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COMPACTIFICATION OF THE MODULI SPACE OF HYPERPLANE ARRANGEMENTS

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Abstract

Consider the moduli space M^0 of arrangements of n hyperplanes in general position in projective $(r-1)$-space. When $r = 2$ the space has a compactification given by the moduli space of stable curves of genus 0 with n marked points. In higher dimensions, the analogue of the moduli space of stable curves is the moduli space of stable pairs: pairs (S, B) consisting of a variety S (possibly reducible) and a divisor $B = B_1 + \cdots + B_n$, satisfying various additional conditions. We identify the closure of M^0 in the moduli space of stable pairs as Kapranov's Hilbert quotient compactification of M^0, and give an explicit description of the pairs at the boundary. We also construct additional irreducible components of the moduli space of stable pairs.

1. Introduction

Let M^0 denote the moduli space of arrangements of n hyperplanes in $\mathbb{P}_k^{r-1}$ in linear general position (i.e., ordered n-tuples of hyperplanes in linear general position modulo the diagonal action of $\mathrm{PGL}(r)$). When $r = 2$, the space, usually denoted $M_{0,n}$, has a celebrated compactification due to Grothendieck and Knudsen, $M_{0,n} \subset \overline{M}_{0,n}$, the moduli of stable n-pointed rational curves. The point of this paper is to generalize the construction to higher dimensions. Of course, $\overline{M}_{0,n}$ is the genus 0 instance of $\overline{M}_{g,n}$, the moduli space of stable n-pointed curves of genus g. From the point of view of Mori theory the correct generalisation of $\overline{M}_{g,n}$ is the moduli of semi-log canonical pairs [KSB88], [Alexeev96a], [Alexeev96b], pairs (S, B) of a variety with a boundary (a reduced Weil divisor) satisfying certain singularity assumptions generalizing toroidal (we will not need the precise definition here). Such a space is expected to exist in all dimensions, but known constructions depend on the minimal model program and so currently apply only to varieties of dimension two or less (note, however, that certain compact moduli spaces

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of pairs with group action have been constructed without using the minimal model program; see, e.g., [Alexeev02]). In this paper we offer an alternative construction for hyperplane arrangements (i.e., for generalizing $\overline{M}_{0,n}$) which is quite elementary and which holds in all dimensions. We will construct a projective scheme M, containing M^0 as an open subset, and a flat projective family $p : (\mathbb{S}, \mathbb{B}) \to M$ of (possibly reducible) $(r-1)$-dimensional varieties with boundary extending the universal family over M^0 of ordered n-tuples of hyperplanes in $\mathbb{P}^{r-1}$. The family has very nice properties (here, and throughout the paper, we work over a fixed algebraically closed field k of arbitrary characteristic):

Theorem 1.1. *Let $(S, B = B_1 + \cdots + B_n)$ be a fibre of $(\mathbb{S}, \mathbb{B})$ over a closed point of M.*

(1) *(S, B) has stable toric singularities (in the sense of* [Alexeev02]*; see Definition* 4.4*). The log canonical sheaf $\omega_S(B)$ is a very ample line bundle, and the cohomology groups $H^i(S, \omega_S(B))$ vanish for $i > 0$.*

(2) *For each subset $I \subset \{1, 2, 3, \ldots, n\}$ with $|I| = r-1$, the scheme-theoretic intersection $\mathbb{B}_I := \bigcap_{i \in I} \mathbb{B}_I \subset \mathbb{S}$ is a section of p, and the family $(\mathbb{S}, \mathbb{B})$ is semi-stable in a neighborhood of this section (i.e., near the corresponding point of the fibre, S and the B_i are smooth and $B_1 + \cdots + B_n$ has normal crossings).*

(3) *The map given by taking residues along the sections $\mathbb{B}_I$,*

$$\mathrm{res} : p_* \omega_p(\mathbb{B}) \to \bigoplus_I p_* \mathcal{O}_{\mathbb{B}_I} = \bigwedge^{r-1} k^n \otimes \mathcal{O}_M,$$

is an isomorphism onto $\bigwedge^{r-1} h^* \otimes \mathcal{O}_M \subset \bigwedge^{r-1} k^n \otimes \mathcal{O}_M$, where $h = k^n/(k \cdot (1, \ldots, 1))$. In particular, $p_* \omega_p(\mathbb{B})$ is locally free of rank $\binom{n-1}{r-1}$. Its formation commutes with all base-extensions. In particular, the above residue map determines a basis of $H^0(S, \omega_S(B))$ canonically associated to the pair (S, B).

(4) *The global sections given by* res *induce a canonical embedding*

$$\mathbb{S} \subset M \times G(r-1, h) \subset M \times \mathbb{P}(\bigwedge^{r-1} h)$$

where $G(r-1, h) \subset \mathbb{P}(\bigwedge^{r-1} h)$ is the Plücker embedding of the Grassmannian of $(r-1)$-planes in h. The closure of $M^0 \subset M$ is identified with Kapranov's Hilbert quotient $G(r, n)///H$ of the Grassmannian of r-planes in k^n by its maximal torus [Kapranov93].

(5) *p is a flat family of log canonically polarised semi-log canonical pairs and so defines a map from M to the moduli stack of semi-log canonical pairs. This is a closed immersion.*

Furthermore, the family $(\mathbb{S}, \mathbb{B})$ is universal and identifies M as a natural moduli space of pairs satisfying properties as in the theorem—what we call "very stable pairs". See Section 6 for the precise statement.

Unfortunately, our M will not in general be irreducible (see Section 7), and thus is not precisely a compactification of M^0. We do not know a functorial characterisation of the closure of M^0 (i.e., of the Hilbert quotient $G(r,n)///H$).

1.1. General philosophy. Before turning to the technical details let us outline the general idea, which is adapted from ideas of [Kapranov93] and [Lafforgue03]. First begin with a pair $(S, B = B_1 + \dots B_n)$ of $\mathbb{P}^{r-1}$ together with n hyperplanes in linear general position. The main observation is that moduli of such pairs can be identified with moduli of equivariant embeddings of a fixed toric variety—the normal projective toric variety associated to the so-called hypersimplex $\Delta(r,n)$—in the Grassmannian, $G(r,n)$.

