \mathcal{I}_Z \otimes \mathcal{F} \to \mathcal{F} \to i_* \mathcal{F}_Z \to 0,
\]

obtained by tensoring the short exact sequence (1) with \( \mathcal{F} \).

\(^{17}\) Or, more generally, a coherent sheaf.
4.7.2. Example. Suppose \( \pi : X \to B \) is a morphism with a section \( A = \sigma(B) \) such that all fibers are reduced curves. We can apply the previous definition either to the fibers of \( \pi \) or to the section. The restriction of an invertible sheaf \( O_X(kA) \) (its local sections are rational functions on \( X \) with poles of order at most \( k \) along \( A \)) to every fiber is the sheaf

\[
\mathcal{O}_{X_b}(kA) = \mathcal{O}_{X_b}(kP)
\]

(its local sections are rational functions on \( X_b \) with poles of order at most \( k \) at \( P = \sigma(b) \).)

Another very useful short exact sequence for us will be the sequence (3) with \( Z = A \). It goes as follows:

\[
0 \to \mathcal{O}_X((k - 1)A) \to \mathcal{O}_X(kA) \xrightarrow{\psi} i_*\mathcal{O}_A(kA) \to 0.
\]

(4)

What is the meaning of the last map \( \psi \)? At any point \( x \in A \), a local section \( \alpha \) of \( \mathcal{O}_X(kA) \) looks like \( g/f^k \), where \( f \) is a local equation of \( A \). The local trivialization of \( \mathcal{O}_X(kA) \) identifies \( g/f^k \in \mathcal{O}_X(kA) \) with \( g \in \mathcal{O}_X \). Thus

\[
\psi(g/f^k) = g|_A.
\]

It is called the principal part of \( \alpha \).

4.7.3. Definition. We would like to compare \( \pi_*\mathcal{F} \), which is a sheaf on \( B \), with the vector space of global sections \( H^0(X_b, \mathcal{F}|_{X_b}) \). For every neighborhood \( b \in U \), we have a restriction homomorphism

\[
\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}U) \to H^0(X_b, \mathcal{F}|_{X_b}).
\]

These homomorphisms commute with further restrictions \( b \in V \subset U \), and therefore give a homomorphism from the stalk of the push-forward

\[
(\pi_*\mathcal{F})_b \to H^0(X_b, \mathcal{F}|_{X_b}).
\]

This stalk \( (\pi_*\mathcal{F})_b \) is a module over the local ring \( \mathcal{O}_{B,b} \). If \( f \in \mathcal{m}_{B,b} \) then every section in \( f(\pi_*\mathcal{F})_b \) restricts to \( X_b \) trivially. Thus we have a canonical homomorphism

\[
i^0_{b} : (\pi_*\mathcal{F})_b \otimes \mathbb{C} \to H^0(X_b, \mathcal{F}|_{X_b}),
\]

where \( \mathbb{C} \simeq \mathcal{O}_{B,b}/\mathcal{m}_{B,b} \). More generally, we have canonical homomorphisms

\[
i^k_b : (R^k\pi_*\mathcal{F})_b \otimes \mathbb{C} \to H^k(X_b, \mathcal{F}|_{X_b}).
\]

An ideal situation would be if \( R^k\pi_*\mathcal{F} \) were a locally free sheaf of sections of some vector bundle \( F^k \) \( B \). Then \( (R^k\pi_*\mathcal{F})_b \otimes \mathbb{C} \) would be identified with the fiber of \( F^k \) at \( b \in B \). The canonical homomorphism would give a linear map from the fiber of \( F^k \) to \( H^k(X_b, \mathcal{F}|_{X_b}) \). If that linear map were an isomorphism, we would be able to interpret \( R^k\pi_*\mathcal{F} \) as a sheaf of sections of a vector bundle with fibers given by cohomologies of fibers \( H^k(X_b, \mathcal{F}|_{X_b}) \). This is not always the case (for example, dimensions of these cohomology groups can jump in special fibers), but there is a powerful cohomology and base change theorem, which gives a necessary condition.

4.7.4. Theorem. Let \( \pi : X \to B \) be a proper flat morphism of algebraic varieties with reduced fibers and let \( \mathcal{F} \) be an invertible or locally free sheaf on \( X \).

1. If \( i^k_b \) is surjective for some \( k \) and \( b \in B \) then it is bijective.
(2) Suppose (1) is satisfied. Then $R^k \pi_* \mathcal{F}$ is locally free in a neighborhood of $b$ if and only if $i_b^{k-1}$ is surjective.

(3) If $H^{k+1}(X_b, \mathcal{F}|_{X_b}) = 0$ then $i_b^k$ is an isomorphism.

The following corollary is the most often used form of cohomology and base change:

4.7.5. COROLLARY. If $H^1(X_b, \mathcal{F}|_{X_b}) = 0$ for some $b \in B$ then $\pi_* \mathcal{F}$ is locally free in a neighborhood of $b$.

Proof. Apply parts (3) and then (2) of the theorem.

§4.8. Riemann-Roch analysis. Consider an elliptic fibration $\pi : X \to B$ with a section $A = \sigma(B)$. The key players will be invertible sheaves $\mathcal{O}_X(kA)$ for $k \geq 0$ and their push-forwards

\[ \mathcal{F}_k := \pi_* \mathcal{O}_X(kA). \]

4.8.1. LEMMA. $\mathcal{F}_k$ is a locally free sheaf of rank $k$ for every $k \geq 1$. Also,

\[ \mathcal{F}_0 = \pi_* \mathcal{O}_X \simeq \mathcal{O}_B \]

(a canonical isomorphism via pull-back $\pi^*$).

Proof. The first statement follows from Corollary 4.7.5.

To prove the second statement, note that we have an injection of sheaves $\pi^* : \mathcal{O}_B \to \pi_* \mathcal{O}_X$. It is also clear that $H^0(X_b, \mathcal{F}|_{X_b}) = \mathbb{C}$ for every $b \in B$. Thus $i_b^0$ is surjective for every $b \in B$ and we apply parts (1) and (2) of the cohomology and base change theorem.

Pushing an exact sequence (4) forward to $B$ gives a long exact sequence

\[ 0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \xrightarrow{\psi} \mathcal{L}^\otimes k \to R^1 \pi_* \mathcal{O}_X((k-1)A), \]

where

\[ \mathcal{L} := \pi_* i_* \mathcal{O}_A(A) \simeq \sigma^* \mathcal{O}_X(A) \]

is an invertible sheaf and $\psi$ is the principal parts map.

4.8.2. LEMMA. For $k \geq 2$, this gives a short exact sequence

\[ 0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \to \mathcal{L}^\otimes k \to 0 \]

For $k = 1$, we get is an isomorphism $\mathcal{F}_0 \simeq \mathcal{F}_1 \simeq \mathcal{O}_B$.

Proof. The first part follows from Theorem 4.7.4. For the second part, Theorem 4.7.4 shows that $R^1 \pi_* \mathcal{O}_X$ is an invertible sheaf, say $\mathcal{M}$. Thus we get a sequence

\[ 0 \to \mathcal{O}_B \to \mathcal{F}_1 \to \mathcal{L} \to \mathcal{M} \to 0. \]

It remains to show that $\delta : \mathcal{L} \to \mathcal{M}$ is a surjective map of invertible sheaves then it is also injective. Locally, after choosing trivializations, the map $\delta$ is the map of $R$-modules $R \to R$, where $B = \mathcal{O}_{B,b}$. This map has to be given by multiplication with $x \in R$. Since the map is surjective, $x$ is a unit. Thus the map is also injective.
Let $B = \text{Spec } R$. Then
\[ H^0(B, \mathcal{F}_0) = H^0(B, \mathcal{F}_1) = R. \]

Since $\mathcal{L}$ is invertible, we can shrink $B$ to a smaller affine neighborhood so that $\mathcal{L} \simeq \mathcal{O}_B$. We are going to fix a trivialization of $\mathcal{L}$. Everything that follows will be determined up to making a different choice of trivialization, i.e. up to multiplying it by an invertible function $\lambda \in R^*$.

Shrink $B$ further to get short exact sequences
\[ 0 \to \mathcal{F}_{k-1}(B) \to \mathcal{F}_k(B) \to \mathcal{L} \otimes \mathcal{O}_B \to 0 \]
for every $2 \leq k \leq 6$, or, equivalently, given our trivialization of $\mathcal{L}$,
\[ 0 \to \mathcal{F}_{k-1}(B) \to \mathcal{F}_k(B) \xrightarrow{\psi_k} R \to 0, \]
where $\psi$ is the principal parts map.

The following easy lemma is left as a homework exercise.

4.8.3. LEMMA. If $0 \to M \to N \to K \to 0$ is an exact sequence of $R$-modules and $M$ and $K$ are free then $N$ is also free.

By induction, in our case we see that $\mathcal{F}_k(B)$ is a free $R$-module of rank $k$ for every $2 \leq k \leq 6$. Its generators can be obtained by choosing any generators in $\mathcal{F}_{k-1}(B)$ along with any element which maps to $1$ by $\psi_k$.

Thus $\mathcal{F}_2(B)$ is a free $R$-module generated by $1 \in \mathcal{F}_1(B) = R$ and some element $x$ such that $\psi_2(x) = 1$. Also, $\mathcal{F}_3(B)$ is a free $R$-module generated by $1, x \in \mathcal{F}_2(B)$ and some $y$ such that $\psi_3(y) = 2$. An annoying renormalization “$2$” is here by historical reasons. As functions on $\mathcal{X}$, $x$ has a pole of order $2$ and $y$ has a pole of order $3$ along $A$. Notice that $x$ and $y$ are not uniquely determined: we can add to $x$ any $R$-multiple of $1$ and we can add to $y$ any linear combination of $1$ and $x \in \mathcal{F}_2(B)$.

Arguing as in the case of a single elliptic curve, we see that

4.8.4. CLAIM. $1, x, y, x^2, xy, x^3$ freely generate $\mathcal{F}_6(B)$.

But $y^2$ is also a section of $\mathcal{F}_6(B)$, thus it can be expressed as a linear combination of these generators (with coefficients in $R$). Since $\psi_6(y^2) = 4$ and $\psi_6(x^3) = 1$, and $\psi_6$ sends other generators to $0$, the linear combination will look like
\[ y^2 = 4x^3 + a + bx + cy + dx^2 + exy, \]
where $a, b, c, d, e \in R$. By completing the square with $y$ and then the cube with $x$, we can bring this expression into the Weierstrass form
\[ y^2 = 4x^3 - g_2x - g_3, \]
where $g_2, g_3 \in R$. This eliminates any ambiguities in choices of $x$ and $y$.

Functions $x$ and $y$ map $X \setminus A$ to $\mathbb{A}_x^2 \times B$, and the image lies on a hypersurface given by equation (5). Projectivizing, we get a morphism
\[ X \to \mathbb{P}_x^2 \times B, \]
and the image lies on a hypersurface $\mathcal{E}$ with equation
\[ y^2z = 4x^3 - g_2xz^2 - g_3z^3. \]
It remains to show that the induced morphism \( \alpha : X \to \mathcal{E} \) is an isomorphism. It restricts to an isomorphism on every fiber \( X_b \to \mathcal{E}_b \), in particular it is bijective. Let’s use one of the versions of Zariski main theorem:

4.8.5. Theorem. A morphism of algebraic varieties \( X \to \mathcal{E} \) with finite fibers can be factored as an open embedding \( i : X \hookrightarrow U \) and a finite morphism \( g : U \to \mathcal{E} \).

In our case, \( U \) must be equal to \( X \). This follows from the following basic property of proper morphisms:

4.8.6. Lemma. Let \( f : X \to B \) be a proper morphism and suppose it can be factored as \( X \to U \to B \). Then the image of \( X \) in \( U \) is closed.

Thus \( X \to \mathcal{E} \) is a finite morphism. To show that it is an isomorphism, it is enough to check that the map of local rings

\[ \alpha^* : \mathcal{O}_{\mathcal{E}, \alpha(p)} \to \mathcal{O}_{X, p} \]

is surjective (and hence an isomorphism) for every \( p \in X \). Since we know that this is true on every fiber, we have

\[ \mathcal{O}_{X, p} = \alpha^*(\mathcal{O}_{\mathcal{E}, \alpha(p)}) + m_{B, b}\mathcal{O}_{X, p} = \alpha^*(\mathcal{O}_{\mathcal{E}, \alpha(p)}) + \alpha^*(m_{\mathcal{E}, \alpha(p)})\mathcal{O}_{X, p}. \]

Thus we can finish by Nakayama’s lemma which applies, because \( X \to \mathcal{E} \) is finite, and therefore \( \mathcal{O}_{X, p} \) is a finitely generated \( \mathcal{O}_{\mathcal{E}, \alpha(p)} \)-module.
§5. **Icosahedron, \(E_8\), and quotient singularities**

It is time to study invariant theory and orbit spaces more systematically. We will start with a finite group \(G\) acting linearly\(^{18}\) on a vector space \(V\) and discuss the orbit space \(V/G\) and the quotient morphism \(\pi: V \to V/G\). There are several reasons to isolate this case:

- The space \(V/G\) is typically singular, and singularities of this form (called quotient or orbifold singularities) form a very common and useful class of singularities.
- Globally, moduli spaces can be often constructed as quotients of algebraic varieties by reductive group actions. It will be easy to generalize most of the results to this set-up but new subtleties will arise.
- Locally, near some point \(p\), most moduli spaces can be modeled on the quotient \(V/G\), where \(V\) is a vector space (a versal deformation space of the geometric object that corresponds to \(p\)) and \(G\) is an automorphism group of this object, which is typically finite.

Let’s start with some examples.

5.0.7. **Example.** Let \(G = S_n\) be a symmetric group acting on \(\mathbb{C}^n\) by permuting the coordinates. Recall that our recipe for computing the orbit space calls for computing the ring of invariants

\[
\mathbb{C}[x_1, \ldots, x_n]^{S_n}.
\]

\(S_n\)-invariant polynomials are called symmetric polynomials. By the classical **Theorem on Symmetric Polynomials**, the ring is generated by elementary symmetric polynomials

\[
\sigma_1 = x_1 + \ldots + x_n, \\
\sigma_k = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}, \\
\sigma_n = x_1 \ldots x_n.
\]

Thus the candidate for the quotient map is

\[
\pi: \mathbb{A}^n \to \mathbb{A}^n, \quad (x_1, \ldots, x_n) \mapsto (\sigma_1, \ldots, \sigma_n).
\]

This map is surjective and its fibers are the \(S_n\)-orbits. Indeed, we can recover \(x_1, \ldots, x_n\) (up to permutation) from \(\sigma_1, \ldots, \sigma_n\) because they are roots of the polynomial

\[
T^n - \sigma_1 T^{n-1} + \ldots + (-1)^n \sigma_n = 0.
\]

5.0.8. **Example.** A linear action on \(\mathbb{C}\) is given by a character, i.e. a homomorphism \(G \to \mathbb{C}^*\). Its image is a subgroup \(\mu_d\) of \(d\)-th roots of unity. Thus

\[
\mathbb{C}[x]^G = \mathbb{C}[x]^{\mu_d} = \mathbb{C}[x^d].
\]

It is clear that \(x^d\) separates orbits because every non-zero orbit has \(d\) elements \(x, \zeta x, \ldots, \zeta^{d-1} x\). The quotient morphism in this case is just

\[
\pi: \mathbb{A}^1 \to \mathbb{A}^1, \quad x \mapsto x^d.
\]

---

\(^{18}\)Recall that a linear action is given by a homomorphism \(G \to \text{GL}(V)\). In this case we also say that \(V\) is a representation of \(G\).
5.0.9. Example. Let \( \mathbb{Z}^2 \) act on \( \mathbb{A}^3 \) by \( (x, y) \mapsto (-x, -y) \). Invariant polynomials are just polynomials of even degree, and so \( \mathbb{C}[x, y]^{\mathbb{Z}^2} = \mathbb{C}[x^2, y^2, xy] \).

The quotient morphism is
\[
\pi : \mathbb{A}^2 \to \mathbb{A}^3, \quad (x, y) \mapsto (x^2, y^2, xy).
\]
It is clear that invariants separate orbits. It is also clear that the quotient map is surjective onto the quadratic cone \( (uv = w^2) \subset \mathbb{A}^3 \).

The quadratic cone is the simplest du Val singularity called \( A_1 \).

§5.1. Chevalley–Shephard–Todd theorem. One can ask: when is the algebra of invariants a polynomial algebra? The answer is pretty but a bit hard to prove:

5.1.1. Theorem. Let \( G \) be a finite group acting linearly and faithfully on \( \mathbb{C}^n \). The algebra of invariants \( \mathbb{C}[x_1, \ldots, x_n]^G \) is a polynomial algebra if and only if \( G \) is generated by pseudo-reflections, i.e. by elements \( g \in G \) such that the subspace of fixed points \( \{ v \in \mathbb{C}^n | gv = v \} \) has codimension 1.

In other words, \( g \) is a pseudo-reflection if and only if its matrix is some basis is equal to \( \text{diag}[\zeta, 1, 1, \ldots, 1] \), where \( \zeta \) is a root of unity. If \( \zeta = -1 \) then \( g \) is called a reflection. For example, if \( S_n \) acts on \( \mathbb{C}^n \) then any transposition \( (ij) \) acts as a reflection with the mirror \( x_i = x_j \). Further examples of groups generated by reflections are Weyl groups of root systems. On the other hand, the action of \( \mu_d \) on \( \mathbb{C} \) is generated by a pseudo-reflection \( z \mapsto \zeta z \). This is not a reflection (for \( d > 2 \)) but the algebra of invariants is still polynomial. The action of \( \mathbb{Z}_2 \) on \( \mathbb{C}^2 \) by \( \pm 1 \) is not a pseudoreflection (a fixed subspace has codimension 2).

Groups generated by pseudo-reflections were classified by Shephard and Todd. There is one infinite family which depends on 3 integer parameters (and includes \( S_n \)) and 34 exceptional cases.

Sketch of proof in one direction. Suppose \( \mathbb{C}[x_1, \ldots, x_n]^G \) is a polynomial algebra. Then the quotient morphism is the morphism \( \pi : X \to Y \), where both \( X \) and \( Y \) are isomorphic to \( \mathbb{C}^n \). We will prove later that \( \pi \) separates orbits. We will also prove that \( \pi \) is the quotient in the usual topological sense if we endow complex algebraic varieties \( X \) and \( Y \) with Euclidean instead of Zariski topology. Let \( U \subset X \) be the complement of the union of subspaces of fixed points of all elements of \( G \) that are not pseudo-reflections. Then \( U \) is clearly \( G \)-invariant, and its complement has codimension at least 2. Let \( V = \pi(U) \subset Y \). Since its complement in \( \mathbb{C}^n \) has codimension at least 2, \( V \) is simply connected. Now we can use a lemma from geometric group theory:

5.1.2. Lemma. Let \( \Gamma \) be a discrete group of homeomorphisms of a linearly connected Hausdorff topological space \( U \). Suppose the quotient space \( U/\Gamma \) is simply connected. Then \( \Gamma \) is generated by elements having a fixed point in \( U \).

In our case, having a fixed point in \( U \) is equivalent to being a pseudoreflection, because fixed points of all other elements were removed by construction of \( U \).
§5.2. Finite generation.

5.2.1. Theorem. Let $G$ be a finite group acting linearly on a vector space $V$. The algebra of invariants $\mathcal{O}(V)^G$ is finitely generated.

We split the proof into Lemma 5.2.3 and Lemma 5.2.4. The second Lemma will be reused later to prove finite generation for reductive groups.

5.2.2. Definition. Let $G$ be a group acting on a $\mathbb{C}$-algebra $A$ by automorphisms. A linear map

$$R : A \rightarrow A^G$$

is called a Reynolds operator if

- $R(1) = 1$ and
- $R(fg) = fR(g)$ for any $f \in A^G$ and $g \in A$.

In particular, the Reynolds operator is a projector onto $A^G$:

$$R(f) = R(f \cdot 1) = fR(1) = f \quad \text{for every } f \in A^G.$$

5.2.3. Lemma. The Reynolds operator $R : \mathcal{O}(V) \rightarrow \mathcal{O}(V)^G$ exists for any linear action of a finite group.

Proof. Since $G$ acts on $V$, it also acts on the polynomial algebra $\mathcal{O}(V)$. The right way to do it is as follows: if $f \in \mathcal{O}(V)$ then

$$(g \cdot f)(x) = f(g^{-1}x).$$

This is how the action on functions is defined: if you try $g$ instead of $g^{-1}$, the group action axiom will be violated (why?) We define the Reynolds operator $R$ as an averaging operator:

$$R(p) = \frac{1}{|G|} \sum_{g \in G} g \cdot p.$$ 

It is clear that both axioms of the Reynolds operator are satisfied. This works over any field as soon as its characteristic does not divide $|G|$. □

5.2.4. Lemma. Let $G$ be a group acting linearly on a vector space $V$ and possessing a Reynolds operator $R : \mathcal{O}(V) \rightarrow \mathcal{O}(V)^G$. Then $\mathcal{O}(V)^G$ is finitely generated.

Proof. This ingenious argument belongs to Hilbert. First of all, the action of $G$ on $\mathcal{O}(V)$ preserves degrees of polynomials. So $\mathcal{O}(V)^G$ is a graded subalgebra of $\mathcal{O}(V)$. Let $I \subset \mathcal{O}(V)$ be the ideal generated by homogeneous invariant polynomials $f \in \mathcal{O}(V)^G$ of positive degree. By the Hilbert’s basis theorem (proved in the same paper as the argument we are discussing), $I$ is finitely generated by homogeneous invariant polynomials $f_1, \ldots, f_r$ of positive degrees. We claim that the same polynomials generate $\mathcal{O}(V)^G$ as an algebra, i.e. any $f \in \mathcal{O}(V)^G$ is a polynomial in $f_1, \ldots, f_r$. Without loss of generality, we can assume that $f$ is homogeneous and argue by induction on its degree. We have

$$f = \sum_{i=1}^r a_if_i,$$
where \( a_i \in \mathcal{O}(V) \). Now apply the Reynolds operator:

\[
 f = R(f) = \sum_{i=1}^{r} R(a_i) f_i.
\]

Each \( R(a_i) \) is an invariant polynomial, and if we let \( b_i \) be its homogeneous part of degree \( \deg f - \deg f_i \), then we still have

\[
 f = \sum_{i=1}^{r} b_i f_i.
\]

By inductive assumption, each \( b_i \) is a polynomial in \( f_1, \ldots, f_r \). This shows the claim. \( \square \)

§5.3. Quotients and their basic properties.