By the Gel'fand-MacPherson transform, M^0 is identified with the set of orbits $G^0(r,n)/H$, where $G^0 \subset G(r,n)$ is the open subset where all Plücker coordinates are nonzero and $H = \mathbb{G}_m^n/\mathbb{G}_m \subset \mathrm{PGL}(n)$ is the standard maximal torus. In [Kapranov93] this correspondence is formulated elegantly as follows: A choice of linear equations for the hyperplanes yields an embedding $S \subset \mathbb{P}^{n-1}$ so that the configuration B is the restriction of the coordinate hyperplanes. H acts freely on the orbit of $[S] \in G(r,n)$ so we have an isomorphism

$$m : H \to H \cdot [S],\ h \mapsto h^{-1}[S].$$

Observe that $S \setminus B \subset H$ is identified with

$$\{P \in H \cdot [S] \mid e \in P\} = H \cdot [S] \cap G(r-1, n-1)_e$$

where $e = (1, \dots, 1) \in \mathbb{P}^{n-1}$, and $G(r-1,n-1)_e \subset G(r,n)$ is the sub Grassmannian of r-planes that contain the fixed vector e. This identification is easily seen to extend to the closure (and indeed to degenerations), see Section 4, $S = \overline{H \cdot [S]} \cap G(r-1, n-1)_e$. This realizes S as a complete intersection inside the orbit closure $\overline{H \cdot [S]}$, the normal projective toric variety corresponding to the polytope $\Delta(r,n)$. Kapranov calls the orbit closure a *Lie complex* and $S \subset \overline{H \cdot [S]}$ its *visible contour*. This realizes M^0 as a locus in $\mathrm{Hilb}(G(r,n))$ of generic orbit closures. The closure of this locus is Kapranov's Hilbert quotient compactification $M^0 \subset G(r,n)///H$. By definition $G(r,n)///H$ carries a flat family, with generic fibres these orbit closures. The advantage of the approach is that the degenerate fibres are quite easy to understand: The generic fibres are closures of generic H-orbits and are embeddings of the normal projective toric variety associated to $\Delta(r,n)$, special fibres are reduced unions of (top dimensional) orbit closures, which are normal projective toric varieties associated to cells in certain tilings (called matroid decompositions) of $\Delta(r,n)$; see Corollary 3.11. In particular, we have a flat family of pairs $(\mathbb{T}, \mathbb{B}_T)$ of *broken* toric varieties and their toric boundaries. A simple but clever observation of Lafforgue shows that the visible contour construction extends to all of $(\mathbb{T}, \mathbb{B}_T)$, and yields exactly as above a flat family $(\mathbb{S}, \mathbb{B}) \subset (\mathbb{T}, \mathbb{B}_T)$ of complete intersections, transverse to the toric boundary, and, in particular, $(\mathbb{S}, \mathbb{B})$ a flat family of pairs with stable toric singularities, compactifying the universal family of hyperplane arrangements over M^0. See Section 4.1. We observe that for each fibre (S, B) of $(\mathbb{S}, \mathbb{B})$ the Plücker embedding

$$S \subset G(r-1, n-1)_e \subset G(r,n) \subset \mathbb{P}(\bigwedge^r k^n)$$

(and so the Hilbert point $[S] \in \operatorname{Hilb}(G(r,n))$) is given by a canonical basis of global log canonical forms, and, in particular, is canonically determined by the isomorphism class of the pair (S, B); see Theorem 5.2. In this way $(\mathbb{S}, \mathbb{B}) \to G(r,n)/\!/\!/H$ induces a closed immersion of $G(r,n)/\!/\!/H$ into the moduli stack of semi-log canonical pairs; thus $G(r,n)/\!/\!/H$ is a submoduli space of pairs. Unfortunately, we cannot identify the image—we do not know precisely which semi-log canonical pairs are limits of generic hyperplane arrangements. Here we use an alternative construction: Instead of $G(r,n)/\!/\!/H$ (which, defined as a closure, does not (as far as we can see) have any natural functorial meaning) we make use of $M \subset \operatorname{Hilb}(G(r,n))$, a closed subscheme of the so-called toric Hilbert scheme; see [HS04]. M parametrises $\mathbb{G}_m^n$-equivariant closed subschemes of $\tilde{G}(r,n)$ (the cone over the Grassmannian in its Plücker embedding) with a prescribed multigraded Hilbert function; see Section 2. M^0 immerses in M as an open subset, with closure $G(r,n)/\!/\!/H$, and, because the toric Hilbert scheme represents a natural functor, M admits a functorial description as a moduli space of pairs with stable toric singularities (satisfying various other properties), which we call very stable pairs. See Section 6 for the precise statement.

2. The log canonical model of the complement of a hyperplane arrangement

This short section is not logically required for the proof of the main theorem; everything we do here we'll redo in later sections in greater generality. As we think the construction is of independent interest, we have written the section so that it can be read on its own, at the cost of some subsequent repetition.

We describe an explicit compactification (S, B) of the complement U of a hyperplane arrangement, following [Kapranov93]. We show that (S, B) is the log canonical model of U, i.e., the canonical compactification of the algebraic variety U obtained via the minimal model program. These compactifications occur as the components of the fibres of the universal family $(\mathbb{S}, \mathbb{B})/M$.

Let $\mathcal{A} = (H_1, \ldots, H_n)$ be an (ordered) arrangement of hyperplanes in $\mathbb{P}^{r-1}$. Let $U = \mathbb{P}^{r-1} \setminus \bigcup \mathcal{A}$, the complement. Assume that the stabiliser of $\mathcal{A}$ in $\mathrm{PGL}(r)$ is finite. Equivalently, the matroid of $\mathcal{A}$ is *connected* [GS87], i.e., there does not exist a decomposition $k^r = V_1 \oplus V_2$ such that for each i either $\mathbb{P}(V_1) \subset H_i$ or $\mathbb{P}(V_2) \subset H_i$.

Choose homogeneous equations F_i for the H_i, and consider the linear embedding

$$F = (F_1 : \ldots : F_n) : \mathbb{P}^{r-1} \subset \mathbb{P}^{n-1}.$$

Let $H = \mathbb{G}_m^n/\mathbb{G}_m \subset \mathbb{P}^{n-1}$ be the usual torus embedding. Observe that the embedding F is determined up to translation by an element of H, and restricts to a (closed) embedding $U \subset H$.

Let $G(r, n)$ denote the Grassmannian of r-planes in k^n. Let V denote the H-orbit in $G(r, n)$ determined by F. The *matroid polytope* of $\mathcal{A}$ is by definition the weight polytope of V. It has full dimension $n-1$ since by assumption the matroid of $\mathcal{A}$ is connected (see [GS87]), and its vertices affinely generate the lattice (see, e.g., [Kapranov93], p. 47, Proof of Proposition 1.2.15). Hence H acts freely on V. The embedding $U \subset V$ given by

$$u \mapsto F(u)^{-1}[F(\mathbb{P}^{r-1})]$$

is canonical (it does not depend on the choice of F).

Let $G(r-1, n-1)_e \subset G(r, n)$ denote the locus of subspaces containing the vector $e = (1, \ldots, 1) \in k^n$. Note that $G(r-1, n-1)_e$ is identified with the Grassmannian of $(r-1)$-planes in $h = k^n/k \cdot e$, the Lie algebra of H. Observe that the locus $U \subset V$ in $G(r, n)$ equals $V \cap G(r-1, n-1)_e$.

Let S and T denote the closures of U and V in $G(r, n)$, respectively. The variety T is isomorphic to the normal toric variety associated to the matroid polytope of $\mathcal{A}$. Write $B = S \setminus U$ and $B_T = T \setminus V$, the toric boundary of T.

Lemma 2.1 (Lafforgue)**.** *S is equal to the scheme-theoretic intersection $T \cap G(r-1, n-1)_e$. The multiplication map $H \times S \to T$ is smooth.*

Proof. This is an application of Lemma 4.1; cf. Theorem 4.5. $\square$

Theorem 2.2. *(S,B) is the log canonical model of U. Moreover,*

1. *(S,B) has toric singularities (i.e., looks étale locally like the pair of a normal toric variety and its toric boundary);*
2. *K_S+B is very ample;*
3. *the embedding $S\subset G(r-1,n-1)_e$ is given by the locally free sheaf $\Omega_S(\log B)$ and the map*

$$h^*\to H^0(\Omega_S(\log B)),\ (\lambda_1,\dots,\lambda_n)\mapsto\sum\lambda_i\frac{dF_i}{F_i}.$$

Proof. (S,B) has toric singularities by the lemma. Assuming (3), $\Omega_S(\log B)$ is identified with the restriction of the dual of the universal subbundle $\mathcal{U}_e\subset\mathcal{O}_{G_e}\otimes h$ on $G(r-1,n-1)_e$. So $\omega_S(B)=\bigwedge^{r-1}\Omega_S(\log B)$ is identified with the restriction of the Plücker line bundle on G_e. Hence K_S+B is very ample.

For $P\in H$, let $\mu_P:H\to H$ be the map given by multiplication by P. The embedding $U\subset G(r-1,h)$ is the Gauss map associated to the embedding $U\subset H$, i.e., the map

$$g:U\to G(r-1,h),P\mapsto[d(\mu_P^{-1})T_PU].$$

Indeed, since $U\subset H$ is the restriction of the *linear* embedding $\mathbb{P}^{r-1}\subset\mathbb{P}^{n-1}$, all the tangent spaces T_PU are equal to $\mathbb{P}^{r-1}\subset\mathbb{P}^{n-1}$ (when regarded as subspaces of $\mathbb{P}^{n-1}$). An explicit computation shows that the embedding $U\subset G(r-1,h)$ is given by the surjection

$$h^*\otimes\mathcal{O}_U\to\Omega_U,\ (\lambda_1,\dots,\lambda_n)\mapsto\sum\lambda_i\frac{dF_i}{F_i}.$$

This map extends to the surjection

$$h^*\otimes\mathcal{O}_S=\Omega_T(\log B_T)|_S\to\Omega_S(\log B),$$

given by the embedding $S\subset T$. Statement (3) follows. □

If $k=\mathbb{C}$, part (3) may be explained conceptually as follows. The exponential map

$$\exp:h\to H,\ (\lambda_1,\dots,\lambda_n)\mapsto(\exp(\lambda_1),\dots\exp(\lambda_n))$$

identifies H with the quotient $h/(2\pi i)N$, where $N=\mathbb{Z}^n/\mathbb{Z}e\subset h=\mathbb{C}^n/\mathbb{C}e$, the cocharacters of H. Assume for simplicity that the hyperplanes $H_1,\dots,H_n$ are distinct, then the map $h^*\to H^0(\Omega_S(\log B))$ is an isomorphism. The embedding $U\subset H$ is identified with the (generalised) Albanese map

$$U\to H^0(\Omega_S(\log B))^*/H_1(U,\mathbb{Z}),\ P\mapsto\left(\omega\mapsto\int_{P_0}^P\omega\right),$$

where $P_0\in U$ is a fixed basepoint. Recall that $g:U\to G(r-1,h)$ is the Gauss map for $U\subset H$. Using the integral formula for the embedding $U\subset H$

and the fundamental theorem of calculus, we deduce that $U \subset G(r-1,h)$ is given by the locally free sheaf Ω_U and the surjection

$$h^* \otimes \mathcal{O}_U = H^0(\Omega_S(\log B)) \otimes \mathcal{O}_U \to \Omega_U.$$

The result follows as above.

3. Construction of the moduli space of pairs

3.1. Multigraded Hilbert schemes. M is a multigraded Hilbert scheme as defined in [HS04]. We briefly review the definition and basic properties.

Let $T = \bigoplus_{a \in A} T_a$ be a k-algebra graded by an Abelian group A. Fix a function $h : A \to \mathbb{N}$. For R a k-algebra, let $H_T^h(R)$ be the set of A-homogeneous ideals $I \subset T \otimes R$ such that, for each $a \in A$, $T_a \otimes R / I_a$ is a locally free R-module of rank $h(a)$. This defines a functor $H_T^h : (k - \text{algebras}) \to (\text{Sets})$. It is represented by a quasiprojective scheme over k, the *multigraded Hilbert scheme* H_T^h. If T is a polynomial ring and the multigrading is positive (i.e., $T_0 = k$), then H_T^h is projective.

Let $S = k[x_1, \ldots, x_N]$ and $\mathbb{A} = \operatorname{Spec} S$. Fix a map

$$\phi : \mathbb{Z}^N \to \mathbb{Z}^n, \ e_i \mapsto a_i$$

corresponding to a homomorphism of tori $\mathbb{G}_m^n \to \mathbb{G}_m^N$, where $\mathbb{G}_m^N$ is the big torus acting on $\mathbb{A}$. Let $\mathcal{A} = \{a_1, \ldots, a_N\}$, the set of weights for the torus action $\mathbb{G}_m^n \curvearrowright \mathbb{A}$, and $A = \mathbb{Z}\mathcal{A} \subset \mathbb{Z}^n$ the lattice generated by $\mathcal{A}$. The map ϕ defines an A-grading of S such that the A-homogeneous ideals $I \subset S$ are the ideals defining $\mathbb{G}_m^n$-invariant closed subschemes in $\mathbb{A}$.