5.3.1. Definition. Let \( G \) be an algebraic group acting algebraically on an affine variety \( X \). Suppose \( \mathcal{O}(X)^G \) is finitely generated. The categorical quotient \( X/\!/G \) is defined as an affine variety such that

\[
 \mathcal{O}(X/\!/G) = \mathcal{O}(X)^G
\]

and the quotient morphism \( \pi : X \to X/\!/G \) is a morphism with the pullback of regular functions given by the inclusion \( \pi^* : \mathcal{O}(X)^G \subset \mathcal{O}(X) \). More concretely, we choose a system of generators \( f_1, \ldots, f_r \) of \( \mathcal{O}(X)^G \) and write

\[
 \mathbb{C}[x_1, \ldots, x_r]/I \cong \mathcal{O}(V)^G, \quad x_i \mapsto f_i.
\]

We define \( X/\!/G \) as an affine subvariety in \( \mathbb{A}^r \) given by the ideal \( I \) and let

\[
 \pi : X \to X/\!/G \hookrightarrow \mathbb{A}^r, \quad v \mapsto f_1(v), \ldots, f_r(v).
\]

A different system of generators gives an isomorphic affine variety.

For example, we can define the quotient of a vector space by a finite group action, due to the finite generation theorem. To show that this definition is reasonable, let’s check two things:

- Fibers of \( \pi \) are exactly the orbits, i.e. any two orbits are separated by polynomial invariants, and
- All points of \( V/\!/G \) correspond to orbits, i.e. \( \pi \) is surjective.

5.3.2 (Separation of Orbits). The first property relies significantly on finiteness of the group. Take two orbits, \( S_1, S_2 \subset V \). Since they are finite, it is easy to see (why?) that there exists a polynomial \( f \in \mathcal{O}(V) \) such that \( f|_{S_1} = 0 \) and \( f|_{S_2} = 1 \). Then the average

\[
 F = R(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f
\]

is an invariant polynomial but we still have \( F|_{S_1} = 0 \) and \( F|_{S_2} = 1 \).

Now surjectivity:

5.3.3. Theorem. Let \( G \) be a group acting linearly on a vector space \( V \) and possessing a Reynolds operator. Then the quotient map \( \pi : V \to V/\!/G \) is surjective.

5.3.4. Lemma. A regular map \( \pi : X \to Y \) of affine varieties is surjective if and only if \( \mathcal{O}(X)\pi^*(n) \neq \mathcal{O}(X) \) for any maximal ideal \( n \subset \mathcal{O}(Y) \).
Proof. For any point \( y \in Y \) (i.e. a maximal ideal \( n \subset \mathcal{O}(Y) \)) we have to show existence of a point \( x \in X \) (i.e. a maximal ideal \( m \subset \mathcal{O}(X) \)) such that \( f(x) = y \) (i.e. \((\pi^*)^{-1}(m) = n\)). So we have to show that there exists a maximal ideal \( m \subset \mathcal{O}(X) \) that contains \( \pi^*(n) \). The image of an ideal under homomorphism is not necessarily an ideal, so the actual condition is that the ideal \( \mathcal{O}(X)\pi^*(n) \) is a proper ideal. \( \square \)

Proof of Theorem 5.3.3. Let \( n \subset \mathcal{O}(V)^G \) be a maximal ideal. We have to show that
\[
\mathcal{O}(V)n \neq \mathcal{O}(V)
\]
(recall that a pull-back of functions for the quotient map \( \pi : V \to V/G \) is just the inclusion \( \mathcal{O}(V)^G \subset \mathcal{O}(V) \)). Arguing by contradiction, suppose that \( \mathcal{O}(V)n = \mathcal{O}(V) \). Then we have
\[
\sum a_if_i = 1,
\]
where \( a_i \in \mathcal{O}(V) \) and \( f_i \in n \). Applying the Reynolds operator, we see that
\[
\sum b_if_i = 1,
\]
where \( b_i \in \mathcal{O}(V)^G \). But \( n \) is a proper ideal of \( \mathcal{O}(V)^G \), contradiction. \( \square \)

This argument only uses the existence of a Reynolds operator, but for finite groups we can do a little bit better:

5.3.5. Lemma. \( \mathcal{O}(V) \) is integral over \( \mathcal{O}(V)^G \). In other words, the quotient morphism \( \pi : V \to V/G \) is finite (and in particular surjective) for finite groups.

Proof. Indeed, any element \( f \in \mathcal{O}(V) \) is a root of the monic polynomial
\[
\prod_{g \in G} (T - g \cdot f).
\]
Coefficients of this polynomial are in \( \mathcal{O}(V)^G \) (by Vieta formulas). \( \square \)

§5.4. Quotient singularity \( \frac{1}{r}(1, a) \) and continued fractions. How to compute the algebra of invariants? In general it can be quite complicated but things become much easier if the group is Abelian. Let’s compute an amusing example of a cyclic quotient singularity \( \frac{1}{r}(1, a) \). It is defined as follows: consider the action of \( \mu_r \) on \( \mathbb{C}^2 \), where the primitive generator \( \zeta \in \mu_r \) acts via the matrix
\[
\begin{bmatrix}
\zeta & 0 \\
0 & \zeta^a
\end{bmatrix}
\]
The cyclic quotient singularity is defined as the quotient
\[
\mathbb{C}^2/\mu_r = \text{MaxSpec} \mathbb{C}[x, y]^\mu_r.
\]
How to compute this algebra of invariants? Notice that the group acts on monomials diagonally as follows:
\[
\zeta \cdot x^i y^j = \zeta^{-i-ja} x^i y^j.
\]
So a monomial \( x^iy^j \) is contained in \( \mathbb{C}[x, y]^\mu_r \) if and only if
\[
i + ja \equiv 0 \mod r.
\]
There are two cases when the answer is immediate:
5.4.1. EXAMPLE. Consider $\frac{1}{r}(1, r - 1)$. Notice that this is the only case when $\mu_r \subset \text{SL}_2$. The condition on invariant monomials is that

$$i \equiv j \mod r$$

(draw). We have

$$\mathbb{C}[x, y]^{\mu_r} = \mathbb{C}[x^r, xy, y^r] = \mathbb{C}[U, V, W]/(V^r - UW).$$

So we see that the singularity $\frac{1}{r}(1, r - 1)$ is a hypersurface in $\mathbb{A}^3$ given by the equation $V^r = UW$. It is called an $A_{r-1}$-singularity.

5.4.2. EXAMPLE. Consider $\frac{1}{r}(1, 1)$. The condition on invariant monomials is

$$i + j \equiv 0 \mod r$$

(draw). We have

$$\mathbb{C}[x, y]^{\mu_r} = \mathbb{C}[x^r, x^{r-1}y, x^{r-2}y^2, \ldots, y^r].$$

The quotient morphism in this case is

$$\mathbb{A}^2 \to \mathbb{A}^{r+1}, \quad (x, y) \mapsto (x^r, x^{r-1}y, x^{r-2}y^2, \ldots, y^r).$$

The singularity $\frac{1}{r}(1, 1)$ with a cone over a rational normal curve

$$[x^r : x^{r-1}y : x^{r-2}y^2 : \ldots : y^r] \subset \mathbb{P}^{r-1}.$$

5.4.3. DEFINITION. Let $r > b > 0$ be coprime integers. The following expression is called the Hirzebruch–Jung continued fraction:

$$\frac{r}{b} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots}} = [a_1, a_2, \ldots, a_k].$$

Hirzebruch–Young continued fractions are similar to ordinary continued fractions but have minuses instead of pluses. For example,

$$\frac{5}{1} = [5],$$

$$\frac{5}{4} = [2, 2, 2, 2],$$

$$\frac{5}{2} = [3, 2].$$

Here’s the result:

5.4.4. THEOREM. Suppose $\mu_r$ acts on $\mathbb{A}^2$ with weights 1 and $a$, where $(a, r) = 1$. Let $\frac{r}{r-a} = [a_1, a_2, \ldots, a_k]$ be the Hirzebruch–Jung continued fraction expansion\(^{19}\). Then $\mathbb{C}[x, y]^{\mu_r}$ is generated by

$$f_0 = x^r, \quad f_1 = x^{r-a}y, \quad f_2, \ldots, f_k, \quad f_{k+1} = y^r,$$

where the monomials $f_i$ are uniquely determined by the following equations:

$$f_{i-1}f_{i+1} = f_i^{a_i} \quad \text{for} \quad i = 1, \ldots, k.$$  \hspace{1cm} (5.4.5)

5.4.6. EXAMPLE. An example is in Figure 2.

\(^{19}\)Notice that we are expanding $r/(r - a)$ and not $r/a.$
5.4.7. REMARK. We see that the codimension of $\mathbb{A}^2/\Gamma$ in the ambient affine space $\mathbb{A}^{k+2}$ is equal to the length of the Hirzebruch–Young continued fraction. This is a good measure of the complexity of the singularity. From this perspective, $\frac{1}{x}(1, 1)$ (the cone over a rational normal curve) is the most complicated singularity: the Hirzebruch–Young continued fraction

$$\frac{r}{(r - 1)} = [2, 2, 2, \ldots, 2] \ (r - 1 \text{ times})$$

uses the smallest possible denominators. It is analogous to the standard continued fraction of the ratio of two consecutive Fibonacci numbers, which has only 1’s as denominators.

Proof of Theorem 5.4.4. Invariant monomials in $\mathbb{C}[x, y]^\mu$ are indexed by vectors of the first quadrant

$$\{(i, j) \mid i, j \geq 0\} \subset \mathbb{Z}^2$$

which are in the lattice

$$L = \{(i, j) \mid i + aj \equiv 0 \mod r\} \subset \mathbb{Z}^2.$$

This intersection is a semigroup and we have to find its generators. We note for future use that $L$ contains the sublattice $r\mathbb{Z}^2$ and can be described as the lattice generated by

$$\begin{bmatrix} 0 \\ r \end{bmatrix}, \begin{bmatrix} r-a \\ 1 \end{bmatrix}, \begin{bmatrix} r \\ 0 \end{bmatrix}.$$

The semigroup of invariant monomials is generated by $\begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ r \end{bmatrix}$, and by the monomials inside the square $\{(i, j) \mid 0 < i, j < r\}$, which are precisely the monomials

$$((r - a)j \mod r, j), \ j = 1, \ldots, r - 1.$$
Of course many of these monomials are unnecessary. The first monomial in the square that we actually need is \( \begin{bmatrix} r - a \\ 1 \end{bmatrix} \). Now take multiples of \( \begin{bmatrix} r - a \\ 1 \end{bmatrix} \).

The next generator will occur when \((r - a)j\) goes over \(r\), i.e. when

\[
 j = \left\lceil \frac{r}{r - a} \right\rceil = a_1
\]

(in the Hirzebruch–Young continued fraction expansion for \(\frac{r}{r - a}\)). Since \((r - a)a_1 \mod r = (r - a)a_1 - r\), the next generator is

\[
 \begin{bmatrix} (r - a)a_1 - r \\ a_1 \end{bmatrix}
\]

Notice that so far this confirms our formula (5.4.5). We are interested in the remaining generators of \(L\) inside the \(r \times r\) square. Notice that they all lie above the line spanned by \( \begin{bmatrix} r - a \\ 1 \end{bmatrix} \). So we can restate our problem: find generators of the semigroup obtained by intersecting \(L\) with points lying in the first quadrant and above the line spanned by \( \begin{bmatrix} r - a \\ 1 \end{bmatrix} \).

Next we notice that

\[
 \begin{bmatrix} r \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} r - a \\ 1 \end{bmatrix} - \begin{bmatrix} (r - a)a_1 - r \\ a_1 \end{bmatrix}.
\]

It follows that lattice \(L\) is also spanned by \( \begin{bmatrix} 0 \\ r \end{bmatrix} \), \( \begin{bmatrix} (r - a)a_1 - r \\ a_1 \end{bmatrix} \), and \( \begin{bmatrix} r - a \\ 1 \end{bmatrix} \).

We are interested in generators of the semigroup obtained by intersecting this lattice with the “angle” spanned by vectors \( \begin{bmatrix} 0 \\ r \end{bmatrix} \) and \( \begin{bmatrix} r - a \\ 1 \end{bmatrix} \).

Consider the linear transformation \(\psi : \mathbb{R}^2 \to \mathbb{R}^2\) such that

\[
\psi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{r} \end{bmatrix}, \quad \psi \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{r - a}{r} \end{bmatrix}.
\]

Then we compute

\[
\psi \begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r - a \end{bmatrix}, \quad \psi \begin{bmatrix} r - a \\ 1 \end{bmatrix} = \begin{bmatrix} r - a \\ 0 \end{bmatrix}, \quad \psi \begin{bmatrix} (r - a)a_1 - r \\ a_1 \end{bmatrix} = \begin{bmatrix} (r - a)a_1 - r \\ 1 \end{bmatrix}.
\]

So we get the same situation as before with a smaller lattice. Notice that if

\[
\frac{r}{r - a} = a_1 - \frac{1}{q}
\]

then

\[
q = \frac{r - a}{(r - a)a_1 - r},
\]

so we will recover all denominators in the Hirzebruch–Jung continued fraction as we proceed inductively. \(\square\)
§5.5. *E*₈-singularity. Let’s try to understand the *du Val* singularity *E*₈ related to the icosahedron. Recall that finite subgroups in SO₃ correspond to platonic solids. For example, *A*₅ is a group of rotations of the icosahedron. Thus *A*₅ acts on the circumscribed sphere of the icosahedron. This action is obviously *conformal* (preserves oriented angles), and so if we think about *S*² as a Riemann sphere *P*¹ (by using the stereographic projection), we get an embedding *A*₅ ⊂ PSL₂ (since it is proved in complex analysis that conformal maps are holomorphic). The preimage of *A*₅ in SL₂ is called the *binary icosahedral group* *Γ*. It has *2 × 60 = 120* elements. The orbit space *C*²//*Γ*, which we are going to construct, is called an *E*₈ singularity.

We have to compute *C*[[*x,y*]]*Γ*. There is a miraculously simple way to write down some invariants: *A*₅ has three special orbits on *S*²:

• 20 vertices of the icosahedron,
• 12 midpoints of faces (vertices of the dual dodecahedron),
• and 30 midpoints of the edges.

Let *f*₁₂, *f*₂₀, and *f*₃₀ be polynomials in *x*, *y* that factor into linear forms that correspond to these special points.

We claim that these polynomials are invariant. Since *Γ* permutes their roots, they are clearly *semi-invariant*, i.e. any *γ* ∈ *Γ* can only multiply them by a scalar, which will be a character of *Γ*. Since they all have even degree, the element −1 ∈ *Γ* does not change these polynomials. But *Γ*/{±1} ∼ *A*₅ is a simple group, hence has no characters at all, hence the claim.

5.5.1. THEOREM. *C*[[*x,y*]]*Γ* = *C*[[*f*₁₂,*f*₂₀,*f*₃₀]] ∼ *C*[[*U,V,W*]]/(*U*⁵ + *V*³ + *W*²).

Proof. Let’s try to prove this using as few explicit calculations as possible. The key is to analyze a chain of algebras

*C*[[*x,y*]] ⊃ *C*[[*x,y*]]*Γ* ⊃ *C*[[*f*₁₂,*f*₂₀,*f*₃₀]] ⊃ *C*[[*f*₁₂,*f*₂₀]].

We claim that *C*[[*f*₁₂,*f*₂₀]] ⊂ *C*[[*x,y*]] (and hence all other inclusions in the chain) is an integral extension. In other words, we claim that a regular map

*A*² → *A*²,  (*x*, *y*) ↦ (*f*₁₂, *f*₂₀)  (5.5.2)

is finite. Since *f*₁₂ and *f*₂₀ have no common zeros in *P*¹, Nullstellensatz implies that

\[ \sqrt{(*f*₁₂, *f*₂₀)} = (*x*, *y)*, \]

i.e. *x*ⁿ, *y*ⁿ ∈ (*f*₁₂, *f*₂₀) for some large *n*. Thus *C*[[*x,y*]] is generated as an algebra by elements (*x* and *y*) which are integral over *C*[[*f*₁₂,*f*₂₀]].

Now let’s consider the corresponding chain of fraction fields

*C*[[*x,y*]] ⊃ Quot *C*[[*x,y*]]*Γ* ⊃ *C*[[*f*₁₂,*f*₂₀,*f*₃₀]] ⊃ *C*[[*f*₁₂,*f*₂₀]].  (6)

Here are some basic definitions, and a fact.

5.5.3. DEFINITION. Let *f* : *X* → *Y* be a dominant map of algebraic varieties of the same dimension. It induces an embedding of fields

\[ *f*^* : *C*(*Y*) ⊂ *C*(*X*). \]
We define the degree of \( f \) as follows:\(^{20}\)
\[
\deg f = [\mathbb{C}(X) : \mathbb{C}(Y)].
\]

5.5.4. Definition. An irreducible affine variety \( X \) is called normal if \( \mathcal{O}(X) \) is integrally closed in \( \mathbb{C}(X) \). More generally, an algebraic variety is called normal if every local ring \( \mathcal{O}_{X,x} \) is integrally closed in \( \mathbb{C}(X) \).

5.5.5. Theorem. Let \( f : X \to Y \) be a finite map of algebraic varieties. Suppose that \( Y \) is normal. Then any fiber \( f^{-1}(y) \) has at most \( \deg f \) points. Let
\[
U = \{ y \in Y \mid f^{-1}(y) \text{ has exactly } \deg f \text{ points } \}.
\]
Then \( U \) is open and non-empty.

This theorem was previously suggested as a homework exercise. Let’s see how we can use it. First of all, any UFD is integrally closed, hence \( \mathbb{C}[x,y] \) and \( \mathbb{C}[f_{12}, f_{20}] \) are integrally closed.

Secondly, \( \mathbb{C}[x,y]^\Gamma \) is integrally closed. Indeed, if \( f \in \text{Quot} \mathbb{C}[x,y]^\Gamma \) is integral over \( \mathbb{C}[x,y]^\Gamma \) then it is also integral over \( \mathbb{C}[x,y] \), but the latter is integrally closed, hence \( f \in \mathbb{C}[x,y] \), and so \( f \in \mathbb{C}[x,y]^\Gamma \).

It follows that \( \{ \mathbb{C}[x,y] : \text{Quot} \mathbb{C}[x,y]^\Gamma = 120 \) because fibers of the quotient morphism are orbits and the general orbit has 120 points.\(^{21}\) The fibers of the map (5.5.2) are level curves of \( f_{12} \) and \( f_{20} \), and therefore contain at most 240 points by Bezout theorem. One can show geometrically that general fibers contain exactly 240 points or argue as follows: if this is not the case then we can conclude from (6) that \( \text{Quot} \mathbb{C}[x,y]^\Gamma = \mathbb{C}(f_{12}, f_{20}) \) and therefore \( f_{30} \in \mathbb{C}(f_{12}, f_{20}) \). But \( f_{30} \) is integral over \( \mathbb{C}[f_{12}, f_{20}] \), and the latter is integrally closed, so \( f_{30} \in \mathbb{C}[f_{12}, f_{20}] \). But this can’t be true because of the degrees! So in fact we have
\[
\text{Quot} \mathbb{C}[x,y]^\Gamma = \mathbb{C}(f_{12}, f_{20}, f_{30}) \quad \text{and} \quad [\mathbb{C}(f_{12}, f_{20}, f_{30}) : \mathbb{C}(f_{12}, f_{20})] = 2.
\]

The latter formula implies that the minimal polynomial of \( f_{30} \) over \( \mathbb{C}(f_{12}, f_{20}) \) has degree 2. The second root of this polynomial satisfies the same integral dependence as \( f_{20} \), and therefore all coefficients of the minimal polynomial are integral over \( \mathbb{C}[f_{12}, f_{20}] \), by Vieta formulas. But this ring is integrally closed, and therefore all coefficients of the minimal polynomial are in fact in \( \mathbb{C}[f_{12}, f_{20}] \). So we have an integral dependence equation of the form
\[
f_{30}^2 + af_{30} + b = 0,
\]
where \( a, b \in \mathbb{C}[f_{12}, f_{20}] \). Looking at the degrees, there is only one way to accomplish this (modulo multiplying \( f_{12}, f_{20}, \) and \( f_{30} \) by scalars), namely
\[
f_{12}^2 + f_{20}^2 + f_{30}^2 = 0.
\]

It remains to prove that \( \mathbb{C}[x,y]^\Gamma = \mathbb{C}[f_{12}, f_{20}, f_{30}] \). Since they have the same quotient field, it is enough to show that the latter algebra is integrally closed, and this follows from the following extremely useful theorem that we are not going to prove, see [M3, page 198]. \( \square \)

\(^{20}\)The dimension is equal to the transcendence degree of the field of functions, so \( \mathbb{C}(X)/\mathbb{C}(Y) \) is an algebraic extension, hence finite (because \( \mathbb{C}(X) \) is finitely generated).

\(^{21}\)If we knew that the second field is \( \mathbb{C}(x,y)^\Gamma \), the formula follows from Galois theory.
5.5.6. **Theorem.** Let $X \subset \mathbb{A}^n$ be an irreducible affine hypersurface\(^{22}\) such that its singular locus has codimension at least 2. Then $X$ is normal.

For example, a surface $S \subset \mathbb{A}^3$ with isolated singularities is normal. It is important that $S$ is a surface in $\mathbb{A}^3$, it is easy to construct examples of non-normal surfaces with isolated singularities in $\mathbb{A}^4$.