Let $\mathbb{N}\mathcal{A} \subset A$ be the semigroup generated by $\mathcal{A}$. Define $h : A \to \mathbb{N}$ by $h(a) = 1$ if $a \in \mathbb{N}\mathcal{A}$ and $h(a) = 0$ otherwise. The multigraded Hilbert scheme H_S^h is the *toric Hilbert scheme* for the torus action $\mathbb{G}_m^n \curvearrowright \mathbb{A}$ [HS04, Sec. 5]. Roughly speaking, H_S^h parametrises generic $\mathbb{G}_m^n$-orbit closures in $\mathbb{A}$ and their toric degenerations. More precisely, let $X_{\mathcal{A}}$ denote the orbit closure $\overline{\mathbb{G}_m^n \cdot e} \subset \mathbb{A}$, where $e = (1, \ldots, 1) \in \mathbb{A}$. Then $X_{\mathcal{A}}$ defines a distinguished point $[X_{\mathcal{A}}] \in H_S^h$, and the orbit closure $\overline{\mathbb{G}_m^N \cdot [X_{\mathcal{A}}]} \subset H_S^h$ is an irreducible component of H_S^h.

If $X = \operatorname{Spec} T \subset \mathbb{A}$ is a $\mathbb{G}_m^n$-invariant closed subscheme, then T is A-graded and H_T^h is the closed subscheme of H_S^h parametrising subschemes of X.

3.2. Stable toric varieties. A subscheme $Z \subset \mathbb{A}$ defining a point of the toric Hilbert scheme H_S^h is an affine stable toric variety as defined in [Alexeev02] (assuming Z is seminormal and reduced and the multigrading is positive). We review the construction of stable toric varieties.

Let A be a lattice and Ω a subdivision of a rational polyhedral cone ω in $A_{\mathbb{R}}$. For $\sigma \in \Omega$ let R_σ denote the semigroup algebra $k[\sigma \cap A]$ and $T_\sigma \subset X_\sigma$ the associated torus embedding. Fix gluing data $t_{\sigma\tau} \in T_\tau$ for each $\tau \subset \sigma$ satisfying the compatibility condition $t_{\tau\upsilon} \cdot t_{\sigma\tau} = t_{\sigma\upsilon}$ in T_υ for each triple $\upsilon \subset \tau \subset \sigma$. Define $p_{\sigma\tau} = t_{\sigma\tau} \circ \mathrm{pr}_{\sigma\tau}$ for $\tau \subset \sigma$, where $\mathrm{pr}_{\sigma\tau}$ is the canonical surjection $R_\sigma \to R_\tau$. Finally, let $R[\Omega, t]$ be the inverse limit of the system $(R_\sigma, p_{\sigma\tau})$.

Remark 3.1. Equivalently, $R[\Omega, t]$ is the equaliser of the maps $\bigoplus R_\sigma \rightrightarrows \bigoplus R_\tau$, where the direct sums are over maximal cones $\sigma \in \Omega$ and codimension 1 interior cones $\tau \in \Omega$, respectively. That is, $R[\Omega, t]$ is the subalgebra of $\bigoplus R_\sigma$ consisting of elements $f = (f_\sigma)$ such that $p_{\sigma_1\tau}(f_{\sigma_1}) = p_{\sigma_2\tau}(f_{\sigma_2})$ for each pair σ_1, σ_2 of maximal cones meeting in a common facet τ.

The variety $X = X(\Omega, t) := \operatorname{Spec} R[\Omega, t]$ has irreducible components $X_\sigma = \operatorname{Spec} R_\sigma$ for $\sigma \in \Omega$ a maximal cone. Combinatorially, the X_σ are glued to form X in the same way that the cones σ are glued to form ω. That is, for each maximal cone σ, the facets of the cone σ correspond to the irreducible components of the toric boundary $X_\sigma \backslash T_\sigma$ of X_σ, and if σ_1 and σ_2 meet in a common facet, then X_{σ_1} and X_{σ_2} are glued along the corresponding divisor. Note that there are also continuous gluing parameters determined by t. There is an action of the torus $T = \operatorname{Hom}(A, \mathbb{G}_m)$ on X extending the action on each component. The algebra $R[\Omega, t]$ with its corresponding A-grading has Hilbert function $h(a) = 1$ for $a \in \omega \cap A$ and $h(a) = 0$ otherwise.

Definition 3.2. An affine stable toric variety is a variety with torus action of the form $T \curvearrowright X(\Omega, t)$ for some Ω, t.

Remark 3.3. If $t_{\sigma\tau} = 1$ for each $\tau \subset \sigma$, then $R[\Omega, t]$ can be alternatively described as follows; cf. [Stanley87]. As a k-vector space, $R = \bigoplus\ k \cdot \chi^a$ where the sum is over the semigroup $\omega \cap A$. The ring structure on R is defined by $\chi^a \cdot \chi^b = \chi^{a+b}$ if a and b are contained in some cone $\sigma \in \Omega$, and $\chi^a \cdot \chi^b = 0$ otherwise.

Let M be a lattice, $P \subset M_{\mathbb{R}}$ a polytope with integral vertices, and $\underline{P}$ a subdivision of P. Let $A = M \oplus \mathbb{Z}$, and embed P in the affine hyperplane $M_{\mathbb{R}} \oplus 1 \subset A_{\mathbb{R}}$. Let Ω be the fan of cones over faces of $\underline{P}$. Fix gluing data t as above and define $Y = Y(\underline{P}, t) := \operatorname{Proj} R[\Omega, t]$. The irreducible components of Y are the polarised projective toric varieties $Y_{P'} = \operatorname{Proj} R_{\mathrm{Cone}(P')}$ associated to the maximal polytopes $P' \in \underline{P}$. The combinatorics of the gluing of the $Y_{P'}$ is encoded by $\underline{P}$. There is an action of the torus $H = \operatorname{Hom}(M, \mathbb{G}_m)$ on Y, and the polarisation $\mathcal{O}(1)$ on Y has a natural H-linearisation.

Definition 3.4. A polarised stable toric variety is a projective variety with a torus action together with a linearised ample sheaf of the form $H \curvearrowright (Y(\underline{P}, t), \mathcal{O}(1))$

Remark 3.5. In [Alexeev02] the definition of stable toric varieties is more general, and the special case above is referred to as the "convex 1-sheeted case".

3.3. The construction. Let $G(r,n) \subset \mathbb{P} = \mathbb{P}(\bigwedge^r k^n)$ be the Plücker embedding of the Grassmannian of r-planes in k^n. Let $\tilde{G}(r,n) \subset \mathbb{A}$ be the cone over the Plücker embedding, and S and T the coordinate rings of $\mathbb{A}$ and $\tilde{G}(r,n)$, respectively. Let $\mathbb{G}_m^n \curvearrowright \mathbb{A}$ be the standard $\mathbb{G}_m^n$-action and H_S^h the associated toric Hilbert scheme.

Definition 3.6. Let $M = H_T^h$, the closed subscheme of the toric Hilbert scheme H_S^h parametrising subschemes of $\tilde{G}(r,n)$.

Note immediately that we have an open immersion $M^0 \subset M$ given by the Gel'fand–MacPherson correspondence $M^0 = G^0(r,n)/H$.

The set of weights of $\mathbb{G}_m^n \curvearrowright \mathbb{A}$ is

$$\mathcal{A} = \left\{ e_{i_1} + \cdots + e_{i_r} \;\middle|\; i_1 < \cdots < i_r \right\} \subset \mathbb{Z}^n$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. The set $\mathcal{A}$ is the set of vertices of the *hypersimplex*

$$\Delta(r,n) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum x_i = r,\ 0 \le x_i \le 1 \right\}.$$

The polytope $\Delta(r,n)$ has $2n$ facets $(x_i = 0)$ and $(x_i = 1)$, $i = 1, \ldots, n$. Write $P = \Delta(r,n)$.

We consider polytopes $P' \subset P$ which are the convex hull of a subset of the vertices $\mathcal{A}$ of P. We regard the coordinates of $\mathbb{A}$ as labelled by $\mathcal{A}$. For $P' \subset P$, let $x_{P'} \in \mathbb{A}$ be the point with coordinates 1 for $a \in P' \cap \mathcal{A}$ and 0 otherwise, and $X_{P'}$ the orbit closure $\overline{\mathbb{G}_m^n \cdot x_{P'}}$. $X_{P'}$ is the affine toric variety (possibly non-normal) associated to the semigroup $\mathbb{N}(P' \cap \mathcal{A}) \subset A$ generated by $P' \cap \mathcal{A}$.

Let $\tilde{\mathbb{T}} \subset \tilde{G}(r,n) \times M$ denote the universal family over M.

Theorem 3.7. *Each fibre of $\tilde{\mathbb{T}}/M$ is a reduced affine stable toric variety associated to a subdivision of* $\mathrm{Cone}(P)$ *induced by a subdivision of P into matroid polytopes.*

Proof. Let Z be a fibre of $\tilde{\mathbb{T}}/M$. By [Sturmfels95, 10.10] there is a polyhedral subdivision $\underline{P}$ of P such that $\mathrm{red}\, Z = \bigcup_{P'} Z_{P'}$ where the union is over maximal polytopes $P' \in \underline{P}$, and $Z_{P'}$ is a translate of $X_{P'}$ by the big torus acting on $\mathbb{A}$.

Each P' is a matroid polytope since $Z \subset \tilde{G}(r,n)$. Hence the set $P' \cap \mathcal{A}$ generates the saturated semigroup $\mathrm{Cone}(P') \cap A$ by [White77], so $Z_{P'}$ is normal. It also follows that Z is reduced. Indeed, we have the surjections of coordinate rings

$$k[Z] \to k[\mathrm{red}\, Z] \to k[Z_{P'}]$$

and $\dim_k k[Z]_a = \dim_k k[Z_{P'}]_a = 1$ for $a \in \operatorname{Cone}(P') \cap A$. Thus $k[Z]_a = k[\operatorname{red} Z]_a$ for each $a \in A$ and $Z = \operatorname{red} Z$ as claimed.

If P'_1 and P'_2 intersect in a common facet, the corresponding boundary divisors of $Z_{P'_1}$ and $Z_{P'_2}$ coincide with the scheme-theoretic intersection $Z_{P'_1} \cap Z_{P'_2}$. Indeed, the ideal of $Z_{P'} \subset Z$ is the direct sum of the graded pieces $k[Z]_a$ of $k[Z]$ for $a \notin \operatorname{Cone} P'$. We deduce that $k[Z]$ is the equaliser of the maps

$$\bigoplus k[Z_{P'}] \rightrightarrows \bigoplus k[Z_{P''}],$$

where the $Z_{P''}$ are the strata of Z corresponding to interior codimension 1 faces $P'' \in \underline{P}$. Hence Z is an affine stable toric variety. □

Corollary 3.8. *The natural map $M \to \operatorname{Hilb}(G(r,n))$ obtained by projectivising $\tilde{G}(r,n) \subset \mathbb{A}$ is a closed embedding.*

Proof. Let $Z \subset \tilde{G}(r,n) \times \operatorname{Spec} R$ be an R-valued point of H^h_T. The family Z/R is flat and has reduced fibres by Theorem 3.7. It follows by [Matsumura89, 2.32] that the ideal $I \subset S \otimes R$ of $Z \subset \mathbb{A} \times \operatorname{Spec} R$ is saturated. Hence the map $H^h_T \to \operatorname{Hilb}(G(r,n))$ is an injection on R-points for each R. □

Corollary 3.9. *The closure of $M^0 \subset M$ is the Hilbert quotient $G(r,n)///H$.*

Proof. By definition $G(r,n)///H$ is the closure of M^0 in $\operatorname{Hilb}(G(r,n))$. □

Remark 3.10. When $k = \mathbb{C}$, the Hilbert quotient $G(r,n)///H$ is identified with the Chow quotient $G(r,n)//H$ (the closure of the locus of generic orbit closures in the Chow variety) via the Hilbert–Chow morphism [Kapranov93, 1.5.2]. We do not use this fact in this paper.

Let $\mathbb{T} \subset G(r,n) \times M$ denote the family obtained by projectivising $\tilde{\mathbb{T}} \subset \tilde{G}(r,n) \times M$.

Corollary 3.11. *Each fibre of $\mathbb{T}/M$ is a reduced projective stable toric variety associated to a subdivision of P into matroid polytopes.*

3.4. Relation to Lafforgue's space. Lafforgue defines a projective scheme $\overline{\Omega} = \overline{\Omega}^{\Delta(r,n)}$ with an open immersion $M^0 \subset \overline{\Omega}$. It may be constructed as follows (see [KT04, 2.9]). Let $\mathbb{P}///_n H \to \mathbb{P}///H \subset \operatorname{Hilb}(\mathbb{P})$ be the normalisation of the Hilbert quotient of $\mathbb{P}(\bigwedge^r k^n)$, i.e., the closure in $\operatorname{Hilb}(\mathbb{P})$ of the locus of generic H-orbit closures. The space $\overline{\Omega}$ is the inverse image in $\mathbb{P}///_n H$ of $\mathbb{P}///H \cap \operatorname{Hilb}(G(r,n))$. This construction induces a finite map $\overline{\Omega} \to M$ such that the family over $\overline{\Omega}$ (coming from $\operatorname{Hilb}(\mathbb{P})$) is the pullback of $\mathbb{T}$. It is an isomorphism over $M^0 \subset M$.