**Proof of Theorem 5.5.5.** Let $y \in Y$ and choose a function $a \in \mathcal{O}(X)$ that takes different values on points in $f^{-1}(y)$. The minimal polynomial $F(T)$ of $a$ over $\mathbb{C}(Y)$ has degree at most $\deg f$. Since $Y$ is normal, all coefficients of the minimal polynomial are in fact in $\mathcal{O}(Y)$. Thus $f^{-1}(y)$ has at most $n$ points. Since we are in characteristic 0, the extension $\mathbb{C}(X)/\mathbb{C}(Y)$ is separable, and hence has a primitive element. Let $a \in \mathcal{O}(X)$ be an element such that its minimal polynomial (=integral dependence polynomial) has degree $n$:

$$F(T) = T^n + b_1 T^{n-1} + \ldots + b_n, \quad b_i \in \mathcal{O}(Y).$$

Let $D \in \mathcal{O}(Y)$ be the discriminant of $F(T)$ and let $U = \{y \in Y \mid D \neq 0\}$ be the corresponding principal open set. We claim that $f$ has exactly $n$ different fibers over any point of $U$. Indeed, the inclusion $\mathcal{O}(Y)[a] \subset \mathcal{O}(X)$ is integral, hence induces a finite map, hence induces a surjective map. But over a point $y \in Y$, the fiber

$$\text{MaxSpec} \mathcal{O}(Y)[a] = \{(y, t) \in Y \times \mathbb{A}^1 \mid t^n + b_1(y)t^{n-1} + \ldots + b_n(y) = 0\}$$

is just given by the roots of the minimal polynomial, and hence consists of $n$ points. Thus the fiber $f^{-1}(y)$ also has $n$ points. $\square$

\(^{22}\)More generally, $X$ can be a complete intersection, or Cohen–Macaulay, or just satisfy Serre’s property $S_2$
§5.6. Homework.

**Problem 1.** (2 points). Prove Claim 4.8.4.

**Problem 2.** (2 points). Prove Lemma 4.8.3.

**Problem 3.** (3 points). Prove Claim ??.

**Problem 4.** For the cyclic quotient singularity $\frac{1}{7}(1, 3)$, compute generators of $\mathbb{C}[x, y]^{\mu_7}$, realize $\mathbb{A}^2/\mu_7$ as an affine subvariety of $\mathbb{A}^3$, and compute the ideal of this subvariety. (2 points)

**Problem 5.** For the cyclic quotient singularity $\frac{1}{5}(1, 4)$, show how to write down explicitly 5 affine charts $Y_i \simeq \mathbb{A}^2$, regular morphisms $Y_i \to \mathbb{A}^2/\mu_5$, and gluing maps between affine charts such that $Y = \cup Y_i$ is a resolution of singularities of $\mathbb{A}^2/\mu_5 \subset \mathbb{A}^3$. (2 points)

**Problem 6.** Let $G$ be a finite group acting on an affine variety $X$ by automorphisms. (a) Show that there exists a closed embedding $X \subset \mathbb{A}^r$ such that $G$ acts linearly on $\mathbb{A}^r$ inducing the original action on $X$. (b) Show that the restriction homomorphism $\mathcal{O}(V)^G \to \mathcal{O}(X)^G$ is surjective. (c) Show that $\mathcal{O}(X)^G$ is finitely generated (2 points).

**Problem 7.** Let $G$ be a group acting linearly on a vector space $V$. Let $L \subset V$ be a linear subspace. Let

$$Z = \{g \in G \mid g|_L = \text{Id}|_L\}, \quad N = \{g \in G \mid g(L) \subset L\}, \quad \text{and } W = N/Z.$$

(a) Show that there exists a natural homomorphism $\pi : \mathcal{O}(V)^G \to \mathcal{O}(L)^W$. (b) Suppose there $G \cdot L = V$. Show that $\pi$ is injective. (2 points).

**Problem 8.** Let $G = SO_n(\mathbb{C})$ be an orthogonal group preserving a quadratic form $f = x_1^2 + \ldots + x_n^2$. Show that $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[f]$. (Hint: apply the previous problem to $L = \mathbb{C}e_1$). (1 point).

**Problem 9.** Let $G = GL_n$ be a general linear group acting on $\text{Mat}_n$ by conjugation. (a) Let $L \subseteq \text{Mat}_n$ be the space of diagonal matrices. Show that $G \cdot L = \text{Mat}_n$. (b) Show that $\mathcal{O}(\text{Mat}_n)^G$ is generated by coefficients of the characteristic polynomial (2 points).

**Problem 10.** (a) In the notation of the previous problem, describe all fibers of the quotient morphism $\pi : \text{Mat}_n \to \text{Mat}_n / G$. (b) Show that not all orbits are separated by invariants and find all orbits in the fiber $\pi^{-1}(0)$. (c) Describe all fibers of $\pi$ that contain only one orbit (3 points).

**Problem 11.** (a) Let $G$ be a finite group acting linearly on a vector space $V$. Show that $\mathcal{C}(V)^G$ (the field of invariant rational functions) is equal to the quotient field of $\mathcal{O}(V)^G$. (b) Show that (a) can fail for an infinite group. (c) Show that if $G$ is any group acting linearly on a vector space $V$ then any invariant rational function $f \in \mathcal{C}(V)^G$ can be written as a ratio of two semi-invariant functions of the same weight. (3 points).

**Problem 12.** Consider the standard linear action of the dihedral group $D_n$ in $\mathbb{R}^2$ (by rotating the regular $n$-gon) and tensor it with $\mathbb{C}$. Compute generators of the algebra of invariants $\mathbb{C}[x, y]^{D_n}$ (2 points).

**Problem 13.** Show that a finite map between affine varieties takes closed sets to closed sets. (1 point).
Problem 14. Let \( f : X \rightarrow Y \) be a regular map of affine varieties such that every point \( y \in Y \) has an affine neighborhood \( U \subset Y \) such that \( f^{-1}(U) \) is affine and the restriction \( f : f^{-1}(U) \rightarrow U \) is affine. Show that \( f \) itself is finite. (2 points)

Problem 15. Let \( R = \mathbb{C}[x_1, \ldots, x_n]^{S_n} \). Show that \( P(t) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \). (1 point)

Problem 16. (a) Compute Poincare series of the algebra \( \mathbb{C}[x, y]^{\mu_r} \), where \( \mu_r \) acts as \( \frac{1}{r}(1, r - 1) \). (b) Compute Poincare series of \( \mathbb{C}[x, y]^{\Gamma} \), where \( \Gamma \) is a binary icosahedral group. (3 points).

Problem 17. (a) Let \( F : \mathbb{A}^n \rightarrow \mathbb{A}^n \) be a morphism given by homogeneous polynomials \( f_1, \ldots, f_n \) such that \( V(f_1, \ldots, f_n) = \{0\} \). Show that \( F \) is finite. (b) Give example of a dominant morphism \( \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) which is not finite (2 points).

Problem 18. Let \( G \) be a group acting by automorphisms on a normal affine variety \( X \). Show that the algebra of invariants \( \mathcal{O}(X)^G \) is integrally closed. (1 point)

Problem 19. Let \( A \) be an integrally closed domain with field of fractions \( K \) and let \( A \subset B \) be an integral extension of domains. Let \( b \in B \) and let \( f \in K[x] \) be its minimal polynomial. Show that in fact \( f \in A[x] \). (1 point)

Problem 20. Consider the action of \( SL_2 \) on homogeneous polynomials in \( x \) and \( y \) of degree 6 written as follows:
\[
\zeta_0 x^6 + 6 \zeta_1 x^5 y + 15 \zeta_2 x^4 y^2 + 20 \zeta_3 x^3 y^3 + 15 \zeta_4 x^2 y^4 + 6 \zeta_5 x y^5 + \zeta_6 y^6.
\]
Show that the function
\[
\det \begin{bmatrix}
\zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 \\
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
\zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 \\
\zeta_3 & \zeta_4 & \zeta_5 & \zeta_6
\end{bmatrix}
\]
belongs to the algebra of invariants \( \mathbb{C}[\zeta_0, \zeta_1, \ldots, \zeta_6]^{SL_2} \). (2 points)

Problem 21. Consider the action of \( A_n \) on \( \mathbb{A}^n \) by permutations of coordinates. Show that \( \mathbb{C}[x_1, \ldots, x_n]^{A_n} \) is generated by elementary symmetric polynomials \( \sigma_1, \ldots, \sigma_n \) and the discriminant \( D = \prod_{1 \leq i < j} (x_i - x_j) \) (2 points).
§6. Weighted projective spaces

§6.1. First examples. Let’s move from quotient spaces by finite groups to the actions of the easiest infinite group: $\mathbb{C}^*$. We will work out the following extremely useful example: fix positive integers $a_0, \ldots, a_n$ (called weights) and consider the action of $\mathbb{C}^*$ on $\mathbb{A}^{n+1}$ defined as follows:

$$\lambda \cdot (x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$$

for any $t \in \mathbb{C}^*$. The quotient (which we are going to construct) is called the weighted projective space. Notation:

$$\mathbb{P}(a_0, \ldots, a_n).$$

For example, we have $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$.

6.1.1. Example. We have met $\mathbb{P}(4, 6)$ before. Recall that any elliptic curve has a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$, $\Delta = g_3^3 - 27g_2^2 \neq 0$ and this is an extremely useful fact for studying elliptic fibrations (and elliptic curves defined over rings of algebraic integers). Coefficients $g_2$ and $g_3$ are defined not uniquely but only up to admissible transformations $g_2 \mapsto t^4g_2$, $g_3 \mapsto t^6g_3$.

So the moduli space of elliptic curves is $\mathbb{P}(4, 6)$ with a point “at infinity” removed (which corresponds to the $\mathbb{C}^*$-orbit $|\Delta = 0|$). So it should come at no surprise that

$$\mathbb{P}(4, 6)_{[g_2:g_3]} \simeq \mathbb{P}^1_{[j:1]},$$

where $j$ is given by the usual formula (3.2.7). Notice however that thinking about $\mathbb{P}(4, 6)$ has a lot of advantages: it encompasses the idea of Weierstrass families better and it emphasizes the role of special elliptic curves with many automorphisms. In general, we will see that weighted projective spaces are different from usual ones: they have singularities.

6.1.2. Example. Let’s construct $\mathbb{P}(1, 1, 2)$ by hand. Take the map

$$\pi : \mathbb{A}^3_{x,y,z} \setminus \{0\} \rightarrow \mathbb{P}^3_{[A:B:C:D]} , \quad (x, y, z) \mapsto [x^2 : xy : y^2 : z].$$

It is easy to see that it separates orbits, i.e. $\pi(x, y, z) = (x', y', z')$ if and only if there exists $t \in \mathbb{C}^*$ such that

$$x' = tx, \quad y' = ty, \quad z' = t^2z.$$

The image is a quadratic cone $AB = C^2$ in $\mathbb{P}^3$.

Now let’s discuss the general construction of the weighted projective space. Remember the drill: we have to find all semi-invariants. Here this is exceptionally easy: any monomial $x_0^{i_0} \ldots x_n^{i_n}$ is a semi-invariant for the $\mathbb{C}^*$-action of weight $w = i_0a_0 + \ldots + i_na_n$ (i.e. $t \in \mathbb{C}^*$ acts by multiplying this monomial by $t^w$). So the algebra of semi-invariants is just the full polynomial algebra

$$\mathbb{C}[x_0, \ldots, x_n].$$

However, we have to introduce a different grading on this algebra, where each variable $x_i$ has degree $a_i$. Here are some basic observations:
• There are no non-constant invariants. So we can not produce a quotient by our method of taking MaxSpec of the algebra of invariants (by taking the image of the map to \( \mathbb{A}^r \) given by \( r \) basic invariants). Here is an “explanation”: all orbits contain zero in their closure. So any invariant polynomial is just a constant equal to the value of this polynomial in \( 0 \). This is the reason we have to remove zero, just like in the \( \mathbb{P}^n \) case. Notice that \( \mathbb{A}^{n+1} \setminus \{0\} \) is not an affine variety anymore. The procedure of taking MaxSpec won’t work after removing the origin.

• The algebra of semi-invariants is generated by variables, which have different degrees. So the situation is different from our experience of writing the Grassmannian \( G(2, n) \) as a quotient \( \text{Mat}(2, n)/GL_2 \), where basic semi-invariants \( (2 \times 2 \) minors) all had the same degree.

So we need a new approach. The idea is simple: \( \mathbb{A}^{n+1} \setminus \{0\} \) is covered by principal open sets \( D(x_i) \). We will take their quotients by \( \mathbb{C}^* \) first and then glue them, just like in the definition of the usual projective space.

In the case of \( \mathbb{P}^n \) we don’t even notice the \( \mathbb{C}^* \) action because we kill it by setting \( x_i = 1 \). So we quite naturally identify \( D(x_i)/\mathbb{C}^* \simeq \mathbb{A}^n \). Let’s denote the corresponding chart \( D_{x_i} \subset \mathbb{P}^n \) to distinguish it from \( D(x_i) \subset \mathbb{A}^{n+1} \).

What will happen in a more general case? Setting \( x_i = 1 \) does not quite eliminate \( t \); it just implies that \( t^{\mu_i} = 1 \). This is still an achievement: it shows that the action of \( \mathbb{C}^* \) on \( D(x_i) \subset \mathbb{A}^{n+1} \) is reduced to the action of \( \mu_{a_i} \) on \( \mathbb{A}^n \). This is a familiar ground: the quotient will be

\[
D_{x_i} = D(x_i)/\mathbb{C}^* \simeq \mathbb{A}^n / \mu_{a_i} = \text{MaxSpec} \mathbb{C}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]^{\mu_{a_i}},
\]

where \( \mu_{a_i} \) acts with weights \( a_0, \ldots, \hat{a}_i, \ldots, a_n \). So for example, a projective quadratic cone \( \mathbb{P}(1, 1, 2) \) is covered by three charts: two copies of \( \mathbb{A}^2 \) and one copy of \( \frac{1}{2}(1, 1) \), which is isomorphic to an affine quadratic cone.

Here is another way of thinking about this. Notice that

\[
\mathcal{O}(D(x_i)) = \mathbb{C} \left[ x_0, \ldots, x_n; \frac{1}{x_i} \right]
\]

and that \( \mathbb{C}^* \) now acts on the affine variety \( D(x_i) \). We can use our old recipe for computing the quotient: take the algebra of invariants and compute its spectrum. So we set

\[
\mathcal{O}(D_{x_i}) = \mathcal{O}(D(x_i))^{\mathbb{C}^*} = \left\{ \frac{p}{x_i^k} \mid p \in \mathbb{C}[x_0, \ldots, x_n], \deg p = ka_i \right\}
\]

(here and after the degree \( \deg \) is our funny weighted degree). There are two cases: if \( a_i = 1 \) then we just have

\[
\mathcal{O}(D_{x_i}) = \mathbb{C} \left[ \frac{x_0}{x_i^{a_0}}, \ldots, \frac{x_n}{x_i^{a_n}} \right] \simeq \mathbb{C}[y_1, \ldots, y_n].
\]

The chart is an affine space, just like for the standard \( \mathbb{P}^n \). To figure out the general case, for simplicity let’s restrict to the weighted projective plane \( \mathbb{P}(a_0, a_1, a_2) \). What will be the first chart? Consider the cyclic field extension

\[
\mathbb{C}(x_0, x_1, x_2) \subset \mathbb{C}(z_0, x_1, x_2),
\]
where $x_0 = z_0^{a_0}$. Then we have

$$\mathcal{O}(D_{x_i}) = \left\{ \frac{p}{z_0^{a_0 k}} \mid p \in \mathbb{C}[x_0, x_1, x_2], \deg p = ka_0 \right\} =$$

$$\left\{ \sum_{i,j} a_{ij} \left( \frac{x_1}{x_0} \right)^i \left( \frac{x_2}{x_0} \right)^j \mid a_1 i + a_2 j \equiv 0 \mod a_0 \right\} \subset \mathbb{C} \left[ \frac{x_1}{x_0}, \frac{x_2}{x_0} \right].$$

So we get a subalgebra in $\mathbb{C}[y_1, y_2]$ spanned by monomials $y_1^i y_2^j$ such that $a_1 i + a_2 j \equiv 0 \mod a_0$. This is our old friend, the cyclic quotient $\frac{1}{a_0} (a_1, a_2)$.

§6.2. Proj (projective spectrum). Let’s generalize this even further. Let $R$ be any finitely generated graded integral domain such that $R_0 = \mathbb{C}$. We can write $R$ as a quotient of $\mathbb{C}[x_0, \ldots, x_n]$ (with grading given by degrees $a_0, \ldots, a_n$ of homogeneous generators of $R$) by a homogeneous (in this grading!) prime ideal. Functions in this ideal are constant along $\mathbb{C}^*$-orbits in $\mathbb{A}^{n+1}$. As a set, we simply define

$$\text{Proj } R \subset \mathbb{P}(a_0, \ldots, a_n)$$

as a set of $\mathbb{C}^*$-orbits where all functions in the ideal vanish.

Rational functions on Proj $R$ are defined as ratios of polynomials of the same (weighted) degree, i.e.

$$\mathbb{C}(\text{Proj } R) = (\text{Quot } R)_0,$$

where the subscript means that we are only taking fractions of degree 0. We call a function regular at some point if it has a presentation as a fraction with a denominator non-vanishing at this point. It is clear that Proj $R$ is covered by affine charts $D_f$ for each homogeneous element $f \in R$ of positive degree, where

$$\mathcal{O}(D_f) = R[1/f]_0.$$

What is the gluing? Given $D_f$ and $D_g$, notice that

$$D_f \cap D_g = D_{fg},$$

is a principal open subset in both $D_f$ (where it is a complement of a vanishing set of a regular function $\frac{f\deg f}{\deg g}$) and $D_g$ (where we use $\frac{\deg g}{\deg f}$). Formally speaking, we have to check that in $\mathbb{C}(\text{Proj } R)$ we have

$$R[1/fg]_0 = R[1/f]_0 \left[ \frac{\deg g}{\deg f} \right]. \quad (6.2.1)$$

This kind of formulas are proved by tinkering with fractions with a sole purpose to balance degrees of the numerator and the denominator. We leave it as an exercise.
6.3. Abstract algebraic varieties. To continue this discussion, we have to ask ourselves: what is it that we are trying to prove? We will later see that Proj $R$ is in fact a projective variety, but at this point it would be useful to give a definition of an abstract algebraic variety.

6.3.1. Definition. For simplicity, we will only define an irreducible algebraic variety $X$. We need

- A finitely generated field extension $K$ of $\mathbb{C}$. This will be a field of rational functions on $X$.
- Topology on $X$.
- For each open subset $U \subset X$ we need a subalgebra $O_X(U) \subset K$. It should satisfy the condition

$$O_X \left( \bigcup_{i \in I} U_i \right) = \bigcap_{i \in I} O_X(U_i).$$

$O_X$ is called the structure sheaf.
- Finally, $X$ should admit a finite cover $\{U_i\}$ such that each $U_i$ (with an induced topology) is an irreducible affine variety (with Zariski topology) with function field $K$ and for each open subset $V \subset U_i$, $O_X(V) \subset K$ is the algebra of rational functions regular on $V$.

In practice, algebraic varieties are constructed by gluing affine varieties. Suppose $A$ and $B$ are irreducible affine varieties with the same function field $K$. Suppose, in addition, that there exists another affine variety $C$ and open immersions

$$i_A : C \hookrightarrow A \quad \text{and} \quad i_B : C \hookrightarrow B.$$ 

Then we define the topological space $X = A \cup_C B$ by identifying points $i_A(x)$ with $i_B(x)$ for any $x \in C$ and by declaring a subset $U \subset X$ open if $U \cap A$ and $U \cap B$ is open. Finally, we set

$$O_X(U) = O_A(U \cap A) \cap O_B(U \cap B)$$

It is easy to generalize this to several affine charts: we need irreducible affine varieties

$$U_0, \ldots, U_r,$$

with the same function field. For each pair $U_i, U_j$ we have affine open subsets

$$U_{ij} \subset U_i, \quad U_{ji} \subset U_j$$

and an isomorphism

$$\phi_{ij} : U_{ij} \to U_{ji}.$$ 

This isomorphism should satisfy (draw pictures)

- $\phi_{ij} = \phi_{ji}^{-1}$,
- $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and
- $\phi_{jk} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$.

6.3.2. Lemma. Proj $R$ is an algebraic variety.
Proof. We have \( K = (\text{Quot } R)_0 \). For any homogeneous \( f \in R \) we have an affine variety
\[
D_f = \text{MaxSpec } R[1/f]_0.
\]
To get a finite atlas, take only homogeneous generators of \( R \). To see the gluing condition, notice that \( D_{fg} \) is a principal open subset in both \( D_f \) and \( D_g \). The compatibility conditions on triple overlaps are of set-theoretic nature, and are clearly satisfied. \( \square \)

§6.4. Separatedness. There is one annoying phenomenon that we can discuss now and then safely ignore later on. One can take two copies of \( A^{1} \) and glue them along \( D(x) = A^{1} \setminus \{0\} \). This produces a famous "line with two origins" (draw). What’s happening here is that diagonally embedded \( D(x) \) is not closed in the product of charts (draw), compare with how \( P^{1} \) is glued (draw). So we give

6.4.1. Definition. An algebraic variety is called separated if it has an affine atlas such that for any pair \( A, B \) of charts with \( C = A \cap B \), the diagonal inclusion of \( C \) in \( A \times B \) is a closed subset of the product.

How to check this in practice?

6.4.2. Lemma. Suppose any two affine charts \( A \) and \( B \) with \( C = A \cap B \) have the following property: there exists \( f \in \mathcal{O}(A) \) and \( g \in \mathcal{O}(B) \) such that
\[
\mathcal{O}(C) = \mathcal{O}(A)_f = \mathcal{O}(B)_g \subset K.
\]
Then \( X \) is separated iff for any \( A \) and \( B \), we have
\[
\mathcal{O}(C) \text{ is generated by } \mathcal{O}(A) \text{ and } \mathcal{O}(B).
\]
In particular, \( \text{Proj } R \) is separated.

Proof. We have \( \mathcal{O}(A \times B) = \mathcal{O}(A) \otimes_k \mathcal{O}(B) \) (why?), and the diagonal map \( \Delta : C \to A \times B \) is given by a homomorphism
\[
\Delta^* : \mathcal{O}(A) \otimes_k \mathcal{O}(B) \to \mathcal{O}(C), \quad f \otimes g \mapsto \frac{f}{1} \cdot \frac{g}{1}.
\]
The closure \( \overline{\Delta(C)} \) of the diagonal is defined by the kernel of \( \Delta^* \). In particular, its algebra of functions is \( \mathcal{O}(A)\mathcal{O}(B) \subset \mathcal{O}(C) \). So \( X \) is separated iff this inclusion is an equality for any pair of charts.