Roughly speaking, the space $\overline{\Omega}$ is a moduli space of varieties with log structures; see [Lafforgue03, Ch. 5] for the precise statement. Our space M is a submoduli space of stable pairs; see Section 6. Given a k-point $[(S,B)] \in M$, a point of $\overline{\Omega}$ over $[(S,B)]$ is given by a smooth log structure on S/k which is nontrivial over the divisors $B_i \subset S$ and the singular locus. Such log structures do not always exist; see Section 7. Moreover, we expect that the log structure is not unique in general, i.e., the map $\overline{\Omega} \to M$ is not injective on k-points.

The spaces $\overline{\Omega}$ and M are, in general, reducible by [KT04, 3.13]. M even has components outside (the image) of $\overline{\Omega}$; see Section 7. Ideally, we would like M to be a connected component of the moduli space of stable pairs, but we do not know if this is the case.

4. Construction of the universal family of pairs

4.1. Lafforgue transversality. Section 4.1 and Theorem 4.5 are based on [Lafforgue03, 5.1]. Let G (which in our application will be a Grassmannian) be a scheme on which an algebraic group Γ acts. Let $\mathcal{V} \subset G \times \Gamma$ be a closed Γ-equivariant subscheme. Define $\mathcal{V}_e := \mathcal{V} \cap (G \times \{e\})$, where $e \in \Gamma$ is the identity element. Note the first projection $\mathcal{V}_e \to G$ is a closed embedding. Let $G_{e,\mathcal{V}} \subset G$ be the image.

Lemma 4.1. *The multiplication map $\mathcal{V}_e \times \Gamma \to \mathcal{V}$ is an isomorphism, and identifies the multiplication map $G_{e,\mathcal{V}} \times \Gamma \to G$ with the first projection $\mathcal{V} \to G$.*

Let $G' \to G$ be a Γ-equivariant map, and let $\mathcal{V}' \subset G' \times \Gamma$ be the pullback. Then $G'_{e,\mathcal{V}'} \subset G'$ in the pullback of $G_{e,\mathcal{V}} \subset G$.

Proof. The map $\mathcal{V} \to \mathcal{V}_e \times \Gamma$ given by $(g,\gamma) \to ((g\gamma^{-1}, e), \gamma)$ is easily seen to be inverse to right multiplication. The rest is easy to check. □

Remark 4.2. Of course, if $\mathcal{V} \to G$ is smooth, then by the lemma so is the map $G_{e,\mathcal{V}} \times \Gamma \to G$.

4.2. Visible contours. Now let $G_e = G(r-1, n-1)_e \subset G = G(r,n)$ be the locus of subspaces containing $e = (1,\dots,1)$. Let $H = \mathbb{G}_m^n/\mathbb{G}_m$ be the standard maximal torus in $\mathrm{PGL}(n)$ and $h = k^n/k \cdot e$ the Lie algebra of H. Note that G_e is identified with $G(r-1, h)$.

Definition 4.3. Following [Kapranov93], we define the *family of visible contours* $p : (\mathbb{S}, \mathbb{B}) \to M$ as follows. Let $\mathbb{S}$ denote the scheme-theoretic intersection $\mathbb{T} \cap (G_e \times M)$. Let $\mathbb{B}_T$ denote the relative toric boundary of $\mathbb{T}/M$ and $\mathbb{B}$ its restriction to $\mathbb{S}$.

There is a decomposition $\mathbb{B}_T = \sum_{i=1}^n \mathbb{B}^+_{i,T} + \sum_{i=1}^n \mathbb{B}^-_{i,T}$, where $\mathbb{B}^+_{i,T}$ and $\mathbb{B}^-_{i,T}$ are the components of the $\mathbb{B}_T$ corresponding to the facets $(x_i = 1)$ and $(x_i = 0)$ of $\Delta(r,n)$, respectively. The components $\mathbb{B}^-_{i,T}$ are disjoint from $\mathbb{S}$, so $\mathbb{B} = \sum_{i=1}^n \mathbb{B}_i$ where $\mathbb{B}_i := \mathbb{B}^+_{i,T}|_{\mathbb{S}}$.

The family $(\mathbb{S}, \mathbb{B}_1 + \cdots + \mathbb{B}_n)/M$ extends the universal family of hyperplane arrangements over M^0 by [Kapranov93, 3.2.3] or Section 2.

Definition 4.4. A *stable toric singularity* $(p \in S, B)$ is a germ of a variety S together with a reduced divisor $B \subset S$ which is isomorphic to a germ of a stable toric variety with its toric boundary.

Theorem 4.5. *The multiplication map* $\mathbb{S} \times H \to \mathbb{T}$ *is smooth with image* $\mathbb{T} \setminus \bigcup \mathbb{B}^{-}_{i,T}$. *The family* $\mathbb{S}$ *and the* $\mathbb{B}_i$ *are flat over* M. *The embedding* $\mathbb{S} \subset \mathbb{T}$ *is the pullback of* $G(r-1, n-1)_e \subset G(r,n)$. *It is a regular embedding with normal bundle the restriction of the universal rank* $n-r$ *quotient bundle on* $G(r-1,n-1)_e$.

Proof. Let $\mathcal{U} \subset G(r,n) \times k^n$ be the universal rank r bundle, and $\mathcal{V} \subset \mathcal{U}$ the intersection of $\mathcal{U}$ with the diagonal torus $\Gamma \subset k^n$ (the locus where all coordinates are nonzero). Then by definition $\mathbb{S} \subset \mathbb{T}$ is the pullback of $G(r-1,n-1)_e \subset G(r,n)$ and, following the notation of Section 4.1, $G_e = G_{e,\mathcal{V}}$. Now it follows from Lemma 4.1 that $\mathbb{S} \times \Gamma \to \mathbb{T}$ is identified with the pullback of $\mathcal{V} \to G(r,n)$, and, in particular, is smooth. Since the scalar matrices act trivially on $G(r,n)$, and thus on $\mathbb{S}$, it follows that $\mathbb{S} \times H \to \mathbb{T}$ is smooth as well. The image of $\mathbb{S} \times H \to \mathbb{T}$ is, by the identification above, the inverse image of the open locus in $G(r,n)$ where the fibre of $\mathcal{V} \to G(r,n)$ is nonempty, that is, the locus of r-planes not contained in a coordinate hyperplane. For T a closed fibre of $\mathbb{T}/M$, the divisor $B^{-}_{i,T}$ corresponding to the facet $(x_i = 0)$ of $\Delta(r,n)$ is given by $T \cap G_i$, where $G_i \subset G$ is the locus of r-planes contained in the ith coordinate hyperplane [Kapranov93, Proposition 1.6.10]. Hence the image of $\mathbb{S} \times H \to \mathbb{T}$ equals $\mathbb{T} \setminus \bigcup \mathbb{B}^{-}_{i,T}$ as claimed. Since H acts trivially on M, flatness of $\mathbb{S}$ and the $\mathbb{B}_i$ (over M) now follow from flatness of $\mathbb{T}$ and the components of $\mathbb{B}_T$.

Finally, the closed subscheme $G_e \subset G$ is the zero locus of the section $\bar{e}$ of the quotient bundle $\mathcal{Q}$ given by $e \in k^n$, thus $G_e \subset G$ is a local complete intersection with normal bundle $\mathcal{N}_{G_e/G} = \mathcal{Q}|_{G_e} = \mathcal{Q}_e$. Now by the previous results $\mathbb{S} \subset \mathbb{T}$ is also a local complete intersection with normal bundle $\mathcal{N}_{\mathbb{S}/\mathbb{T}} = \mathcal{Q}_e|_{\mathbb{S}}$. $\square$

Corollary 4.6. *Let* (T, B_T) *be a fibre of* $(\mathbb{T}, \mathbb{B}_T)/M$ *and* (S,B) *its visible contour.*

(1) (S,B) *has stable toric singularities.*
(2) *Consider the stratification of* S *induced by the stratification of* T *by orbit closures. A stratum* $S' = S \cap T'$ *is nonempty if and only if* $T' \not\subset \bigcup B^{-}_{i,T}$. *In this case,* S' *is irreducible and normal of the expected dimension* $\dim T' - (n-r)$.

Remark 4.7. The stratification of S coincides with that defined by arbitrary intersections of components of S and B. In particular, it is obviously intrinsic. Let $\underline{P}$ be the polyhedral subdivision of $P = \Delta(r,n)$ associated to the stable toric variety T. The poset of orbit closures in T is identified with the poset of faces of $\underline{P}$. The poset of strata of S is therefore identified with the poset of faces of $\underline{P}$ which are not contained in the union of facets $\bigcup(x_i = 0) \subset \Delta(r,n)$ corresponding to $\bigcup B^-_{i,T} \subset T$.

Remark 4.8. Let S' be a component of S and B' the divisor on S' given by the restriction of B and the double locus. Then, by Section 2, (S', B') is the log canonical model of the complement of a hyperplane arrangement.

Let ω_p denote the relative dualising sheaf of $p : \mathbb{S} \to M$.

Theorem 4.9. *The line bundle $\omega_p(\mathbb{B})$ is the restriction of the Plücker line bundle on $G_e \times M$.*

Lemma 4.10. *Let $T \curvearrowright X/S$ be a flat family of reduced stable toric varieties of dimension d. Let B be the relative toric boundary of X/S and $M = \operatorname{Hom}(T, \mathbb{G}_m)$. There is a canonical isomorphism $\omega_{X/S} \cong \mathcal{O}_X(-B) \otimes \bigwedge^d M$.*

Proof of Lemma 4.10. Let $X^0 \subset X$ be the smooth locus of X/S. The torus action induces a map $\Omega_{X^0/S} \to \mathcal{O}_{X^0} \otimes_k \operatorname{Lie}(T)^* = \mathcal{O}_{X^0} \otimes_{\mathbb{Z}} M$ which extends to an isomorphism $\Omega_{X^0/S}(\log B) \to \mathcal{O}_{X^0} \otimes M$ (cf. [Oda88, p. 116, Proposition 3.1]). Taking top exterior powers we obtain an isomorphism $\omega_{X^0/S}(B) \to \mathcal{O}_{X^0} \otimes \bigwedge^d M$, and twisting by $\mathcal{O}_X(-B)$ an isomorphism $\omega_{X^0/S} \to \mathcal{O}_{X^0}(-B) \otimes \bigwedge^d M$. We claim this extends to an isomorphism $\omega_{X/S} \to \mathcal{O}_X(-B) \otimes \bigwedge^d M$. Since $\omega_{X/S}$ is flat over S and has S_2 fibres it satisfies a relative S_2 property, namely $\omega_{X/S} = j_\star \omega_{X^1/S}$ for $j : X^1 \subset X$ an open subscheme such that the complement has fibres of codimension at least 2 (see [Hacking04, Lem. A.3]). Similarly for $\mathcal{O}_X(-B)$. So, it is enough to check the claim on the open locus $X^1 \subset X$ given by the complement of the torus orbits of codimension at least 2 in the fibres. At a point $P \in X^1$, either X/S is smooth, or $P \notin B$ and the fibre is étale locally isomorphic to $(xy = 0) \subset \mathbb{A}^{d+1}$. In the second case, there is a T-invariant affine open neighbourhood $U \subset X$ of P such that, working étale locally on S, the family $T \curvearrowright U/S$ is of the form

$$\mathbb{G}_m^d \curvearrowright ((xy = f) \subset \mathbb{A}^2_{x,y} \times \mathbb{G}_m^{d-1} \times S),$$

where $f \in \mathcal{O}_S$ and the $\mathbb{G}_m^d$ action on $\mathbb{A}^2_{x,y} \times \mathbb{G}_m^{d-1}$ is given by

$$\mathbb{G}_m \times \mathbb{G}_m^{d-1} \ni (t_0, t) : (x, y, t') \mapsto (t_0 x, t_0^{-1} y, tt').$$

We reduce to the case $d = 1$, $S = \mathbb{A}^1_u$, $f = u$, where the result is well known. $\square$

Let $M = \operatorname{Hom}(H, \mathbb{G}_m) = \sum(x_i = 0) \subset \mathbb{Z}^n$, the characters of H, and $N = M^* = \mathbb{Z}^n/\mathbb{Z}e$.