The last remark follows from the formula
\[
(R_{fg})_0 = (R_f)_0(R_g)_0, \quad (6.4.3)
\]
which we leave as an exercise. \( \square \)

§6.5. Veronese embedding. We now have two models for \( \mathbb{P}(1, 1, 2) \): as a weighted projective plane defined by charts and as a quadratic cone in \( \mathbb{P}^3 \). What is the relationship between these models? We are going to show that in fact any \( \text{Proj } R \) is a projective variety.

6.5.1. Definition. If \( R \) is a graded ring then its subring \( R^{(d)} = \sum_{d|n} R_n \) is called a \( d \)-th Veronese subring.

For example, for \( \mathbb{P}(1, 1, 2) \) the second Veronese subring is generated by \( x^2, xy, y^2, \) and \( z \), subject to a single quadratic relation. So \( \text{Proj } R^{(2)} \) is a quadratic cone in \( \mathbb{P}^3 \) in this case. The basic fact is:
6.5.2. Proposition. \( \text{Proj } R = \text{Proj } R^{(d)} \) for any \( d \).

Proof. First of all, we have \((\text{Quot } R)_0 = (\text{Quot } R^{(d)})_0\). Indeed, any fraction \( a/b \in (\text{Quot } R)_0 \) can be written as \( ab^{d-1}/b^d \in (\text{Quot } R^{(d)})_0 \).

Let \( f_1, \ldots, f_r \) be homogeneous generators of \( R \), so that \( \text{Proj } R \) is covered by charts \( D_{f_i} \). Then \( f_1^{(d)}, \ldots, f_r^{(d)} \in R^{(d)} \) are not necessarily generators, however \( \text{Proj } R^{(d)} \) is still covered by charts \( D_{f_i^{(d)}} \). Indeed, if all \( f_i^{(d)} \) vanish at some point \( p \in \text{Proj } R^{(d)} \) then also any function in the ideal generated by them (and hence any function in its radical) vanishes at \( p \). But any generator \( g \) of \( R^{(d)} \) can be expressed as a polynomial in \( f_1, \ldots, f_r \), and therefore a sufficiently high power of \( g \) belongs to the ideal (in \( R^{(d)} \)) generated by \( f_1^{(d)}, \ldots, f_r^{(d)} \). So we have

\[
\text{Proj } R = \bigcup_{i=1}^r D_{f_i} \quad \text{and} \quad \text{Proj } R^{(d)} = \bigcup_{i=1}^r D_{f_i^{(d)}}
\]

The basic local calculation we need is that charts \( D_{f_i} \) of \( \text{Proj } R \) and \( D_{f_i^{(d)}} \) of \( \text{Proj } R^{(d)} \) can be identified, i.e. that

\[
R^{(d)}[1/f_1^{(d)}]_0 \simeq R[1/f]_0
\]

for any homogeneous element \( f \) of \( R \). But indeed,

\[
\frac{g}{f_i} = \frac{f_j^{d-1}g}{f_j^{d}}
\]

as soon as \( dj > i \). So \( \text{Proj } R \) and \( \text{Proj } R^{(d)} \) have the same charts glued in the same way. \( \Box \)

Now another basic algebraic fact is:

6.5.3. Lemma. For a sufficiently large \( d \), \( R^{(d)} \) is generated by \( R_d \).

Proof. Let \( a_1, \ldots, a_r \) be degrees of homogeneous generators \( f_1, \ldots, f_r \) of \( R \). Let \( a = \text{l.c.m.}(a_1, \ldots, a_r) \) and let \( d = ra \). We claim that this \( d \) works. For each \( i \), let \( a = a_i b_i \); then

\[
\text{deg } f_i^{b_i} = a.
\]

Now take any element \( f \in R_{kd} \). We claim that it can be written as a polynomial in elements of \( R_d \). It suffices to consider a monomial \( f = f_1^{n_1} \cdots f_r^{n_r} \). For inductive purposes, notice that \( \text{deg } f = kd = (kr)a \). If \( n_i < b_i \) for each \( i \) then

\[
\text{deg } f < ra = d,
\]

a contradiction. So we can write \( f = f_i^{b_i} g \), where \( \text{deg } g = \text{deg } f - a \). Continuing inductively, we will write

\[
f = [f_1^{b_1} \cdots f_r^{b_r}] g,
\]

where \( \text{deg } g = d \) and degree of the first term is a multiple of \( d \). Since \( \text{deg } f_i^{b_i} = a \) for each \( i \), we can group elements of the first term into groups of \( r \) powers each of degree \( d \). This shows that \( f \) can be written as a polynomial in elements of \( R_d \). \( \Box \)
By the lemma we can realize \( \text{Proj} R \) as a subvariety in \( \mathbb{P}^N \) for a sufficiently large \( N \). Indeed, \( \text{Proj} R \simeq \text{Proj} R^{(d)} \) and
\[
R^{(d)} = \mathbb{C}[y_0, \ldots, y_N]/I,
\]
where \( I \) is a homogeneous ideal (in the usual sense). So
\[
\text{Proj} R^{(d)} = V(I) \subset \mathbb{P}^N.
\]

6.5.4. **Corollary.** \( \text{Proj} R \) is a projective variety.

We are going to spend a considerable amount of time studying the moduli space $M_2$ of algebraic curves of genus 2. Incidentally, this will also give us the moduli space $A_2$ of principally polarized Abelian surfaces: those are algebraic surfaces isomorphic to $\mathbb{C}^2/\Lambda$, where $\Lambda \cong \mathbb{Z}^4$ is a lattice. So Abelian surfaces are naturally Abelian groups just like elliptic curves. We will see that $M_2$ embeds in $A_2$ as an open subset (via the Jacobian construction) and the complement $A_2 \setminus M_2$ parametrizes split Abelian surfaces of the form $E_1 \times E_2$, where $E_1$ and $E_2$ are elliptic curves. The map $M_g \hookrightarrow A_g$ can be constructed in any genus (its injectivity is called the Torelli theorem) but the dimensions are vastly different:

$$\dim M_g = 3g - 3 \quad \text{and} \quad \dim A_g = \frac{g(g + 1)}{2}.$$  

The characterization of $M_g$ as a sublocus of $A_g$ is called the Shottky problem.

§7.1. Genus 2 curves: analysis of the canonical ring. Let’s start with a basic Riemann–Roch analysis of a genus 2 curve $C$. We fix a canonical divisor $K$. We have

$$\deg K = 2 \times g - 2 = 2 \quad \text{and} \quad l(K) = g = 2.$$  

So we can assume that

$$K \geq 0$$

is an effective divisor. by Riemann–Roch, for any point $P \in C$,

$$l(K - P) - l(K - (K - P)) = 1 - 2 + \deg(K - P) = 0.$$  

Since $l(P) = 1$ (otherwise $C$ is isomorphic to $\mathbb{P}^1$), we have $l(K - P) = 1$. So $|K|$ has no fixed part, and therefore gives a degree 2 map

$$\phi_{|K|} : C \to \mathbb{P}^1.$$  

By Riemann–Hurwitz, it has 6 ramification points called Weierstrass points. We also see that $C$ admits an involution permuting two branches of $\phi_{|2K|}$. It is called the hyperelliptic involution.

Now consider $|3K|$. By Riemann–Roch, we have $l(3K) = 5$ and $l(3K - P - Q) = 3$ for any points $P, Q \in C$. It follows that $|3K|$ is very ample and gives an embedding

$$C \hookrightarrow \mathbb{P}^4.$$  

To get a bit more, we observe that most of geometry of $C$ is nicely encoded in the canonical ring

$$R(K) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nK).$$  

We can give a more general definition:

7.1.1. Definition. Let $D \geq 0$ be an effective divisor on a curve $C$. Its graded algebra is defined as follows:

$$R(D) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nD).$$
This is a graded algebra: notice that if $f \in \mathcal{L}(aD)$ and $g \in \mathcal{L}(bD)$ then

$$(fg) + (a+b)D = (f) + aD + (g) + bD \geq 0,$$

so $fg \in \mathcal{L}(a+b)D$.

7.1.2. REMARK. We have only defined divisors on curves in this class, but in principle it is no harder to defined a graded algebra of any divisor on an algebraic variety of any dimension. The canonical ring $R(K)$ of a smooth variety of dimension $n$ was a subject of a really exciting research in the last 30 years which culminated in the proof of a very important theorem of Siu and Birkar–Cascini–Hacon–McKernan: $R(K)$ is a finitely generated algebra. This does not sound like much, but it allows us to define $\text{Proj } R(K)$, the so-called canonical model of $X$. It is easy to see that it depends only on the field of rational functions $\mathbb{C}(X)$. In the curve case, $C$ is uniquely determined by its field of functions, by in dimension $> 1$ it is easy to modify a variety without changing its field of rational functions (e.g. by blow-ups). So it is very handy to have this canonical model of the field of rational functions. There exists a sophisticated algorithm, called the Minimal Model Program, which (still conjecturally) allows one to construct the canonical model by performing a sequence of basic “surgeries” on $X$ called divisorial contractions and flips.

We can compute the Hilbert function of $R(K)$ by Riemann–Roch:

$$h_n(R(K)) = l(nK) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ 5 & \text{if } n = 3 \\ 2n - 1 & \text{if } n \geq 2. \end{cases}$$

Let’s work out the generators. $\mathcal{L}(0) = \mathbb{C}$ is generated by 1. This is a unity in $R(K)$. Let $x_1, x_2$ be generators of $\mathcal{L}(K)$. One delicate point here is that we can (and will) take $x_1$ to be $1 \in \mathbb{C}(C)$, but it should not be confused with a previous 1 because it lives in a different degree in $R(K)$! In other words, $R(K)$ contains a graded polynomial subalgebra $\mathbb{C}[x_1]$, where any power $x_1^n$ is equal to 1 as a rational function on $C$.

Any other element of first degree has pole of order 2 at $K$ (because if it has a pole of order 1, it would give an isomorphism $C \simeq \mathbb{P}^1$.

A subalgebra $S = \mathbb{C}[x_1, x_2]$ of $R$ is also a polynomial subalgebra: if we have some homogeneous relation $f(x_1, x_2)$ of degree $d$ then we have

$$f(x_1, x_2) = \prod_{i=1}^{d} (\alpha_i x_1 + \beta_i x_2) = 0 \quad \text{in } \mathbb{C}(C),$$

which implies that $\alpha_i x_1 + \beta_i x_2 = 0$ for some $i$, i.e. that $x_1$ and $x_2$ are not linearly independent, contradiction.
The Hilbert function of $S$ is

$$h_n(S) = \begin{cases} 
1 & \text{if } n = 0 \\
2 & \text{if } n = 1 \\
3 & \text{if } n = 2 \\
4 & \text{if } n = 3 \\
n & \text{if } n \geq 2.
\end{cases}$$

So the next generator we need for $R(K)$ is a generator $y$ in degree 3.

What happens in degree 4? We need 7 elements and we have 7 elements

$$x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4, yx_1, yx_2.$$ 

We claim that they are indeed linearly independent, and in fact we claim:

7.1.3. LEMMA. There is no linear relation in $\mathbb{C}(C)$ of the form

$$yf_k(x_1, x_2) = f_{k+3}(x_1, x_2),$$

where the lower index is the degree. In particular, $R(K)$ is generated by $x_1, x_2, y$.

Proof. Suppose the linear relation of the form above exists. Then $y$, as a rational function on $C$, is a rational function $f(x_1, x_2)$. One can show that this is impossible either by an elementary analysis of possible positions of roots of $y$ and this rational function $f(x_1, x_2)$ or by simply invoking the fact that as we already know $3K$ is very ample, and in particular functions in $|3K|$ separate points of $C$. But if $y$ is a rational function in $x_1$ and $x_2$ then $y$ takes the same values on two points from each fiber of $\phi_{|2K|}$. \hfill $\Box$

It follows that

7.1.4. LEMMA. $R(K)$ is isomorphic to a polynomial algebra in $x_1, x_2, y$ modulo a relation

$$y^2 = f_6(x_1, x_2),$$

where $f_6$ is a polynomial of degree 6.

Proof. We already know that $R(K)$ is generated by $x_1, x_2, y$, and that $y \notin \mathbb{C}(x_1, x_2)$. It follows that $y^2, y\mathbb{C}[x_1, x_2]_3$, and $\mathbb{C}[x_1, x_2]_6$ are linearly dependent in $R(K)_6$ and this gives the only relation in $R(K)$:

$$y^2 = yf_3(x_1, x_2) + f_6(x_1, x_2).$$

We can make a change of variables $y' = y - \frac{1}{2}f_3$ to complete the square, which brings the relation in the required form. \hfill $\Box$

§7.2. Graded algebra of an ample divisor. Now let’s interpret these algebraic results geometrically. The basic fact is:

7.2.1. LEMMA. If $D$ is an ample divisor on a curve $C$ then Proj $R(D) = C$.

Proof. If $D$ is very ample and $R(D)$ is generated by $R(D)_1$ then $R(D)$ is isomorphic to a polynomial algebra in $x_0, \ldots, x_N \in \mathcal{L}(D)$ modulo the relations that they satisfy, i.e. $R(D) = \mathbb{C}[x_0, \ldots, x_N]/I$, where $I$ is a homogeneous ideal of $C \subset \mathbb{P}^N$. So in this case clearly Proj $R(D) = C$. In general, if $D$ is ample then $kD$ is very ample for some $k > 0$. Also, we know by Lemma 6.5.3 that the Veronese subalgebra $R(ID) = R(D)^{(l)}$ is generated
by its first graded piece for some $l > 0$. So $klD$ is a very ample divisor and $R(klD) = R(\mathcal{O}_C)$ is generated by its first graded piece. Then we have $\text{Proj} R(D) = \text{Proj} R(klD) = C$. We are not using here that $C$ is a curve, so if you know your divisors in higher dimension, everything works just as nicely.

As a corollary, we have

7.2.2. COROLLARY. Let $C$ be a genus 2 curve. Then $R(K)$ induces an embedding

$$C \subset \mathbb{P}(1, 1, 3)$$

and the image is defined by an equation

$$y^2 = f_6(x_1, x_2).$$

(7.2.3)

The embedding misses a singularity of $\mathbb{P}(1, 1, 3)$ (where $x_1 = x_2 = 0$, $y = 1$). In the remaining two charts of $\mathbb{P}(1, 1, 3)$, the curve is given by equations

$$y^2 = f_6(1, x_2) \quad \text{and} \quad y^2 = f_6(x_1, 1).$$

The projection onto $\mathbb{P}_{[x_1 : x_2]}^1$ is a bicanonical map $\phi_{2K}$ and roots of $f_6$ are branch points of this $2 : 1$ cover. In particular, $f_6$ has no multiple roots and any equation of the form (7.2.3) defines a genus 2 curve.

The tricanonical embedding $C \subset \mathbb{P}^4$ factors through the Veronese embedding

$$\mathbb{P}(1, 1, 3) \hookrightarrow \mathbb{P}^4, \quad (x_1, x_2, x_3, y) \mapsto [x_1^3 : x_1^2 x_2 : x_1 x_2^2 : x_2^3 : y],$$

where the image is a projectivized cone over a rational normal curve.

This sets up a bijection between curves of genus 2 and unordered 6-tuples of distinct points $p_1, \ldots, p_6 \in \mathbb{P}^1$ modulo $\text{PGL}_2$. We are going to use this to construct $M_2$. The classical way of thinking about 6 unordered points in $\mathbb{P}^1$ is to identify them with roots of a binary form $f_6(x_1, x_2)$ of degree 6. Let $V_6$ be a vector space of all such forms and let $D \subset \mathbb{P}(V_6)$ be the discriminant hypersurface (which parameterizes binary sextics with multiple roots). Thus we have (set-theoretically):

$$M_2 = (\mathbb{P}(V_6) \setminus D)/\text{PGL}_2.$$
More systematically, the procedure is as follows. Suppose a group $G$ acts on a projective variety $X$. Suppose we can write $X = \text{Proj } R$, where $R$ is some finitely generated graded algebra. This is called a choice of polarization. Suppose we can find an action of $G$ on $R$ that induces an action of $G$ on $X$. This is called a choice of linearization. Then we can form a GIT quotient $X//G = \text{Proj } R^G$.

In the example above, $\mathbb{P}^2 = \text{Proj } \mathbb{C}[x_1, x_2, x_3]$, and

$$\mathbb{P}^2//S_3 = \text{Proj } \mathbb{C}[x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_1x_3, x_1x_2x_3] = \mathbb{P}(1, 2, 3).$$

We will use this construction to describe $M_2$.

§7.4. Classical invariant theory of a binary sextic. We have to describe the algebra $R = O(V_6)^{\text{SL}_2}$ of $\text{SL}_2$-invariant polynomial functions for the linear action of $\text{SL}_2$ on $V_6$. The classical convention for normalizing the coefficients of a binary form is to divide coefficients by the binomial coefficients:

$$f_6 = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$ 

Explicit generators for $R$ were written down in the 19-th century by Clebsch, Cayley, and Salmon. We are not going to prove that they indeed generate the algebra of invariants but let’s discuss them to see how beautiful the answer is. Let $p_1, \ldots, p_6$ denote the roots of the dehomogenized form $f_6(x, 1)$ and write $(ij)$ as a shorthand for $p_i - p_j$. Then we have the following generators (draw some graphs):

$$I_2 = a^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2$$

$$I_4 = a^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$I_6 = a^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$D = I_{10} = a^{10} \prod_{i<j} (ij)^2$$

$$I_{15} = a^{15} \sum_{\text{fifteen}} ((14)(36)(52) - (16)(32)(54)).$$

Here the summations are chosen to make the expressions $S_6$-invariant. In particular, they can all be expressed as polynomials in $\mathbb{C}[a, b, c, d, e, f, g]$, for example

$$I_2 = -240(ag - 6bf + 15ce - 10d^2).$$

(7.4.1)

Here is the main theorem:

7.4.2. THEOREM. The algebra $R = O(V_6)^{\text{SL}_2}$ is generated by invariants $I_2, I_4, I_6, I_{10},$ and $I_{15}$. The subscript is the degree. Here $D = I_{10}$ is the discriminant which vanishes iff the binary form has a multiple root. The unique irreducible relation among the invariants is

$$I_{15}^3 = G(I_2, I_4, I_6, I_{10}).$$

Now we use our strategy to construct $M_2$.
• Compute $V_6/\SL_2 = \MaxSpec R$ first. By 19-th century, this is 
\[ C[I_2, I_4, I_6, I_{10}, I_{15}]/(I_{15}^2 = G(I_2, I_4, I_6, I_{10})). \]
• Now quotient the result by $C^*$, i.e. compute $\Proj R$. Here we have a magical simplification: $\Proj R = \Proj R^{(2)}$ but the latter is generated by $I_2, I_4, I_6, I_{10}$, and $I_{15}^2$. Since $I_{15}^2$ is a polynomial in other invariants, in fact we have 
\[ \Proj R^{(2)} = \Proj C[I_2, I_4, I_6, I_{10}] = \Proj(2, 4, 6, 10) = \Proj(1, 2, 3, 5). \]
• To get $M_2$, remove a hypersurface $D = 0$, i.e. take the chart $D_{I_{10}}$ of $\Proj(2, 4, 6, 10)$. Since $I_{15}$ is a polynomial in other invariants, in fact we have 
\[ \Proj R^{(2)} = \Proj C[I_2, I_4, I_6, I_{10}] = \Proj(2, 4, 6, 10) = \Proj(1, 2, 3, 5). \]
• One can show that $C[A, B, C]^{\mu_5}$ has 8 generators. So as an affine variety, we have 
\[ M_2 = \A^3/\mu_5, \] where $\mu_5$ acts with weights $1, 2, 3$.
• One can show that $C[A, B, C]^{\mu_5}$ has 8 generators. So as an affine variety, we have 
\[ M_2 = \Proj(V_6) \setminus D/\PGL_2 \hookrightarrow \A^3, \]
\[ \{y^2 = f(x)\} \mapsto \left( \frac{I_2^2}{I_{10}}, \frac{I_2 I_4}{I_{10}}, \frac{I_4}{I_{10}}, \frac{I_2 I_6}{I_{10}}, \frac{I_4 I_6}{I_{10}}, \frac{I_6^2}{I_{10}}, \frac{I_6}{I_{10}} \right). \]

This of course leaves more questions then gives answers:

(1) How do we know that points of $M_2$ correspond to isomorphism classes of genus 2 curves? In other words, why is it true that our quotient morphism 
\[ \Proj(V_6) \setminus D \to \A^3/\mu_5 \]
is surjective and separates $\PGL_2$-orbits? It is of course very easy to give examples of quotients by infinite group actions that do not separate orbits.

(2) Can one prove the finite generation of the algebra of invariants and separation of orbits by the quotient morphism without actually computing the algebra of invariants?

(3) Is $M_2$ a coarse moduli space (and what is a family of genus 2 curves)?

(4) Our explicit description of $M_2$ as $\A^3/\mu_5$ shows that it is singular. Which genus 2 curves contribute to singularities?

(5) Our construction gives not only $M_2$ but also its compactification by $\Proj R$. Can we describe the boundary $\Proj R \setminus M_2$?

(6) Are there other approaches to the construction of $M_2$?

§7.5. Homework.

Problem 1. Let $C \subset \P^d$ be a rational normal curve of degree $d$, let $\hat{C} \subset \A^{d+1}$ be the affine cone over it, and let $\bar{C} \subset \P^{d+1}$ be its projective closure. Show that $\hat{C}$ is isomorphic to $\P(1, 1, d)$ (1 point).

Problem 2. Let $P \in \P^1$ be a point. Let $R = \bigoplus_{k \geq 0} \mathcal{L}(kP)$ be the associated ring. Describe the projective embedding of $\P^1$ given by the $d$-th Veronese subalgebra of $R$ (1 point).

Problem 3. Show that $I_2$ (see (7.4.1)) is indeed an $\SL_2$-invariant polynomial. (2 points)
Problem 4. A weighted projective space \( \mathbb{P}(a_0, \ldots, a_n) \) is well-formed if no \( n \) of the weights \( a_0, \ldots, a_n \) have a common factor. For example, \( \mathbb{P}(1, 1, 3) \) is well-formed but \( \mathbb{P}(2, 2, 3) \) is not. Consider the polynomial ring \( R = \mathbb{C}[x_0, \ldots, x_n] \), where \( x_i \) has weight \( a_i \). (a) Suppose that \( d = \gcd(a_0, \ldots, a_n) \). Show that \( R^{(d)} = R \) and that \( \mathbb{P}(a_0, \ldots, a_n) \isom \mathbb{P}(a_0/d, \ldots, a_n/d) \). (b) Suppose that \( d = \gcd(a_1, \ldots, a_n) \) and that \( (a_0, d) = 1 \). Compute \( R^{(d)} \) and show that \( \mathbb{P}(a_0, \ldots, a_n) \isom \mathbb{P}(a_0, a_1/d, \ldots, a_n/d) \). Conclude that any weighted projective space is isomorphic to a well-formed one (2 points).

Problem 5. Compute \( \text{Proj} \mathbb{C}[x, y, z]/(x^3 + y^3 + z^2) \). Here \( x \) has weight 12, \( y \) has weight 20, and \( z \) has weight 30 (1 point).

Problem 6. Using the fact that \( M_2 = \mathbb{A}^3/\mu_5 \), where \( \mu_5 \) acts with weights \( 1, 2, 3 \), construct \( M_2 \) as an affine subvariety of \( \mathbb{A}^5 \) (1 point).

Problem 7. Let \( V_4 \) be the space of degree 4 binary forms. Show that \( \mathcal{O}(V_4)^{\text{SL}_2} \) is a polynomial algebra generated by invariants of degrees 2 and 3 (hint: use Problem 4 from the previous homework). (3 points).

Problem 8. (a) Prove (6.2.1). (b) Prove (6.4.3) (1 point).

Problem 9. Let \( P \in E \) be a point on an elliptic curve. (a) Compute \( \text{Proj} R(P) \) and the embedding of \( E \) in it. (b) Compute \( \text{Proj} R(2P) \) and the embedding of \( E \) in it. (2 points)

Problem 10. Let \( P \in E \) be a point on an elliptic curve. Show that \( \phi_{|4P|} \) embeds \( E \) in \( \mathbb{P}^3 \) as a complete intersection of two quadrics (i.e. the homogeneous ideal of \( E \) in this embedding is generated by two quadrics) (2 points).

Problem 11. Show that any genus 2 curve \( C \) can be obtained as follows. Start with a line \( l \subset \mathbb{P}^3 \). Then one can find a quadric surface \( Q \) and a cubic surface \( S \) containing \( l \) such that \( Q \cap S = l \cup C \) (2 points).

Problem 12. Assuming that \( M_2 = \mathbb{A}^3/\mu_5 \) set-theoretically, define families of curves of genus 2 (analogously to families of elliptic curves), and show that \( M_2 \) is a coarse moduli space (2 points).

Problem 13. Assuming the previous problem, show that \( M_2 \) is not a fine moduli space (2 points).

Problem 14. Show that \( \mathbb{A}^n \setminus \{0\} \) is not an affine variety for \( n > 1 \) (1 point).

Problem 15. Suppose \( X \) and \( Y \) are separated algebraic varieties. Explain how to define \( X \times Y \) as an algebraic variety and show that it is separated (2 points).

Problem 16. (a) Show that an algebraic variety \( X \) is separated if and only if the diagonal \( X \) is closed in \( X \times X \). (b) Show that a topological space \( X \) is Hausdorff if and only if the diagonal \( X \) is closed in \( X \times X \) equipped with a product topology. (c) Explain how (a) and (b) can be both true but \( \mathbb{A}^1 \) is both separated and not Hausdorff (2 points).

Problem 17. Use affine charts to show that \( G(2, n) \) is an algebraic variety without using the Plücker embedding (1 point).

Problem 18. Consider rays \( R_1, \ldots, R_k \subset \mathbb{R}^2 \) emanating from the origin, having rational slopes, going in the counter-clockwise direction, and spanning the angle \( 2\pi \) once. Suppose that each angle \( R_i R_{i+1} \) (and \( R_i R_1 \)) is less than \( \pi \). This is called a (two-dimensional) fan. The angles \( R_i R_{i+1} \) (and
$R_k R_1$) are called (top-dimensional) cones of the fan. Rays themselves are also (one-dimenesional) cones. The origin is a zero-dimensional cone. Now for each cone $\sigma$ of the fan, consider the semigroup $\Lambda = \sigma \cap \mathbb{Z}^2$ and the dual semigroup

$$\Lambda^\perp = \{(u, v) \in \mathbb{Z}^2 \mid u_i + v_j \geq 0 \text{ for any } (i, j) \in \Lambda \} \subset \mathbb{Z}^2.$$ 

Let $K$ be the field $\mathbb{C}(x, y)$. We can think about an element $(i, j) \in \mathbb{Z}^2$ as a Laurent monomial $x^i y^j$. This gives us algebras $\mathbb{C}[\sigma] \subset K$ spanned by monomials in $\Lambda^\perp$. (a) Show that for each inclusion of cones $\tau \subset \sigma$, MaxSpec $\mathbb{C}[\tau]$ is a principal open subset in MaxSpec $\mathbb{C}[\sigma]$. (b) Show that one can glue all MaxSpec $\mathbb{C}[\sigma]$ together. This is called a toric surface. (c) Show that weighted projective planes are toric surfaces (3 points).

Problem 19. An algebraic curve is called bielliptic if it admits a $2 : 1$ morphism $C \to E$ onto an elliptic curve; the covering transformation is called a bielliptic involution. Let $C$ be a genus 2 curve. (a) Show that if $C$ is bielliptic then its bielliptic involution commutes with its hyperelliptic involution. (b) Show that $C$ is bielliptic if and only if the branch locus $p_1, \ldots, p_6 \in \mathbb{P}^1$ of its bi-canonical map has the following property: there exists a $2 : 1$ morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(p_1) = f(p_2)$, $f(p_3) = f(p_4)$, and $f(p_5) = f(p_6)$. (c) Show that (b) is equivalent to the following: if we realize $\mathbb{P}^1$ as a conic in $\mathbb{P}^1$ then lines $p_1 p_2$, $p_3 p_4$, and $p_5 p_6$ all pass through a point (3 points).
§8. GIT quotients and stability.

Let’s summarize where we stand. We want to construct $M_2$ as an orbit space for

$$\text{SL}_2 \text{ acting on } \mathbb{P}(V_6) \setminus D.$$ 

We use our standard approach using invariants. The classical invariant theory tells us that $\mathcal{O}(V_6)^{\text{SL}_2}$ is generated by $I_2, I_4, I_6, I_{10} = D$, and $I_{15}$ with a single quadratic relation $I^2_{15} = g(I_2, I_4, I_6, I_{10})$.

So our natural candidate for the quotient is $\text{Proj } \mathcal{O}(V_6)^{\text{SL}_2}$, and the quotient map is

$$f \mapsto [I_2(f) : \ldots : I_{15}(f)] \in \mathbb{P}(2, 4, 6, 10, 15).$$

Here we got lucky: since $\text{Proj } R = \text{Proj } R^{(2)}$, we can also write the quotient map as

$$f \mapsto [I_2(f) : \ldots : I_{10}(f)] \in \mathbb{P}(2, 4, 6, 10) = \mathbb{P}(1, 2, 3, 5).$$

Since there are no relations between $I_2, \ldots, I_{10}$ we actually expect the quotient to be $\mathbb{P}(1, 2, 3, 5)$.

If we throw away the vanishing locus of the discriminant, we get the affine chart

$$\{D \neq 0\} \subset \mathbb{P}(1, 2, 3, 5).$$

So our hope is that

$$M_2 = \mathbb{A}^3 / \mu_5,$$

where $\mu_5$ acts with weights $1, 2, 3$. We’ve seen that if we want to embed this cyclic quotient singularity in the affine space, we need at least $\mathbb{A}^8$.

Of course this construction alone does not guarantee that each point of $\mathbb{A}^3 / \mu_5$ corresponds to a genus 2 curve and that different points correspond to different curves: this is something we are trying to work out in general.

§8.1. Algebraic representations of reductive groups. Consider any representation

$$G \to \text{GL}(V).$$

We want to define the quotient of $\mathbb{P}(V)$ by $G$, or even more generally a quotient of any projective variety by some action of $G$.

8.1.1. Definition. Suppose that $G$ is a group and an affine algebraic variety such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are regular. Then $G$ is called a linear algebraic group.

Examples:

- $\text{GL}_n = D(\det) \subset \text{Mat}_n$,
- $\text{SL}_n = V(\det -1) \subset \text{Mat}_n$,
- “the maximal torus” (diagonal matrices in $\text{GL}_n$),
- “the Borel subgroup” (upper-triangular matrices in $\text{GL}_n$),
- $\text{SO}_n, \text{Sp}_n$,
- finite groups.
8.1.2. Remark. The terminology “linear algebraic group” can be explained by a theorem of Chevalley: any linear algebraic group is isomorphic to a (Zariski closed) subgroup of $GL_n$ for some $n$. And vice versa, it is clear that any such subgroup is linear algebraic. If we remove “affine” from the definition of an algebraic group, then there are other possibilities, for example an elliptic curve (or an Abelian surface or more generally an Abelian variety) is a projective algebraic group.

Non-examples:

- $SL_2(\mathbb{Z})$ and other non-finite discrete groups.
- $SU_n \subset SL_n(\mathbb{C})$ and other non-finite compact linear Lie groups.

In fact, we can show that

8.1.3. Lemma. $SU_n$ is Zariski dense in $SL_n(\mathbb{C})$.

Proof. Indeed, let $f$ be any regular function on $SL_n(\mathbb{C})$ that vanishes on $SU_n$. We have to show that it vanishes on $SL_n(\mathbb{C})$. It is equally easy to show this for any function holomorphic in the neighborhood of $I \in SL_n(\mathbb{C})$. Consider the exponential map

$$\exp : Mat_n(\mathbb{C}) \to GL_n(\mathbb{C}), \quad A \mapsto \exp(A) = I + A + \frac{A^2}{2} + \ldots$$

This map is biholomorphic in the neighborhood of the origin (the inverse map is given by $\log$) and (locally) identifies $SL_n(\mathbb{C})$ with a complex vector subspace $sl_n$ of complex matrices with trace 0 and $SU_n$ with a real subspace $su_n$ of skew-Hermitian matrices (i.e. matrices such that $A + A^t = 0$). So $g = f(\exp(A))$ is a function holomorphic near the origin which vanishes on $su_n$. But since $su_n + i su_n = sl_n$, this function vanishes on $sl_n$ as well: indeed, the kernel of its differential at any point of $su_n$ contains $su_n$, and therefore contains $sl_n$ (being a complex subspace). So all partial derivatives of $g$ vanish along $su_n$, and continuing by induction all higher-order partial derivatives of $g$ vanish along $su_n$. So $g$ is identically zero by Taylor’s formula. □

8.1.4. Definition. A finite-dimensional representation of a linear algebraic group is called algebraic (or rational) if the corresponding homomorphism $G \to GL(V)$ is a regular morphism. In other words, an algebraic representation is given by a homomorphism

$$G \to GL(V), \quad g \mapsto \begin{pmatrix} a_{11}(g) & \cdots & a_{1n}(g) \\ \vdots & \ddots & \vdots \\ a_{n1}(g) & \cdots & a_{nn}(g) \end{pmatrix},$$

where $a_{ij} \in \mathcal{O}(G)$.

We can generalize this definition to non-linear actions of $G$ on any algebraic variety $X$: the action is called algebraic if the “action” map

$$G \times X \to X$$

is a regular map. Why is this definition the same as above for the linear action? (explain). Finally, we need a notion of a linearly reductive group.
8.1.5. THEOREM. An algebraic group is called linearly reductive if it satisfies any of the following equivalent conditions:

1. Any finite-dimensional algebraic representation $V$ of $G$ is completely reducible, i.e. is a direct sum of irreducible representations.

2. For any finite-dimensional algebraic representation $V$ of $G$, there exists a $G$-equivariant projector $\pi_V : V \to V^G$ (which is then automatically unique).

3. For any surjective linear map $A : V \to W$ of algebraic $G$-representations, the induced map $V^G \to W^G$ is also surjective.

Proof. (1) $\Rightarrow$ (2). Decompose $V = V_1 \oplus \ldots \oplus V_k$. Suppose $G$ acts trivially on the first $r$ sub-representations and only on them. Let $U \subset V$ be an irreducible subrepresentation. By Schur’s lemma, its projection on any $V_i$ is either an isomorphism or a zero map. It follows that if $U$ is trivial, it is contained in $V_1 \oplus \ldots \oplus V_r$ and if it is not trivial, it is contained in $V_{r+1} \oplus \ldots \oplus V_k$. So in fact we have a unique decomposition $V = V^G \oplus V_0$, where $V_0$ is the sum of all non-trivial irreducible subrepresentations. The projector $V \to V^G$ is the projector along $V_0$.

(2) $\Rightarrow$ (3). Suppose that the induced map $V^G \to W^G$ is not onto. Choose $w \in W^G$ not in the image of $V^G$ and choose any projector $W^G \to \langle w \rangle$ that annihilates the image of $V^G$. Then the composition $V \xrightarrow{A} W \xrightarrow{\pi_W} W^G \to \langle w \rangle$ is a surjective $G$-invariant linear map $f : V \to \mathbb{C}$ that annihilates $V^G$. After dualizing, we have a $G$-invariant vector $f \in V^*$ which is annihilated by all $G$-invariant linear functions on $V^*$. However, this is nonsense: we can easily construct a $G$-invariant linear function on $V^*$ which does not annihilate $f$ by composing a $G$-invariant projector $V^* \to (V^*)^G$ (which exists by (2)) with any projector $(V^*)^G \to \langle f \rangle$.

(3) $\Rightarrow$ (1). It is enough to show that any sub-representation $W \subset V$ has an invariant complement. Here we get sneaky and apply (2) to the restriction map of $G$-representations $\text{Hom}(V, W) \to \text{Hom}(W, W)$. The $G$-invariant lift of $\text{Id} \in \text{Hom}(W, W)$ gives a $G$-invariant projector $V \to W$ and its kernel is a $G$-invariant complement of $W$. □

Next we study finite generation of the algebra of invariants. Suppose we have a finite-dimensional representation $G \to \text{GL}(V)$. We are looking for criteria that imply that $\mathcal{O}(V)^G$ is a finitely generated algebra. In fact, we already know (Lemma 5.2.4) that it is enough to show existence of a Reynolds operator $\mathcal{O}(V) \to \mathcal{O}(V)^G$.

8.1.6. LEMMA. The Reynolds operator exists for any algebraic finite-dimensional representation $G \to \text{GL}(V)$ of a linearly reductive group. In particular, $\mathcal{O}(V)^G$ in this case is finitely generated.
Proof. \( \mathcal{O}(V) \) is an algebra graded by degree and each graded piece \( \mathcal{O}(V)_n \) has an induced representation of \( G \). By linear reductivity, there exists a unique \( G \)-invariant linear projector \( R_n : \mathcal{O}(V)_n \to \mathcal{O}(V)_n^G \) for each \( n \). We claim that this gives a Reynolds operator \( R \). The only thing to check is that

\[
R(f g) = f R(g)
\]

for any \( f \in \mathcal{O}(V)^G \). Without loss of generality we can assume that \( f \in \mathcal{O}(V)^G_m \) and \( g \in \mathcal{O}(V)^G_m \). Then \( f \mathcal{O}(V)_n \) is a \( G \)-sub-representation in \( \mathcal{O}(V)^G_{m+n} \), so its \( G \)-invariant projector \( f R(g) \) should agree with a \( G \)-invariant projector \( R(f g) \) on \( \mathcal{O}(V)^G_{m+n} \). \( \square \)

§8.2. Finite Generation Theorem via the unitary trick.

8.2.1. Lemma. Any algebraic representation of \( \text{SL}_n \) is completely reducible, i.e. \( \text{SL}_n \) is a linearly reductive group.

Proof. We will use a unitary trick introduced by Weyl (and Hurwitz). An algebraic representation of \( \text{SL}_n \) induces a continuous representation of \( \text{SU}_n \), and any sub-representation of \( \text{SU}_n \) is in fact a sub-representation for \( \text{SL}_n \) by Lemma 8.1.3 (explain).

So it is enough to show that any continuous complex representation \( \text{SU}_n \to \text{GL}(V) \) is completely reducible. There are two ways to prove this. One is to use the basic lemma above and to construct an equivariant projector \( V \to V^G \) for any finite-dimensional continuous representation. Just like in the case of finite groups, one can take any projector \( p : V \to V^G \) and then take it average

\[
\pi(v) = \frac{\int_{\text{SU}_n} p(g v) \, d\mu}{\int_{\text{SU}_n} \, d\mu}.
\]

Here \( \mu \) should be an equivariant measure on \( \text{SU}_n \) (a so-called Haar measure), and then of course we would have to prove its existence. We will follow a more naive approach.

We claim that \( V \) has an \( \text{SU}_n \)-invariant positive-definite Hermitian form. If this is true then for any complex subrepresentation \( U \subset V \), the orthogonal complement \( U^\perp \) is also \( \text{SU}_n \)-invariant, and we will keep breaking \( V \) into pieces until each piece is irreducible. To show the claim, consider the induced action of \( \text{SU}_n \) on all Hermitian forms \( (\cdot, \cdot) \) on \( V \) by change of variables. Let \( S \) be the set of positive-definite Hermitian forms. The action of \( \text{SU}_n \) preserves the set \( S \), which is convex because any positive linear combination of positive-definite Hermitian forms is positive-definite. So it is enough to prove the following Lemma. \( \square \)

8.2.2. Lemma. Let \( S \subset \mathbb{R}_n \) is a convex set preserved by a compact subgroup \( K \) of the group of affine transformations (i.e. compositions of linear transformations and translations) of \( \mathbb{R}^n \). Then \( K \) has a fixed point on \( S \).

Proof. Without loss of generality we can assume that \( S \) is convex and compact. Indeed, since \( K \) is compact, any \( K \)-orbit in \( S \) is compact as well (being the image of \( K \) under a continuous map). The convex hull \( S' \) of this \( K \)-orbit is therefore a compact, convex, and \( G \)-invariant subset of \( S \). A \( K \)-fixed point in \( S' \) will of course also be a \( K \)-fixed point in \( S \).
If the minimal affine subspace containing $S$ is not the whole of $\mathbb{R}^n$ (draw the picture), then take this linear span $\mathbb{R}^k$ instead of $\mathbb{R}^n$ (since $K$ preserves $S$, it also preserves its affine span). $K$ clearly has an induced action there by affine transformations).

Now let $p$ be the center of mass of $S$ with coordinates

$$p_i = \frac{\int_S x_i \, dV}{\int_S \, dV}$$

(here $dV$ is the standard measure on $\mathbb{R}^k$). Since $S$ is convex, the Riemann sum definition of the integral shows that $p \in S$ and that $p$ is preserved by any affine transformation of $\mathbb{R}^n$ that preserves $S$. So, $p$ is fixed by $K$. □

8.2.3. Corollary. For any finite-dimensional algebraic representation $V$ of $\text{SL}_n$, the algebra of invariants $\mathcal{O}(V)^{\text{SL}_n}$ is finitely generated.

§8.3. Surjectivity of the quotient map. Let’s consider any algebraic finite-dimensional representation of a linearly reductive group $G \to \text{GL}(V)$. The existence of the Reynolds operator alone implies surjectivity of the quotient map $V \to \text{MaxSpec} \mathcal{O}(V)^G$

see Theorem 5.3.3. However, this is not the quotient that we want!

8.3.1. Definition. Let’s fix the following terminology. If $G$ acts on the affine variety $X$ then we call $X//G = \text{MaxSpec} \mathcal{O}(X)^G$ the affine quotient. If $G$ acts on the projective variety $X = \text{Proj} R$ and this action lifts to the action of $G$ on $R$ then we will call $X/G = \text{Proj} R^G$ the GIT quotient. In most of our examples $X = \mathbb{P}(V)$ and the action of $G$ comes from the linear action of $G$ on $V$. Often we will just restrict to this case.

These constructions are related. For example, the principal open subset of $\mathbb{P}(V)/\text{SL}_2$ (isomorphic to $M_2$) is the affine quotient $U//\text{SL}_2$, where $U = \{D \neq 0\} \subset \mathbb{P}(V)$ ($D = I_{10}$ is the discriminant).

Let’s generalize. Take any algebraic representation $G \to \text{GL}(V)$ and the induced action of $G$ on $\mathbb{P}(V)$ (even more generally, we can take any algebraic action of $G$ on a graded algebra $R$ without zero-divisors and the induced action of $G$ on $\text{Proj} R$). Our quotient is given by $\text{Proj} \mathcal{O}(V)^G$ (even more generally, we take $\text{Proj} R^G$). On charts this quotient looks as follows. Let $f_1, \ldots, f_n \in \mathcal{O}(V)^G$ be homogeneous generators (or in fact any homogeneous elements that generate the irrelevant ideal after taking the radical). For example, for $M_2$ we can take $I_2, I_4, I_6, D = I_{10}$. We have quotient maps $D(f_i) = \text{MaxSpec}(\mathcal{O}(V)_{f_i}), 0 \to D'(f_i) = \text{MaxSpec}(\mathcal{O}(V)^G_{f_i}, 0$ induced by the inclusion $\mathcal{O}(V)^G \hookrightarrow \mathcal{O}(V)$.

Notice that many elements of $\mathbb{P}(V)$ go missing in the quotient! 8.3.2. Definition. A point $x \in \mathbb{P}(V)$ (more generally, $x \in \text{Proj} R$) is called unstable if all $G$-invariant polynomials (more generally, functions in $R^G$) of positive degree vanish on $x$. Let $\mathcal{N} \subset \mathbb{P}(V)$ be the locus of unstable points (also known as the null-cone). Caution: points that are not unstable are called semistable.
We see that the GIT quotient is actually the map
\[ \mathbb{P}(V) \setminus \mathcal{N} \to \text{Proj} \mathcal{O}(V)^G \]
(more generally, $\text{Proj} R \setminus \mathcal{N} \to \text{Proj} R^G$) obtained by gluing quotients on charts. We will return to the question of describing the unstable locus later.

But now, let’s finish our discussion of surjectivity of the GIT quotient. On charts, we have affine quotients. Let’s study them first. We start with a very important observation

8.3.3. THEOREM. Consider the regular action of a linear algebraic group $G$ on an affine variety $X$. Then

1. Any function $f \in \mathcal{O}(X)$ is contained in a finite-dimensional sub-representation $U \subset \mathcal{O}(X)$.
2. There exists an algebraic representation of $G$ on a vector space $V$ and a $G$-invariant closed embedding $X \hookrightarrow V$.

Proof. The action $G \times X \to X$ induces a homomorphism (called the co-action)
\[ \phi : \mathcal{O}(X) \to \mathcal{O}(G \times X) = \mathcal{O}(G) \otimes \mathbb{C} \mathcal{O}(X), \]
\[ f \mapsto \sum_{i=1}^{k} \alpha \otimes f_i. \]

Quite literally, this means that $f(gx) = \sum_{i=1}^{k} \alpha(g) f_i(x)$ for $g \in G, x \in X$.

For example, $f$ is in the linear span of $f_i$’s (take $g = 1$). And of course any $f(gx)$ with fixed $g$ is in the linear span of $f_i$’s. But this implies that the linear span $U \subset \mathcal{O}(X)$ of all functions $f(gx)$ for $g \in G$ is finite-dimensional. It is clear that $U$ is $G$-invariant.

For part (b), we first use (a) to show that $\mathcal{O}(X)$ contains a finite-dimensional $G$-invariant subspace $V^*$ that contains generators of $\mathcal{O}(X)$. Realizing $\mathcal{O}(X)$ as the quotient algebra of the polynomial algebra $\mathcal{O}(V)$ gives a required embedding $X \hookrightarrow V$. \qed

Now we can finally work out surjectivity of the quotient map.

8.3.4. THEOREM. Consider the regular action of a linearly reductive group $G$ on an affine variety $X$. Then

1. There exists a Reynolds operator $\mathcal{O}(X) \to \mathcal{O}(X)^G$.
2. $\mathcal{O}(X)^G$ is finitely generated, let $X//G$ be the affine quotient.
3. The quotient map $X \to X//G$ is surjective.

Proof. (1) $\mathcal{O}(X)$ is a sum of finite-dimensional representations of $G$. So the Reynolds operator is defined just like in the case of $\mathcal{O}(V)$, the only difference is that there is no grading to naturally break the algebra into finite-dimensional pieces. But since the projector $U \to U^G$ is unique, we will have a well-defined projector $\mathcal{O}(X) \to \mathcal{O}(X)^G$.

(2) and (3) now follow from the existence of the Reynolds operator just like in the case of the group acting on a vector space (the linearity was not used anywhere). \qed
8.3.5. Corollary. Any point of \( h^3/\mu_5 \) represents a genus 2 curve.

§8.4. Separation of orbits. We look at the affine case first.

8.4.1. Theorem. Let \( G \) be a linearly reductive group acting regularly on the affine variety \( X \). Let \( O, O' \) be two \( G \)-orbits. Then TFAE

1. The closures of \( O \) and \( O' \) have a common point.
2. There exists a sequence of orbits \( O = O_1, \ldots, O_n = O' \) such that the closures of \( O_i \) and \( O_{i+1} \) have a common point for any \( i \).
3. \( O \) and \( O' \) are not separated by \( G \)-invariants in \( \mathcal{O}(X)^G \).

In particular, every fiber of \( \pi : X \to X//G \) contains exactly one closed orbit.

Proof. (1) \( \Rightarrow \)(2) \( \Rightarrow \)(3) are clear (since any \( G \)-invariant regular function takes the same value on any orbit and its closure). We have to show (3) \( \Rightarrow \)(1).

Let \( I, I' \subset \mathcal{O}(X) \) be ideals of \( O, O' \). They are clearly \( G \)-invariant subspaces. Suppose that \( \bar{O} \cap \bar{O}' = \emptyset \). By Nullstellensatz, this implies that

\[ I + I' = \mathcal{O}(X), \]

i.e. we can write \( 1 = f + g \), where \( f \in I \) and \( g \in I' \). Now apply the Reynolds operator: \( R(f) + R(g) = 1 \). We claim that \( R(f) \in I \) (and similarly \( R(g) \in I' \)). This implies that \( R(f) \) is an invariant function which is equal to 0 on \( O \) and 1 on \( O' \), i.e. these orbits are separated by invariants. But the claim is clear: recall that the Reynolds operator \( \mathcal{O}(V) \to \mathcal{O}(V)^G \) is obtained by simply gluing all projectors \( U \to U^G \) for all finite-dimensional subrepresentations \( U \subset \mathcal{O}(V) \). In particular, \( R \) preserves any \( G \)-invariant subspace of \( \mathcal{O}(V) \), for example \( I \) and \( I' \).

\[ \square \]

Now we can finally describe \( M_2 \):

8.4.2. Theorem. There are natural bijections (described previously) between

1. isomorphism classes of genus 2 curves;
2. \( \text{SL}_2 \) orbits in \( \mathbb{P}(V_6) \setminus D \);
3. points in \( h^3/\mu_5 \) acting with weights 1, 2, 3.

Proof. The only thing left to check is that all \( \text{SL}_2 \) orbits in \( \mathbb{P}(V_6) \setminus D \) are closed. But this is easy: for any orbit \( O \) and any orbit \( O' \neq O \) in its closure, \( \dim O' < \dim O \). However, all \( \text{SL}_2 \) orbits in \( \mathbb{P}(V_6) \setminus D \) have the same dimension 3, because the stabilizer can be identified with a group of projective transformations of \( \mathbb{P}^1 \) permuting roots of the binary sextic, which is a finite group if all roots are distinct (or even if there are at least three distinct roots).

This gives a pretty decent picture of the quotient \( \mathbb{P}(V_6)/\text{SL}_2 \), at least in the chart \( D \neq 0 \), which is the chart we mostly care about. To see what’s going on in other charts, let’s experiment with generators \( I_2, I_4, I_6, I_{10} \) (defined in §7.4). Simple combinatorics shows that (do it):

- if \( f \in V_6 \) has a root of multiplicity 4 then \( f \) is unstable.
- if \( f \in V_6 \) has a root of multiplicity 3 then all basic invariants vanish except (potentially) \( I_2 \).

So we should expect the following theorem:
8.4.3. Theorem. Points of \( \mathbb{P}(V_6)/\mathrm{SL}_2 = \mathbb{P}(1, 2, 3, 5) \) correspond bijectively to \( \mathrm{GL}_2 \)-orbits of degree 6 polynomials with at most a double root (there can be several of them) plus an extra point \([1 : 0 : 0 : 0]\), which has the following description. All polynomials with a triple root (but no fourtuple root) map to this point in the quotient. The corresponding orbits form a one-parameter family (draw it), with a closed orbit that corresponds to the polynomial \( x^3y^3 \).

To prove this theorem, it is enough to check the following facts:

1. Any unstable form \( f \) has a fourtuple root (or worse). In other words, semistable forms are the forms that have at most triple roots.
2. A semistable form \( f \in \mathbb{P}(V_6) \) has a finite stabilizer unless \( f = x^3y^3 \) (this is clear: this is the only semistable form with two roots).
3. Any semistable form \( f \) without triple roots has a closed orbit in the semistable locus in \( \mathbb{P}(V_6) \), and hence in any principal open subset \( D_+(I) \), it belongs to, where \( I \) is one of the basic invariants. Notice that we do not expect \( f \) to have a closed orbit in the whole \( \mathbb{P}(V_6) \), in fact one can show that there is only one closed orbit there, namely the orbit of \( x^6 \).
4. If \( f \) has a triple root then it has the orbit of \( x^3y^3 \) in its closure. Indeed, suppose \( f = x^3g \), where \( g = y^3 + ay^2x + byx^2 + cx^3 \) is a cubic form (it has to start with \( y^3 \), otherwise \( f \) has a fourtuple root). Let’s act on \( f \) by a matrix \( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \). We get \( x^3y^3 + at^2y^2x + bt^4yx^5 + ct^6x^6 \). So as \( t \to 0 \), we get \( x^3y^3 \) in the limit.

To check (1) and (3), one can use an exceptionally useful numerical criterion also known as Hilbert–Mumford criterion, which we will discuss in the next section. First some definitions to wrap up our discussion:

8.4.4. Definition. Suppose a linearly reductive group \( G \) acts regularly on an affine variety \( X \). A point \( x \in X \) is called stable if two conditions are satisfied:

- The \( G \)-orbit of \( x \) is closed.
- The stabilizer of \( x \) is finite.

For example, any point in \( \mathbb{P}(V_6) \setminus D \) is stable. On the other hand, \( x^3y^3 \) is an example of a semistable but not stable point in \( \mathbb{P}(V_6) \setminus \{I_2 = 0\} \).

8.4.5. Remark. The concept of stability is very general and applies in many circumstances. In moduli questions, we call objects stable if their isomorphism class gives a point in the moduli space. Unstable objects have to be discarded in the moduli space. Semistable but not stable objects survive in the moduli space but a single point in the moduli space can correspond to several isomorphism classes. The concept of stability is not intrinsic to objects but depends on how we construct the moduli space, i.e. which objects we want to consider as isomorphic and how we define families of objects. One can change “stability conditions” and get a different moduli space (variation of moduli spaces). If we construct a moduli space as an orbit space for some group action, then stability can be defined as Mumford’s GIT-stability discussed above.
§8.5. Unstable locus: Hilbert–Mumford criterion. Let’s focus on linear actions on linearly reductive groups.

8.5.1. Definition. The group $T = (\mathbb{C}^*)^n$ is called an algebraic torus.

8.5.2. Theorem. An algebraic torus $T$ is a linearly reductive group. In fact, any algebraic representation of $T$ is diagonalizable and is isomorphic to a direct sum of one dimensional irreducible representations $V_\chi$, where $\chi : T \to GL_1(\mathbb{C}) = \mathbb{C}^*$ is an algebraic character. Any character has a form

$$ (z_1, \ldots, z_n) \mapsto z_1^{m_1} \cdots z_n^{m_n} $$

for some vector $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$.

Proof. One can prove this just like for $SL_n$. The analogue of $SU_n$ (the maximal compact subgroup) will be the real torus $(S^1)^n \subset (\mathbb{C}^*)^n$, where

$$ S^1 = \{z, \ |z| = 1\} \subset \mathbb{C}^*. $$

Just for fun, let’s give a different proof, although close in spirit. We have $O(T) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is the algebra of Laurent polynomials. Let $\mu \subset \mathbb{C}^*$ be the subgroup of all roots of unity. Being infinite, it is Zariski dense in $\mathbb{C}^*$. And in fact, $\mu^n \subset (\mathbb{C}^*)^n$ (the subgroup of all torsion elements) is also Zariski dense. Indeed, if

$$ f(z_1, \ldots, z_n) = \sum_i g_i(z_1, \ldots, z_{n-1})z_i^n $$

vanishes on $\mu^n$ then all coefficients $g_i$ must vanish on $\mu^{n-1}$ (why?), hence they are identically zero by inductive assumption. Now any representation $T \to GL(V)$ restricts to a representation $\mu^n \to GL(V)$. The image consists of commuting matrices of finite order, hence can be simultaneously diagonalizable. But then the image of $T$ is diagonalizable as well. As for the description of one-dimensional representations, notice that a character $T \to \mathbb{C}^*$ is a non-vanishing regular function on $T$. We can write it as a Laurent monomial multiplied by a polynomial $f(z_1, \ldots, z_n)$ which does not vanish in $(\mathbb{C}^*)^n$. Therefore, its vanishing locus in $\mathbb{A}^n$ is a union of coordinate hyperplanes. By factoriality of the ring of polynomials (and Nullstellensatz), it follows that $f$ is a monomial multiplied by a constant. Since $f(1, \ldots, 1) = 1$, this constant is equal to 1. 

8.5.3. Definition. Let $G$ be a connected linearly reductive group. An algebraic subgroup $T \subset G$ is called a maximal torus if $T \simeq (\mathbb{C}^*)^n$ and $T$ is maximal by inclusion among this kind of subgroups.

8.5.4. Theorem. All maximal tori in $G$ are conjugate.

We won’t need this theorem, so we won’t prove it. But notice that this is clear if $G = SL_n$. Indeed, any algebraic torus $T \subset G$ will be diagonalizable in some basis of $\mathbb{C}^n$, which means that (after conjugation by a change of basis matrix), $T$ is contained in a subgroup $\text{diag}(z_1, \ldots, z_n)$, where $z_1 \cdots z_n = 1$. So this subgroup is the maximal torus and any other maximal torus is conjugate to it.

The Hilbert–Mumford criterion consists of two parts: reduction from $G$ to $T$ and analysis of stability for torus actions. First the reduction part.
8.5.5. Theorem. Consider any finite-dimensional representation \( G \to \text{GL}(V) \) of a linearly reductive group. Let \( T \subset G \) be a maximal torus, and let \( v \in V \). TFAE:

1. \( v \) is unstable, i.e. any homogeneous polynomial \( f \in \mathcal{O}(V)^G \) of positive degree vanishes on \( v \).
2. The \( G \)-orbit \( Gv \) contains \( 0 \) in its closure.
3. There exists \( u \in Gv \) such that the \( T \)-orbit \( Tu \) contains \( 0 \) in its closure.

Proof. It is clear that (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1). Theorem 8.4.1 shows that (1) \( \Rightarrow \) (2).

We will do a difficult implication (2) \( \Rightarrow \) (3) only for \( G = \text{SL}_n \). Recall that any matrix \( A \in \text{SL}_n \) has a polar decomposition

\[ A = UP, \]

where \( U \in K = \text{SU}_n \) and \( P \) is positive-definite Hermitian matrix. By spectral theorem, \( P \) has an orthonormal basis of eigenvectors, so we can write

\[ P = U'D(U')^{-1}, \]

where \( U' \in K \) and \( D \) is a diagonal matrix. So combining these fact, we have a useful decomposition

\[ G = KTK. \]

By hypothesis, \( 0 \in \overline{Gv} \) (Zariski closure). We show in the exercises that in fact we also have \( 0 \in \overline{Gv} = \overline{KTKv} \) (the closure in real topology). Since \( K \) is compact, this implies that

\[ 0 \in \overline{TKv} \]

(the closure in real topology). Consider the quotient map

\[ \pi_T : V \to V//T \]

and let

\[ O = \pi_T(0). \]

\( \pi_T \) is continuous in real topology (since polynomials are continuous functions), so we have

\[ O \in \pi_T(\overline{TKv}) \Rightarrow O \in \overline{\pi_T(\overline{TKv})} = \overline{\pi_T(\overline{Kv})}. \]

But by compactness,

\[ \overline{\pi_T(\overline{Kv})} = \pi_T(\overline{Kv}), \]

and therefore there exists \( g \in K \) such that \( \pi_T(gv) = O \), i.e. \( 0 \in \overline{Tu} \) for \( u = gv \). \( \square \)

Now we have to investigate the unstable locus for the torus action. Consider any finite-dimensional representation \( T = (\mathbb{C}^*)^n \to \text{GL}(V) \). Let \( V = \bigoplus_{m \in \mathbb{Z}^n} V_m \) be the decomposition of \( V \) into \( T \)-eigenspaces.

8.5.6. Definition. For any \( u \in V \), let \( u = \sum u_m \) be the decomposition of \( u \) into \( T \)-eigenvectors. Then

\[ NP(u) = \text{Convex Hull}\{m \in \mathbb{Z}^n | u_m \neq 0\} \]

is called the \textit{Newton polytope} of \( u \).

\[ ^{23}\text{An analogous decomposition holds in any connected complex linearly reductive group with } K \text{ its maximal connected compact subgroup.} \]
Before we state a general theorem, let’s look at some examples of Newton polytopes and its relation to stability.

8.5.7. **Example.** Finish the description of unstable elements in $V_6$.

8.5.8. **Example.** Consider the action of $\text{SL}_3$ on degree 2 polynomials in three variables. Describe unstable Newton polygons, show that they correspond to singular conics. This relates nicely to the fact that $O(V_2)^{\text{SL}_3}$ is generated by a single invariant (discriminant).

8.5.9. **Theorem.** Consider any finite-dimensional algebraic representation $T = (\mathbb{C}^*)^n \to \text{GL}(V)$. Let $v \in V$. TFAE:

- Weight of $v$ is a vector of nonzero coordinates.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r \leq \dim V$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r = \dim V$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r > \dim V$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r < \dim V$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r = \dim V$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r > \dim V$.
- $v$ is a sum of monomials $x_{i_1} \cdots x_{i_r}$ with $r < \dim V$.
(1) \( v \) is unstable.
(2) \( 0 \not\in NP(v) \).

Proof. The game here is based on the fact that there are two ways to describe convexity: using positive linear combinations or using supporting hyperplanes. More precisely, we have the following well-known Lemma, which goes by names of Farkas’ Lemma, Gordan Theorem, etc.

8.5.10. Lemma. Let \( S \subset \mathbb{R}^n \) be a convex hull of finitely many lattice points \( v_1, \ldots, v_k \in \mathbb{Z}^n \). Then

- If \( 0 \not\in S \) if and only if there exists a vector \( u \in \mathbb{Z}^n \) such that \( u \cdot v_i > 0 \) for any \( i \).
- If \( 0 \in S \) if and only if there exist rational numbers \( \alpha_1, \ldots, \alpha_k \geq 0 \) such that \( 0 = \sum \alpha_i v_i \) and \( \sum \alpha_i = 1 \).

Now we can prove the Theorem. If \( 0 \not\in NP(v) \) then by Lemma we can choose a vector \( u = (u_1, \ldots, u_n) \in \mathbb{Z}^n \) such that \( u \cdot v_i > 0 \) for any \( i \). Consider a subgroup \( \chi(t) = (tu_1, \ldots, tu_n) \subset T \). Then we have

\[
\chi(t) \cdot v = \sum_{m \in \mathbb{Z}^n} \chi(t) \cdot v_m = \sum_{m} t^m u_m.
\]

We see that

\[
\lim_{t \to \infty} \chi(t) \cdot v = 0.
\]

On the other hand, let’s suppose that \( 0 \in NP(v) \). By lemma, we can choose rational numbers \( \alpha_m \geq 0 \) indexed by \( m \) such that \( v_m \neq 0 \), and not all of them equal to 0, such that \( 0 = \sum \alpha_m v_m \). By rescaling, we can assume that all these numbers are integers. Choose linear functions \( f_m \) on \( V \) for any \( m \) such that \( v_m \neq 0 \). We can assume that \( f_m(v_m) = 1 \). But now consider the function

\[
I = \prod f_m^{\alpha_m}.
\]

We have \( I(v) = 1 \) and \( I \) is \( T \)-invariant. Therefore, 0 is not in the closure of \( Tv \).

\[\square\]

§8.6. Hypersurfaces. We will discuss some examples when stability is easy to verify. Let \( G \) be a reductive group acting on the affine variety \( X \) with the quotient

\[
\pi : X \to X//G = \text{MaxSpec } \mathcal{O}(X)^G.
\]

Let

\[
X^s \subset X
\]

be the set of stable points and let

\[
Z \subset X
\]

be the subset of points such that \( G_x \) is not finite.

8.6.1. Theorem. \( Z \) is closed, \( X^s \) is open, \( X_s \) is the complement in \( X \) of \( \pi^{-1}(\pi(Z)) \). The quotient \( \pi \) induces a \( 1-1 \) bijection between \( G \)-orbits in \( X^s \) points in \( \pi(X^s) \).
Proof. Consider the map
\[ G \times X \to X \times X, \ (g,x) \mapsto (gx,x). \]
Let \( \tilde{Z} \) be the preimage of the diagonal it is closed. But then \( Z \) is the locus of points in \( X \) where the fibers of \( \pi_2|Z \) have positive dimension. Thus \( Z \) is closed, by semi-continuity of dimension of fibers.

Next we claim that \( \pi(Z) \) is closed. Here we are only going to use the fact that \( Z \) is closed, thus \( Z = V(I) \) for some \( I \subset \mathcal{O}(X) \). Then
\[ \overline{\pi(Z)} = V(\pi^*(I)) = I \cap \mathcal{O}(X)^G = I^G. \]
But the exact sequence
\[ 0 \to I \to \mathcal{O}(X) \to \mathcal{O}(X)/I \to \]
stays exact after taking \( G \)-invariants (this is one of the equivalent definitions of reductivity), so we have
\[ \mathcal{O}(Z)^G = \mathcal{O}(X)^G/I^G. \]
Thus the map \( \pi|_Z : Z \to \overline{\pi(Z)} \) can be identified with the quotient map
\[ Z \to Z//G, \]
which as we know is surjective.

Now suppose \( x \in \pi^{-1}(\pi(Z)) \). Then the fiber of \( \pi \) through \( x \) contains a closed orbit with a positive-dimensional stabilizer. Thus either \( Gx \) is not closed or \( Gx \) is positive-dimensional. In any case \( x \) is not stable.

If \( x \notin \pi^{-1}(\pi(Z)) \) then \( Gx \) is finite. If \( Gx \) is not closed then a closed orbit in the closure of \( Gx \) also does not belong to \( \pi^{-1}(\pi(Z)) \), which is a contradiction. So in fact \( x \) is stable. \( \square \)

We are going to prove a classical theorem of Matsumura, Monsky, and Mumford (MMM) concerning stability of smooth hypersurfaces. On the group-theoretic side, we consider the representation of \( \text{SL}_{n+1} \) in the vector space
\[ V_{n,d} = \text{Sym}^d(\mathbb{C}^{n+1})^* \]
which parametrizes polynomials of degree \( d \) in \( n + 1 \) variables. Let
\[ U_{n,d} \subset V_{n,d} \]
be the locus such that the corresponding hypersurface in \( \mathbb{P}^n \) is non-singular, and let \( D_{n,d} \) be the complement, the discriminant set.

8.6.2. Theorem. \( D_{n,d} \) is an irreducible hypersurface of degree \( (n+1)(d-1)^n \). Its defining equation \( D_{n,d} \) (called the discriminant) belongs to \( \mathcal{O}(V_{n,d})^\text{SL}_{n+1} \).

Proof. The proof is by dimension count. Consider the incidence subset \( Z \subset \mathbb{P}(V_{n,d}) \times \mathbb{P}^n \) of pairs \((F,z)\) such that \( z \in \text{Sing}(F = 0) \). This is a closed subset (defined by vanishing of partials \( F_0, \ldots, F_n \)). All fibers of its projection onto \( \mathbb{P}^n \) are irreducible (in fact they are projective spaces) and have dimension \( \dim \mathbb{P}(V_{n,d}) - n - 1 \) (why?). Therefore \( Z \) is irreducible (in fact smooth) and \( \dim Z = \dim \mathbb{P}(V_{n,d}) - 1 \) by the Theorem on dimension of fibers. Notice that the projectivization of \( D_{n,d} \) is the image of \( Z \). Thus \( D_{n,d} \) is irreducible and to count its dimension it suffices to show that a general hypersurface
singular at $z \in \mathbb{P}^n$ is singular only there: this would imply in fact that the first projection
\[ Z \rightarrow \mathcal{D}_{n,d} \]
is birational (this is a very useful resolution of singularities of the discriminant locus). But this is easy: just take a smooth hypersurface $S \subset \mathbb{P}^{n-1}$ (for example $x_1^d + \ldots + x_n^d$). The cone over it has only one singular point (the vertex).

We won’t need the degree, but here is a quick calculation in case you are wondering. Take a general pencil $aF + bG$ of degree $d$ hypersurfaces. We have to count the number of singular hypersurfaces in this pencil. A general singular hypersurface has a unique singularity, so we might just as well count the total number of singular points of hypersurfaces in the pencil. Those points are intersection points of $n+1$ “partial derivatives hypersurfaces” $aF_i + bG_i = 0$, $i = 0, \ldots, n$ that intersect transversally (why?). Quite generally, if $X$ is a smooth projective algebraic variety of dimension $n$ then it has intersection theory. Its easiest incarnation is that for any $n$ divisors (integral combinations of irreducible hypersurfaces) $D_1, \ldots, D_n$ one can compute the intersection number
\[ D_1 \cdot \ldots \cdot D_n \]
which has two basic properties:

- If the hypersurfaces intersect transversally then this is the number of intersection points.
- If $D_i \equiv D'_i$ are linearly equivalent divisors (i.e. there exists $f \in k(X)$ such that $(f) = D_i - D'_i$) then $D_1 \cdot \ldots \cdot D_n = D'_1 \cdot \ldots \cdot D'_n$.

Using this in our case, each partial derivatives hypersurface is linearly equivalent to a hypersurface
\[ L + (d-1)H = \{az_0^{d-1} = 0\}. \]

So the intersection number is
\[ (L + (d-1)H)^{n+1} = (n+1)(d-1)^n \]
by the binomial formula.

Finally, the discriminant hypersurface is obviously $\text{SL}_{n+1}$-invariant, and so the action of $\text{SL}_{n+1}$ can only multiply it by a character. But $\text{SL}_{n+1}$ has no non-trivial characters.

So we can mimic the construction of $M_2$ and consider

- The GIT quotient $\mathbb{P}(V_{n,d})/\text{SL}_{n+1}$ which compactifies its principal open subset
- $(\mathbb{P}(V_{n,d}) \setminus \mathcal{D}_{n,d})/\text{SL}_{n+1}$, the moduli space of non-singular hypersurfaces.

In fact, in direct analogy with the case of $V_6$, we claim that

8.6.3. Theorem (Matsumura–Monsky–Mumford). Let $d \geq 3$. Then all smooth hypersurfaces are stable, i.e. the principal open subset above parametrizes smooth hypersurfaces in $\mathbb{P}^n$ modulo projective transformations.
Proof. The first step is to reduce the question to a commutative algebra problem. It suffices to prove that any point in \( \mathbb{P}(V_{n,d}) \setminus D_{n,d} \) has a finite stabilizer in \( \text{SL}_{n+1} \), or, which is the same, that any point \( F \) in \( V_{n,d} \setminus D_{n,d} \) has a finite stabilizer in \( \text{GL}_{n+1} \) (why?) . If the stabilizer is infinite then the orbit map 

\[ \text{GL}_{n+1} \to V_{n,d}, \quad A \mapsto AF = F(\sum a_{0i}x_i, \ldots, \sum a_{ni}x_i) \]

has positive dimensional fibers. This implies that its differential is not injective. Computing it by the chain rule, we find linear forms 

\[ l_0 = \sum a_{0i}x_i, \ldots, l_n = \sum a_{ni}x_i \]

such that 

\[ l_0F_0 + \ldots + l_nF_n \]

is a zero polynomial. Without loss of generality we can assume that \( l_0 \neq 0 \). Let \( I \subset \mathbb{C}[x_0, \ldots, x_n] \) be the ideal generated by \( F_1, \ldots, F_n \). Then we have 

\[ \sqrt{(F_0, I)} = (x_0, \ldots, x_n) \]

(because \( F = 0 \) is non-singular), 

\[ L_0 \notin I \]

(by degree reasons, this is where we use \( d \geq 3 \)), and 

\[ L_0F_0 \in I. \]

We claim that this is impossible by some very cool commutative algebra, which we are going to remind. We won’t give a self-contained proof, but everything can be found in [AM] (or even better, in [Ma]). □

8.6.4. Definition. An ideal \( Q \subset R \) is called primary if 

\[ xy \in Q \quad \Rightarrow \quad x \in Q \quad \text{or} \quad y^n \in Q \quad \text{for some } n. \]

It is easy to see (why?) that the radical of a primary ideal is a prime ideal. If \( \sqrt{Q} = P \) then we say that \( Q \) is \( P \)-primary.

8.6.5. Theorem. Any ideal \( I \) in a Noetherian ring \( R \) has a primary decomposition 

\[ I = Q_1 \cap \ldots \cap Q_r, \]

where \( Q_i \)'s are primary, and no \( Q_i \) contains the intersection of the remaining ones (this would make it redundant). 

Intersection of \( P \)-primary ideals is \( P \)-primary, so we can and will assume that the radicals \( P_i \) of \( Q_i \)'s are different. 

The prime ideals \( P_i \)'s are called associated primes of \( I \). One has 

- A prime ideal \( P \subset R \) is associated to \( I \) if and only if \( P = (I : x) \) for some \( x \in R \). 
- The union of associated primes is given by elements which are zero-divisors modulo \( I \): 

\[ P_1 \cup \ldots \cup P_r = \{ x \in R \mid (I : x) \neq I \}. \] (8.6.6)
What is the geometry behind this? We have
\[ \sqrt{I} = P_1 \cap \ldots \cap P_r, \quad \text{where} \quad P_i = \sqrt{Q_i}. \]
This decomposition corresponds to breaking \( V(I) = V(\sqrt{I}) \) into irreducible components with one annoying caveat: it could happen that some \( P_i \)'s are actually redundant, i.e. \( P_i \supset P_j \) for some \( j \neq i \). If this happens then we call \( P_i \) an embedded prime. Otherwise we call \( P_i \) a minimal (or isolated prime). Here is an example of a primary decomposition when this happens:
\[ (x^2, xy) = (x) \cap (x, y)^2. \]
Here \( (x) \) is a minimal prime and \( (x, y) \) is an embedded prime (draw the picture). It’s nice to explore situations when embedded primes don’t appear. We have the following fundamental

8.6.7. Theorem (Macaulay’s unmixedness theorem). Let
\[ I = (f_1, \ldots, f_r) \subset k[x_0, \ldots, x_n] \]
be an ideal such that \( \text{codim} V(I) = r \) (in general \( \text{codim} V(I) \leq r \)). Then \( I \) is unmixed, i.e. there are no embedded primes.

Noetherian rings that satisfy the conclusion of this theorem are called Cohen–Macaulay rings. (For the proof that this condition is equivalent to the standard definition of Cohen-Macaulay rings via depth and height of localizations, see [Ma]). So \( k[x_0, \ldots, x_n] \) is a Cohen–Macaulay ring. In fact if \( X \) is a smooth affine variety then \( \mathcal{O}(X) \) is Cohen–Macaulay.

Final step in the proof of Matsumura–Monsky. Clearly \( V(I) = V(F_1, \ldots, F_r) \) has dimension 1 (if it has a component of dimension greater than 1 then \( V(F_0, \ldots, F_r) \) has a positive dimension but it is a point). By unmixedness, \( I \) has no embedded primes, i.e. all associated primes of \( I \) have height \( r \). However, since \( F_0L_0 \in I, F_0 \) belongs to one of the associated primes of \( I \) by (8.6.5). This implies that the ideal \( (F_0, \ldots, F_r) \) is contained in the ideal of height \( r \), which contradicts the fact that \( V(F_0, \ldots, F_r) \) is a point.

This shows how useful commutative algebra is even when the basic object is just a non-singular hypersurface. I will put more exercises on unmixedness in the homework. Commutative algebra serves as a technical foundation of modern algebraic geometry through the theory of algebraic schemes.

§8.7. Homework.

In problems 1–7, describe the unstable locus for given representations.

Problem 1. The action of \( \text{SL}_n \) on \( \text{Mat}_{n,m} \) by left multiplication (1 point).

Problem 2. Show that a degree 3 form in 3 variables is semistable for the action of \( \text{SL}_3 \) if and only if the corresponding cubic curve is either smooth or has a node (1 point).

Problem 3. The action of \( \text{GL}_n \) on \( \text{Mat}_{n,n} \) by conjugation (1 point).

Problem 4. The action of \( \text{SL}_n \) on quadratic forms in \( n \) variables (1 point).

Problem 5. Show that a degree 3 form in 3 variables is semistable for the action of \( \text{SL}_3 \) if and only if the corresponding cubic curve is either smooth or has a node (1 point).
Problem 6. Show that a degree 4 form in 3 variables is semistable for the action of $\text{SL}_3$ if and only if the corresponding quartic curve in $\mathbb{P}^2$ has no triple points and is not the union of the plane cubic and an inflectional tangent line (2 points).

Problem 7. Show that a degree 3 form in 4 variables is semistable for the action of $\text{SL}_4$ if and only if all points of the corresponding cubic surface in $\mathbb{P}^3$ are either smooth, or ordinary double points, or double points $p$ such that (after a linear change of variables) the quadratic part of $F(x, y, z, 1)$ is $xy$ and the line $x = y = 0$ is not contained in $S$ (3 points).

Problem 8. Let $G$ be a finite group with a representation $\rho : G \to \text{GL}(V)$. Let $P(z) = \sum \dim O(V)^G_k z^k$ be the Poincare series of the algebra of invariants. Show that (2 points)

$$P(z) = \frac{1}{G} \sum_{g \in G} \frac{1}{\det(\text{Id} - z\rho(g))}.$$ 

Problem 9. (1 point). Let $X$ be a quasiprojective algebraic variety and let $x, y \in X$, $x \neq y$. Show that there exists an open subset $U$ of $X$ which contains both $x$ and $y$, and a function $f \in \mathcal{O}(U)$ such that $f(x) \neq f(y)$.

Problem 10. (a) Let $G$ be a finite group with a representation $\rho : G \to \text{GL}(V)$. Show that $\mathcal{O}(V)^G$ is generated by elements of the form

$$\sum_{g \in G} (g \cdot f)^d,$$

where $f$ is a linear form. (b) Show that $\mathcal{O}(V)^G$ is generated by polynomials of degree at most $|G|$ (2 points).

Problem 11. Find a genus 2 curve $C$ such that $\text{Aut} C$ contains $\mathbb{Z}_5$ and confirm (or disprove) my suspicion that this curve gives a unique singular point of $M_2$ (2 points).

Problem 12. In the proof of Theorem 8.3.3, it was left unchecked that the representation of $G$ in $V$ is algebraic. Show this (1 point).

Problem 13. Let $X$ and $Y$ be irreducible affine varieties and let $f : X \to Y$ be a morphism. (a) Show that $f$ factors as $X \to Y \times \mathbb{A}^r \to Y$, where the first map is generically finite and the second map is a projection. (b) Show that if $f$ is generically finite then $Y$ contains a principal open subset $D$ such that $f^{-1}(D) \to D$ is finite. (c) Show that the image of $f$ contains an open subset. (2 points)

Problem 14. Let $G$ be a linear algebraic group acting regularly on an algebraic variety $X$. Using the previous problem, show that any $G$-orbit is open its closure (1 point).

---

24 A hypersurface $F(x_0, \ldots, x_n) \subset \mathbb{P}^n$ has a point of multiplicity $d$ at $p \in \mathbb{P}^n$ if the following holds. Change coordinates so that $p = [0 : \ldots : 0 : 1]$. Then $F(x_0, \ldots, x_{n-1}, 1)$ should have no terms of degree less than $d$. A point of multiplicity 2 (resp. 3) is called a double (resp. triple) point. A point $p$ is called an ordinary double point if $p$ is not a double point and the quadratic part of $F(x_0, \ldots, x_{n-1}, 1)$ is a non-degenerate quadratic form.

25 It is not hard to show that these last singularities are in fact $A_2$ singularities.
Problem 15. Let $H \subset G$ be a linearly reductive subgroup of a linear algebraic group. Show that the set of cosets $G/H$ has a natural structure of an affine algebraic variety. On the other hand, show that the set of cosets $GL_n/B$, where $B$ is a subgroup of upper-triangular matrices, has a natural structure of a projective variety (2 points).

Problem 16. Let $G \to GL(V)$ be a representation of a linearly reductive group and let $\pi : V \to V//G$ be the quotient. Show that the following properties are equivalent (2 points):

- $V//G$ is non-singular at $\pi(0)$.
- $V//G$ is non-singular.
- $\mathcal{O}(V)^G$ is a polynomial algebra.
§9. Jacobians and periods

So far we have focussed on constructing moduli spaces using GIT, but there exists a completely different approach using variations of Hodge structures. I will try to explain the most classical aspect of this theory, namely the map

\[ M_g \to A_g, \quad C \mapsto \text{Jac} \ C. \]

Injectivity of this map is the classical Torelli theorem.

§9.1. Albanese torus. Let \( X \) be a smooth projective variety. We are going to integrate in this section, so we will mostly think of \( X \) as a complex manifold. Recall that we have the first homology group \( H_1(X, \mathbb{Z}) \).

We think about it in the most naive way, as a group generated by smooth oriented loops \( \gamma : S^1 \to X \) modulo relations \( \gamma_1 + \ldots + \gamma_r = 0 \) if loops \( \gamma_1, \ldots, \gamma_r \) bound a smooth oriented surface in \( X \) (with an induced orientation on loops). We then have a first cohomology group \( H^1(X, \mathbb{C}) = \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}) \).

This group can also be computed using de Rham cohomology

\[
H^1_{dR}(X, \mathbb{C}) = \left\{ \text{complex-valued 1-forms } \omega = \sum f_i \, dx_i \text{ such that } d\omega = 0 \right\} / \{ \text{exact forms } \omega = df \}.
\]

Pairing between loops and 1-forms is given by integration

\[
\int_{\gamma} \omega,
\]

which is well-defined by Green’s theorem. The fact that \( X \) is a smooth projective variety has important consequences for the structure of cohomology, most notably one has Hodge decomposition, which in degree one reads

\[
H^1_{dR}(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1},
\]

where \( H^{1,0} = H^0(X, \Omega^1) \) is the (finite-dimensional!) vector space of holomorphic 1-forms, and \( H^{0,1} = H^{1,0}^* \) is the space of anti-holomorphic 1 forms. Integration gives pairing between \( H_1(X, \mathbb{Z}) \) (modulo torsion) and \( H^0(X, \Omega^1) \), and we claim that this pairing is non-degenerate. Indeed, if this is not the case then \( \int_{\gamma} \omega = 0 \) for some fixed non-trivial cohomology class \( \gamma \in H_1 \) (modulo torsion) and for any holomorphic 1-form \( \omega \). But then of course we also have \( \int_{\gamma} \overline{\omega} = 0 \), which contradicts the fact that pairing between \( H_1(X, \mathbb{Z}) \) (modulo torsion) and \( H^1(X, \mathbb{C}) \) is non-degenerate.

It follows that we have a complex torus

\[
\text{Alb}(X) = \frac{H^0(X, \Omega^1)^*}{H_1(X, \mathbb{Z})/\text{Torsion}} = V/\Lambda = \mathbb{C}^q/\mathbb{Z}^{2q}
\]

called the Albanese torus of \( X \). \( \Lambda \) is called the period lattice and

\[
q = \dim H^0(X, \Omega^1)
\]
is called the *irregularity* of \( X \). If we fix a point \( p_0 \in X \), then we have a holomorphic Abel–Jacobi map

\[
\mu : X \to \text{Alb}(X), \quad p \mapsto \int_{p_0}^{p} \mathbf{\cdot}
\]

The dependence on the path of integration is killed by taking the quotient by periods. Moreover, for any 0-cycle \( \sum a_i p_i \) (a formal combination of points with integer multiplicities) such that \( \sum a_i = 0 \), we can define \( \mu(\sum a_i p_i) \) by breaking \( \sum a_i p_i = \sum (q_i - r_i) \) and defining

\[
\mu(\sum a_i p_i) = \sum \int_{r_i}^{q_i} \mathbf{\cdot}
\]

Again, any ambiguity in paths of integration and breaking the sum into differences disappears after we take the quotient by periods.

When \( \dim X > 1 \), we often have \( q = 0 \) (for example if \( \pi_1(X) = 0 \) or at least \( H_1(X, \mathbb{C}) = 0 \)), but for curves \( q = g \), the genus, and some of the most beautiful geometry of algebraic curves is revealed by the Abel–Jacobi map.

§9.2. **Jacobian.** Let \( C \) be a compact Riemann surface (= an algebraic curve). The Albanese torus in this case is known as the **Jacobian**

\[
\text{Jac}(C) = \frac{H^0(C, K)^*}{H^1(C, \mathbb{Z})} = \frac{V}{\Lambda} = \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}
\]

The first homology lattice \( H_1(C, \mathbb{Z}) \) has a non-degenerate skew-symmetric intersection pairing \( \gamma \cdot \gamma' \), which can be computed by first deforming loops \( \gamma \) and \( \gamma' \) a little bit to make all intersections transversal and then computing the number of intersection points, where each point comes with \( + \) or \( - \) depending on orientation of \( \gamma \) and \( \gamma' \) at this point. In the standard basis of \( \alpha \) and \( \beta \) cycles (draw), the intersection pairing has a matrix

\[
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\]

\( H^1_{dR}(C, \mathbb{C}) \) also has a non-degenerate skew-symmetric pairing given by

\[
\int_{C} \omega \wedge \omega'.
\]

We can transfer this pairing to the dual vector space \( H_1(C, \mathbb{C}) \) and then restrict to \( H_1(C, \mathbb{Z}) \). It should come at no surprise that this restriction agrees with the intersection pairing defined above. To see this concretely, let’s work in the standard basis

\( \delta_1, \ldots, \delta_{2g} = \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \)

of \( \alpha \) and \( \beta \) cycles. We work in the model where the Riemann surface is obtained by gluing the \( 4g \) gon \( \Delta \) with sides given by

\( \alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \alpha_2, \ldots \)

Fix a point \( p_0 \) in the interior of \( \Delta \) and define a function \( \pi(p) = \int_{p_0}^{p} \omega \) (integral along the straight segment). Since \( \omega \) is closed, the Green’s formula
shows that for any point \( p \in \alpha_i \), and the corresponding point \( q \in \alpha_i^{-1} \), we have
\[
\pi(q) - \pi(p) = \int_{\alpha_i} \omega.
\]
For any point \( p \in \beta_i \) and the corresponding point \( q \in \beta_i^{-1} \), we have
\[
\pi(q) - \pi(p) = \int_{\beta_i} \omega = -\int_{\alpha_i} \omega.
\]
Then we have
\[
\int_C \omega \wedge \omega' = \int_{\Delta} d\pi \wedge \omega' = \int_{\Delta} d(\pi \omega') \quad \text{(because \( \omega' \) is closed)}
\]
\[
= \int_{\partial \Delta} \pi \omega' \quad \text{(by Green’s formula)}
\]
\[
= \sum \int_{\alpha_i \cup \alpha_i^{-1}} \pi \omega' + \sum \int_{\beta_i \cup \beta_i^{-1}} \pi \omega' =
\]
\[
= - \sum \int_{\beta_i} \omega \int_{\alpha_i} \omega' + \sum \int_{\beta_i} \omega \int_{\alpha_i} \omega',
\]
which is exactly the pairing dual to the intersection pairing\(^{26}\).

Specializing to holomorphic \( 1 \)-forms gives Riemann bilinear relations

9.2.1. PROPOSITION. Let \( \omega \) and \( \omega' \) be holomorphic \( 1 \)-forms. Then
\[
\sum \left( \int_{\alpha_i} \omega \int_{\beta_i} \omega' - \int_{\beta_i} \omega \int_{\alpha_i} \omega' \right) = \int_C \omega \wedge \omega' = 0,
\]
and
\[
\sum \left( \int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega}' - \int_{\beta_i} \omega \int_{\alpha_i} \bar{\omega}' \right) = \int_C \omega \wedge \bar{\omega}'.
\]

We define a Hermitian form \( H \) on \( H^0(C, K) \) by formula
\[
i \int_C \omega \wedge \bar{\omega}'
\]
(notice an annoying \( i \) in front) and we transfer it to the Hermitian form on \( V := H^0(C, K)^* \), which we will also denote by \( H \). The imaginary part \( \text{Im} H \) is then a real-valued skew-symmetric form on \( H^0(C, K) \) (and on \( V \)).

We can view \( V \) and \( H^0(C, K) \) as dual real vector spaces using the pairing \( \text{Re} \nu(\omega) \). Simple manipulations of Riemann bilinear relations give
\[
\sum \left( \text{Re} \int_{\alpha_i} \omega \right) \left( \text{Re} \int_{\beta_i} \bar{\omega}' \right) - \left( \text{Re} \int_{\beta_i} \omega \right) \left( \text{Re} \int_{\alpha_i} \bar{\omega}' \right) = \text{Im} i \int_C \omega \wedge \bar{\omega}',
\]
i.e. we have

9.2.2. COROLLARY. The restriction of \( \text{Im} H \) on \( \Lambda := H_1(C, \mathbb{Z}) \) is the standard intersection pairing.

\(^{26}\)Note that a general rule for computing dual pairing in coordinates is the following: if \( B \) is a non-degenerate bilinear form on \( V \), choose bases \( \{ e_i \} \) and \( \{ \bar{e}_i \} \) of \( V \) such that \( B(e_i, \bar{e}_j) = \delta_{ij} \). Then the dual pairing on \( V^* \) is given by \( B^*(f, f') = \sum f(e_i) f'(\bar{e}_i) \). It does not depend on the choice of bases. In our example, the first basis of \( H_1(C, \mathbb{Z}) \) is given by cycles \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \), and the second basis is then given by \( \beta_1, \ldots, \beta_g, -\alpha_1, \ldots, -\alpha_g \).
The classical way to encode Riemann’s bilinear identities is to choose a basis \( \omega_1, \ldots, \omega_g \) of \( H^0(C, K) \) and consider the period matrix

\[
\Omega = \begin{bmatrix}
\int_{\alpha_1} \omega_1 & \cdots & \int_{\alpha_g} \omega_1 \\
\vdots & \ddots & \vdots \\
\int_{\alpha_1} \omega_g & \cdots & \int_{\alpha_g} \omega_g
\end{bmatrix}.
\]

Since \( H \) is positive-definite, the first minor \( g \times g \) of this matrix is non-degenerate, and so in fact there exists a unique basis \( \{ w_i \} \) such that

\[
\Omega = \begin{bmatrix} \text{Id} & | & Z \end{bmatrix},
\]

where \( Z \) is a \( g \times g \) matrix. Riemann’s bilinear identities then imply that \( Z = Z^t \) and \( \text{Im} Z \) is positive-definite.

9.2.3. Definition. The Siegel upper-half space \( S_g \) is the space of symmetric \( g \times g \) complex matrices \( Z \) such that \( \text{Im} Z \) is positive-definite.

To summarize our discussion above, we have the following

9.2.4. Corollary. Let \( C \) be a genus \( g \) Riemann surface. Let \( \text{Jac}(C) = V/\Lambda \) be its Jacobian. Then \( V = H^0(C, K)^* \) carries a Hermitian form \( H = i \int_C \omega \wedge \overline{\omega}' \), and \( \text{Im} H \) restricts to the intersection pairing on \( \Lambda = H_1(C, \mathbb{Z}) \). Any choice of symplectic basis \( \{ \delta_i \} = \{ \alpha_i \} \cup \{ \beta_i \} \) in \( \Lambda \) determines a unique matrix in \( S_g \).

Different choices of a symplectic basis are related by the action of the symplectic group \( \text{Sp}(2g, \mathbb{Z}) \). So we have a map

\[
M_g \to A_g := S_g / \text{Sp}(2g, \mathbb{Z}).
\]

It turns out that \( A_g \) is itself a moduli space.

§9.3. Abelian varieties.

9.3.1. Definition. A complex torus \( V/\Lambda \) is called an Abelian variety if carries a structure of a projective algebraic variety, i.e. there exists a holomorphic embedding \( V/\Lambda \hookrightarrow \mathbb{P}^N \).

One has the following theorem of Lefschetz:

9.3.2. Theorem. A complex torus is projective if and only if there exists a Hermitian form \( H \) on \( V \) (called polarization) such that \( \text{Im} H \) restricts to an integral skew-symmetric form on \( \Lambda \).

It is easy to classify integral skew-symmetric forms \( Q \) on \( \mathbb{Z}^{2g} \):

9.3.3. Lemma. There exist uniquely defined positive integers \( \delta_1 | \delta_2 | \ldots | \delta_g \) such that the matrix of \( Q \) in some \( \mathbb{Z} \)-basis is

\[
\begin{bmatrix}
-\delta_1 & -\delta_2 & \cdots & -\delta_g \\
\delta_1 & \delta_2 & \cdots & \delta_g \\
-\delta_1 & -\delta_2 & \cdots & -\delta_g \\
\vdots & \vdots & \ddots & \vdots \\
-\delta_1 & -\delta_2 & \cdots & -\delta_g
\end{bmatrix}.
\]
Proof. For each \( \lambda \in \Lambda = \mathbb{Z}^{2g} \), let \( d_\lambda \) be the positive generator of the principal ideal \( \{ Q(\lambda, \bullet) \} \subset \mathbb{Z} \). Let \( \delta_1 = \min(d_\lambda) \), take \( \lambda_1, \lambda_{g+1} \in \Lambda \) such that \( Q(\lambda, \lambda_{g+1}) = \delta_1 \). Those are the first two vectors in the basis. For any \( \lambda \in \Lambda \), we know that \( \delta_1 \) divides \( Q(\lambda, \lambda_1) \) and \( Q(\lambda, \lambda_{g+1}) \), and therefore

\[ \lambda + \frac{Q(\lambda, \lambda_1)}{\delta_1} \lambda_1 + \frac{Q(\lambda, \lambda_{g+1})}{\delta_1} \lambda_{g+1} \in \langle \lambda_1, \lambda_{g+1} \rangle \mathbb{Z}. \]

Now we proceed by induction by constructing a basis in \( \langle \lambda_1, \lambda_{g+1} \rangle \mathbb{Z} \). \( \square \)

9.3.4. Definition. A polarization \( H \) is called principal if we have

\[ \delta_1 = \ldots = \delta_g = 1 \]

in the canonical form above. An Abelian variety \( V/\Lambda \) endowed with a principal polarization is called a principally polarized Abelian variety.

So we have

9.3.5. Corollary. \( A_g \) parametrizes principally polarized Abelian varieties.

In fact \( A_g \) has a natural structure of an algebraic variety. One can define families of Abelian varieties in such a way that \( A_g \) is a coarse moduli space.

§9.4. Abel’s Theorem. Returning to the Abel–Jacobi map, we have the following fundamental

9.4.1. Theorem (Abel’s theorem). The Abel–Jacobi map \( AJ : \text{Div}^0(C) \to \text{Jac}(C) \) induces a bijection

\[ \mu : \text{Pic}^0(C) \simeq \text{Jac}(C). \]

The proof consists of three steps:

1. \( AJ(f) = 0 \) for any rational function \( f \in k(C) \), hence \( AJ \) induces \( \mu \).
2. \( \mu \) is injective.
3. \( \mu \) is surjective.

For the first step, we consider a holomorphic map \( \mathbb{P}^1_{[\lambda, \mu]} \) given by

\[ [\lambda, \mu] \mapsto AJ(\lambda f + \mu). \]

It suffices to show that this map is constant. We claim that any holomorphic map \( r : \mathbb{P}^1 \to V/\Lambda \) is constant. It suffices to show that \( dr = 0 \) at any point. But the cotangent space to \( V/\Lambda \) at any point is generated by global holomorphic forms \( dz_1, \ldots, dz_g \) (where \( z_1, \ldots, z_g \) are coordinates in \( V \). A pull-back of any of them to \( \mathbb{P}^1 \) is a global 1-form on \( \mathbb{P}^1 \), but \( K_{\mathbb{P}^1} = -2[\infty] \), hence the only global holomorphic form is zero. Thus \( dr^*(dz_i) = 0 \) for any \( i \), i.e. \( dr = 0 \).

§9.5. Differentials of the third kind. To show injectivity of \( \mu \), we have to check that if \( D = \sum a_ip_i \in \text{Div}^0 \) and \( \mu(D) = 0 \) then \( D = (f) \). If \( f \) exists then

\[ \nu = \frac{1}{2\pi i} d\log(f) = \frac{1}{2\pi i} \frac{df}{f} \]

has only simple poles, these poles are at \( p_i \)’s and \( \text{Res}_{p_i} \nu = a_i \) (why?) . Moreover, since branches of \( \log \) differ by integer multiples of \( 2\pi i \), any period

\[ \int_\gamma \nu \in \mathbb{Z} \]
for any closed loop \( \gamma \). And it is easy to see that if \( \nu \) with these properties exists then we can define

\[
f(p) = \exp(2\pi i \int_{p_0}^p \nu).
\]

This will be a single-valued meromorphic (hence rational) function with \( (f) = D \). So let’s construct \( \nu \). Holomorphic 1-forms on \( C \) with simple poles are classically known as differentials of the third kind. They belong to the linear system \( H^0(C, K + p_1 + \ldots + p_r) \). Notice that we have an exact sequence

\[
0 \to H^0(C, K) \to H^0(C, K + p_1 + \ldots + p_r) \xrightarrow{\psi} C^r,
\]

where \( \psi \) is given by taking residues. By Riemann–Roch, dimensions of the linear systems are \( g \) and \( g + r - 1 \). By a theorem on the sum of residues, the image of \( \psi \) lands in the hyperplane \( \sum a_i = 0 \). It follows that \( \psi \) is surjective onto this hyperplane, i.e. we can find a differential \( \eta \) of the third kind with any prescribed residues (as long as they add up to zero). The game now is to make periods of \( \eta \) integral by adding to \( \eta \) a holomorphic form (which of course would not change the residues). Since the first \( g \times g \) minor of the period matrix is non-degenerate, we can arrange that \( A \)-periods of \( \eta \) are equal to 0.

Now, for any holomorphic 1-form \( \omega \), arguing as in the proof of Prop. 9.2.1, we have the following identity:

\[
\sum (\int_{\alpha_i} \omega \int_{\beta_i} \eta - \int_{\beta_i} \omega \int_{\alpha_i} \eta) = \sum_{i=1}^r a_i \pi(p_i) = \sum_{i=1}^r a_i \int_{p_i}^{p_0} \omega.
\]

Indeed, we can remove small disks around each \( p_i \) to make \( \eta \) holomorphic in their complement, and then compute \( \int \omega \wedge \eta \) by Green’s theorem as in Prop. 9.2.1. This gives

\[
\sum \int_{\alpha_i} \omega \int_{\beta_i} \eta = \sum_{i=1}^r a_i \int_{p_i}^{p_0} \omega.
\]

Since \( \mu(D) = 0 \), we can write the RHS as \( \int_\gamma \omega \), where \( \gamma = \sum m_i \delta_i \) is an integral linear combination of periods. Applying this to the normalized basis of holomorphic 1-forms gives

\[
\int_{\beta_i} \eta = \int_{\gamma} \omega_i.
\]

Now let

\[
\eta' = \eta - \sum_{k=1}^g m_{g+k} \omega_k.
\]

Then we have

\[
\int_{\alpha_i} \eta' = -m_{g+i}
\]

and

\[
\int_{\beta_i} \eta' = \int_{\gamma} \omega_i - \sum m_{g+k} \int_{\beta_i} \omega_k =
\]
\[ \sum m_k \int_{\alpha_i} \omega_i + \sum m_{g+k} \int_{\beta_k} \omega_i - \sum m_{g+k} \int_{\beta_k} \omega_k = m_i \]

**§9.6. Summation maps.** To show surjectivity, we are going to look at the summation maps

\[ C^d \to \text{Pic}^d \to \text{Jac}(C), \quad (p_1, \ldots, p_d) \mapsto \mu(p_1 + \ldots + p_d - dp_0), \]

where \( p_0 \in C \) is a fixed point. It is more natural to define

\[ \text{Sym}^d C = C^d / S_d, \]

and think about summation maps as maps

\[ \phi_d : \text{Sym}^d C \to \text{Jac} C. \]

It is not hard to endow \( \text{Sym}^d C \) with a structure of a complex manifold in such a way that \( \phi_d \) is a holomorphic map\(^\text{27}\). We endow \( \text{Sym}^d C \) with a quotient topology for the map \( \pi : C^d \to \text{Sym}^d C \), and then define complex charts as follows: at a point \( (p_1, \ldots, p_d) \), choose disjoint holomorphic neighborhoods \( U_i \)'s of \( p_i \)'s (if \( p_i = p_j \) then choose the same neighborhood \( U_i = U_j \)). Let \( z_i \)'s be local coordinates. Then local coordinates on \( \pi(U_1 \times \ldots \times U_d) \) can be computed as follows: for each group of equal points \( p_i, i \in I \), use elementary symmetric functions in \( z_i, i \in I \) instead of \( z_i \)'s themselves.

The main point is absolutely obvious

**9.6.1. Lemma.** For \( D \in \text{Pic}^d \), \( \mu^{-1}(D) = |D| \). Fibers of \( \mu \) are projective spaces.

To show that \( \mu \) is surjective it suffices to show that \( \phi_d \) is surjective. Since this is a proper map of complex manifolds of the same dimension, it suffices to check that a general fiber is a point. In view of the previous Lemma this boils down to showing that if \( (p_1, \ldots, p_g) \in \text{Sym}^g \) is sufficiently general then \( H^0(C, p_1 + \ldots + p_g) = 1 \). For inductive purposes, let's show that

**9.6.2. Lemma.** For any \( k \leq g \), and sufficiently general points \( p_1, \ldots, p_k \in C \), we have \( H^0(C, p_1 + \ldots + p_k) = 1 \).

**Proof.** By Riemann-Roch, we can show instead that

\[ H^0(C, K - p_1 - \ldots - p_k) = g - k \]

for \( k \leq g \) and for a sufficiently general choice of points. Choose an effective canonical divisor \( K \) and choose points \( p_i \) away from it. Then we have an exact sequence

\[ 0 \to \mathcal{L}(K - p_1 - \ldots - p_k) \to \mathcal{L}(K - p_1 - \ldots - p_{k-1}) \to \mathbb{C}, \]

where the last map is the evaluation map at the point \( p_k \). It follows that either \( |K - p_1 - \ldots - p_k| = |K - p_1 - \ldots - p_{k-1}| \) or dimensions of these two projective spaces differ by 1, the latter happens if one of the functions in \( \mathcal{L}(K - p_1 - \ldots - p_{k-1}) \) does not vanish at \( p_k \). So just choose \( p_k \) to be a point where one of these functions does not vanish. \( \Box \)

**9.6.3. Corollary.** We can identify \( \text{Pic}^0 \) and \( \text{Jac} \) by means of \( \mu \).\(^\text{27}\)

\(^{27}\)It is also not hard to show that \( \text{Sym}^d C \) is a projective algebraic variety. Since \( \text{Jac} C \) is projective by Lefschetz theorem, it follows (by GAGA) that \( \phi_d \) is actually a regular map.
§9.7. Theta-divisor.

9.7.1. Corollary. The image of $\phi_{d-1}$ is a hypersurface $\Theta$ in $\text{Jac } C$.

9.7.2. Definition. $\Theta$ is called the theta-divisor.

9.7.3. Example. If $g = 1$, not much is going on: $C = \text{Jac } C$. If $g = 2$, we have $\phi_1 : C \to \text{Jac } C$: the curve itself is a theta-divisor! The map $\phi_2$ is a bit more interesting: if $h^0(C, p + q) > 1$ then $p + q \in |K|$ by Riemann–Roch.

In other words, $p$ and $q$ are permuted by the hyperelliptic involution and these pairs $(p, q)$ are parametrized by $\mathbb{P}^1$ as fibers of the $2 : 1$ map $\phi_{|K|} : C \to \mathbb{P}^1$. So $\phi_2$ is an isomorphism outside of $K \in \text{Pic}^2$, but $\phi_2^{-1}(K) \simeq \mathbb{P}^1$.

Since both $\text{Sym}^2 C$ and $\text{Jac } C$ are smooth surfaces, this implies that $\phi_2$ is a blow-up of the point.

9.7.4. Example. In genus 3, something even more interesting happens. Notice that $\phi_2$ fails to be an isomorphism only if $C$ carries a pencil of degree 2, i.e. if $C$ is hyperelliptic. In this case $\phi_2$ again contracts a curve $E \simeq \mathbb{P}^1$, but this time it is not a blow-up of a smooth point. To see this, I am going to use adjunction formula. Let $\tilde{E} \subset C \times C$ be the preimage. Then $\tilde{E}$ parametrizes points $(p, q)$ in the hyperelliptic involution, i.e. $\tilde{E} \simeq C$ but not a diagonally embedded one. We can write a holomorphic 2-form on $C \times C$ as a wedge product $\text{pr}_1^*(\omega) \wedge \text{pr}_2^*(\omega)$, where $\omega$ is a holomorphic 1-form on $C$. Since $\deg K_C = 2$, the canonical divisor $K$ on $C \times C$ can be chosen as a union of 4 vertical and 4 horizontal rulings. This $K : \tilde{E} = 8$, but

$$(K + \tilde{E}) \cdot \tilde{E} = 2g(\tilde{E}) - 2 = 4$$

by adjunction, which implies that $\tilde{E} \cdot \tilde{E} = -4$. Under the $2 : 1$ map $C \times C \to \text{Sym}^2 C$, $\tilde{E} 2 : 1$ covers our $E \simeq \mathbb{P}^1$. So $E^2 = -2$. This implies that the image of $\phi_2$ has a simple quadratic singularity at $\phi_2(E)$. So the Abel-Jacobi map will distinguish between hyperelliptic and non-hyperelliptic genus 3 curves by appearance of a singular point in the theta-divisor.


Problem 1. Generalizing the action of $\text{SL}(2, \mathbb{Z})$ on the upper-half plane, give formulas for the action of $\text{Sp}(2g, \mathbb{Z})$ on $S_g$ (1 point).

Problem 2. In the proof of Lemma 9.3.3, show that indeed we have

$$\delta_1|\delta_2|\ldots|\delta_g$$

(1 point).

Problem 3. Show that $(\mathbb{A}^2)^n//S_n$ is singular (with respect to the action permuting factors) at the point that corresponds to the orbit $(0, \ldots, 0)$ (2 points).

Problem 4. Show that $\text{Sym}^d \mathbb{P}^1 = \mathbb{P}^d$ (1 point).

Problem 5. Let $C$ be an algebraic curve. Define $\text{Sym}^d C$ as an algebraic variety (1 point).

Problem 6. Show that if $\phi_1(C) \subset \text{Jac } C$ is symmetric (i.e. $\phi_1(C) = -\phi_1(C)$) then $C$ is hyperelliptic. Is the converse true? (1 point).

Problem 7. Show that either the canonical map $\phi_{|K|}$ is an embedding or $C$ is hyperelliptic. (1 point).
Problem 8. Let $C$ be a non-hyperelliptic curve. $C$ is called trigonal if it admits a $3 : 1$ map $C \to \mathbb{P}^1$. (a) Show that $C$ is trigonal if and only if its canonical embedding $\phi_C$ has a trisecant, i.e. a line intersecting it in (at least) three points. (b) Show that if $C$ is trigonal then its canonical embedding is not cut out by quadrics\(^{28}\) (2 points).

Problem 9. Show that the secant lines of a rational normal curve in $\mathbb{P}^n$ are parametrized by the surface in the Grassmannian $G(2, n + 1)$ and that this surface is isomorphic to $\mathbb{P}^2$ (2 points).

Problem 10. Consider two conics $C_1, C_2 \subset \mathbb{P}^2$ which intersect at 4 distinct points. Let $E \subset C_1 \times C_2$ be a curve that parametrizes pairs $(x, y)$ such that the line $L_{xy}$ connecting $x$ and $y$ is tangent to $C_2$ at $y$. (a) Show that $E$ is an elliptic curve. (b) Consider the map $t : E \to E$ defined as follows: send $(x, y)$ to $(x', y')$, where $x'$ is the second point of intersection of $L_{xy}$ with $C_1$ and $L_{x'y'}$ is the second tangent line to $C_2$ through $x'$. Show that $t$ is a translation map (with respect to the group law on the elliptic curve). (c) Show that if there exists a 7-gon inscribed in $C_1$ and circumscribed around $C_2$ then there exist infinitely many such 7-gons, more precisely there is one through each point of $C_1$ (3 points).

Problem 11. Let $C$ be a hyperelliptic curve and let $R = \{p_0, \ldots, p_{2g+1}\}$ be the branch points of the $2 : 1$ map $C \to \mathbb{P}^1$. We choose $p_0$ as the base point for summation maps $\phi_d : \text{Sym}^d \to \text{Jac}$. For any subset $S \subset R$, let $\alpha(S) = \phi_{|S|}(S)$. (a) Show that $\alpha_S \in \text{Jac}[2]$ (the 2-torsion part). (b) Show that $\alpha_S = \alpha_S'$. (c) Show that $\alpha$ gives a bijection between subsets of $B_g$ of even cardinality defined up to $S \leftrightarrow S'$ and points of $\text{Jac}[2]$ (3 points).

Problem 12. A divisor $D$ on $C$ is called a theta-characteristic if $2D \sim K$. A theta-characteristic is called vanishing if $h^0(D)$ is even and positive. Show that a curve of genus 2 has no vanishing theta characteristics but a curve of genus 3 has a vanishing theta characteristic if and only if it is a hyperelliptic curve (1 point).

Problem 13. Show that a nonsingular plane curve of degree 5 does not have a vanishing theta characteristic (3 points).

Problem 14. Let $E = \{y^2 = 4x^3 - g_2x - g_3\}$ be an elliptic curve with real coefficients $g_2, g_3$. Compute periods to show that $E \simeq \mathbb{C}/\Lambda$, where either $\Lambda = \mathbb{Z} + \tau \mathbb{Z}$ or $\Lambda = \mathbb{Z} + \tau(1 + i) \mathbb{Z}$ (with real $\tau$) depending on the number of real roots of the equation $4x^3 - g_2x - g_3 = 0$ (3 points).

Problem 15. Consider a (non-compact!) curve $C = \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$. Since $\mathbb{P}^1$ has no holomorphic 1-forms, lets consider instead differentials of the third kind and define

$$V := H^0(\mathbb{P}^1, K + p_1 + \ldots + p_r)^*.$$  

Show that $\Lambda := H_1(C, \mathbb{Z}) = \mathbb{Z}^{-1}$, define periods, integration pairing, and the "$\text{Jacobian}" V/\Lambda$. Show that $V/\Lambda \simeq (\mathbb{C}^*)^{r-1}$ and that $C$ embeds in $V/\Lambda \simeq (\mathbb{C}^*)^{r-1}$ by the Abel-Jacobi map (2 points).

Problem 16. Let $C$ be an algebraic curve with a fixed point $p_0$ and consider the Abel–Jacobi map $\phi = \phi_1 : C \to \text{Jac}$. For any point $p \in C$, we have

\(^{28}\)This is practically if and only if statement by Petri’s theorem.
a subspace $d\phi(T_pC) \subset T_{\phi(p)}\text{Jac}$. By applying a translation by $\phi(p)$, we can identify $T_{\phi(p)}\text{Jac}$ with $T_0\text{Jac} \simeq \mathbb{C}^g$. Combining these maps together gives a map $C \rightarrow \mathbb{P}^{g-1}$, $p \mapsto d\phi(T_pC)$. Show that this map is nothing but the canonical map $\phi_{|K}$ (2 points).

**Problem 17.** Let $F$, $G$ be homogeneous polynomials in $\mathbb{C}[x, y, z]$. Suppose that curves $F = 0$ and $G = 0$ intersect transversally at the set of points $\Gamma$. (a) Show that associated primes of $(F, G)$ are the homogeneous ideals $I(p_i)$ of points $p_i \in \Gamma$. (b) Show that every primary ideal of $(F, G)$ is radical by computing its localizations at $p_i$'s (c) Conclude that $I(\Gamma) = (F, G)$, i.e. any homogeneous polynomial that vanishes at $\Gamma$ is a linear combination $AF + BG$ (2 points).

§10. **Torelli theorem**

Will type when have some time.

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