Proof of Theorem 4.9. Let $\mathcal{U}_e$ and $\mathcal{Q}_e$ denote the universal sub-bundle and quotient bundle on G_e, respectively. We have canonical isomorphisms

$$\omega_p(\mathbb{B}) \cong \omega_{\mathbb{T}/M}(\mathbb{B}) \otimes \bigwedge^{n-r} \mathcal{N}_{\mathbb{S}/\mathbb{T}} \cong \mathcal{O}_{\mathbb{T}} \otimes \bigwedge^{n-1} M \otimes \bigwedge^{n-r} \mathcal{Q}_e|_{\mathbb{S}}$$

by the adjunction formula, Theorem 4.5, and Lemma 4.10. The exact sequence

$$0 \to \mathcal{U}_e \to \mathcal{O}_{G_e} \otimes h \to \mathcal{Q}_e \to 0$$

on G_e yields the isomorphism

$$\mathcal{O}_{G_e} \otimes \bigwedge^{n-1} h^* \otimes \bigwedge^{n-r} \mathcal{Q}_e \cong \bigwedge^{r} \mathcal{U}_e^* = \mathcal{O}_{G_e}(1),$$

where $\mathcal{O}_{G_e}(1)$ is the Plücker line bundle. Composing with the above isomorphism using the equality $M \otimes_{\mathbb{Z}} k = h^*$, we obtain an isomorphism $\omega_p(\mathbb{B}) \cong \mathcal{O}_{G_e}(1)|_{\mathbb{S}}$, as required. $\square$

5. Special sections

Let $I \subset [n]$ be a subset with $|I| = r - 1$ and let $\mathbb{B}_I$ denote the scheme-theoretic intersection $\bigcap_{i \in I} \mathbb{B}_i$.

Proposition 5.1. *$\mathbb{B}_I \subset \mathbb{S}$ is a section of $p : \mathbb{S} \to M$. For each fibre (S, B) of p, S is smooth and B has normal crossings at B_I.*

Proof. Let (T, B_T) be a fibre of $(\mathbb{T}, \mathbb{B}_T)/M$ and (S, B) its visible contour. Write $I = \{i_1, \ldots, i_{r-1}\}$. The scheme $B_I = \bigcap_{i \in I} B_i \subset S$ is the intersection of the scheme $B_{I,T} = \bigcap_{i \in I} B_{i,T}^+ \subset T$ with G_e. The divisor $B_{i,T}^+$ equals the intersection $T \cap G_{e_i}$, where $G_{e_i} \subset G$ is the locus of subspaces containing e_i, by [Kapranov93, Proposition 1.6.10]. Thus

$$B_{I,T} \subset \bigcap_{i \in I} G_{e_i} = \mathbb{P}(k^n / \langle e_i \mid i \in I \rangle) = \mathbb{P}^{\bar{I}}.$$

The subscheme $B_{I,T} \subset T$ corresponds to the face $\Gamma = \bigcap_{i \in I} (x_i = 1)$ of $\Delta(r, n)$, which equals the $(n - r)$-simplex

$$\operatorname{conv}\{e_{i_1} + \cdots + e_{i_{r-1}} + e_j \mid j \notin I\}.$$

We deduce $B_{I,T}$ and $\mathbb{P}^{\bar{I}}$ have the same dimension, and so (the first being a subscheme of the second) are equal. Hence B_I is equal to the point $\langle e, e_{i_1}, \ldots, e_{i_{r-1}} \rangle \in G(r, n)$. In particular, $\mathbb{B}_I$ is a section of $p : \mathbb{S} \to M$.

Let $\underline{P}$ denote the subdivision of $P = \Delta(r, n)$ associated to T. We show that T is smooth at a general point of $B_{I,T}$ by analysing the subdivision $\underline{P}$ at Γ. The polytope P lies in the affine hyperplane $(\sum x_i = r) \subset \mathbb{R}^n$, an affine space under $M_{\mathbb{R}}$. Let $I' = I \cup \{i_r\}$, some $i_r \notin I$, and fix an embedding $P \subset M_{\mathbb{R}}$ by identifying the vertex $e_{i_1} + \cdots + e_{i_r}$ as the origin. Let $\langle S \rangle$ denote the cone and $\langle S \rangle_{\mathbb{R}}$ the vector space generated by a set $S \subset M_{\mathbb{R}}$. Consider the quotient cone

$$\sigma := (\langle P \rangle + \langle \Gamma \rangle_{\mathbb{R}}) / \langle \Gamma \rangle_{\mathbb{R}}.$$

We have $\langle P\rangle = \langle e_j - e_i \mid j \notin I',\ i \in I'\rangle$ and $\langle \Gamma\rangle = \langle e_j - e_{i_r} \mid j \notin I'\rangle$. So, identifying $M_{\mathbb{R}}/\langle\Gamma\rangle_{\mathbb{R}}$ with $(x_j = 0, j \notin I') \subset M_{\mathbb{R}}$, we have

$$\sigma = \langle e_{i_r} - e_i \mid i \in I\rangle.$$

In particular, σ is simplicial, and the generators of $\langle P\rangle$ yield a minimal set of generators of σ. We claim that there is a unique maximal polytope P' of $\underline{P}$ containing Γ. Indeed, the edges of any such P' are also edges of P (since P' is a matroid polytope, see [GS87]), so the corresponding cone $\sigma' \subset \sigma$ is generated by a collection of edges of σ. Hence $\sigma' = \sigma$ because σ is simplicial, and P' is unique as claimed. So T has a unique component T' containing the stratum $B_{I,T}$, and T' is smooth at a general point of $B_{I,T}$ (because σ is simplicial and its edges generate the lattice). We deduce that S is smooth at B_I by Theorem 4.5. $\square$

Recall that $h = k^n/k \cdot e$, the Lie algebra of H.

Theorem 5.2. *Let*

$$\mathrm{res} : p_*\omega_p(\mathbb{B}) \to \bigoplus_I p_*\mathcal{O}_{\mathbb{B}_I} = \bigwedge^{r-1} k^n \otimes \mathcal{O}_M$$

be the canonical map given by taking residues along the special sections. Let

$$c := \bigwedge^{r-1} h^* \otimes \mathcal{O}_M \to p_*\omega_p(\mathcal{B})$$

be the map defining the embedding $\mathbb{S} \subset G_e \times M \subset \mathbb{P}(\bigwedge^{r-1} h) \times M$. The composition

$$\mathrm{res} \circ c : \bigwedge^{r-1} h^* \otimes \mathcal{O}_M \to \bigwedge^{r-1} k^n \otimes \mathcal{O}_M$$

is induced by the inclusion $h^* \subset k^n$, c is an isomorphism, and res *is an isomorphism onto its image.*

Proof. Let $I \subset [n]$ be a subset of size $r-1$. Write $I = \{i_1, \dots, i_{r-1}\}$ where $i_1 < \cdots < i_{r-1}$. The residue map $\omega_p(\mathbb{B}) \to \mathcal{O}_{\mathbb{B}_I}$ is identified with the restriction of the residue map $\omega_{\mathbb{T}/M}(\mathbb{B}) \otimes \bigwedge^{n-r} \mathcal{Q} \to \mathcal{O}_{\mathbb{B}_I}$ on $\mathbb{T}/M$ via the adjunction $\omega_p(\mathbb{B}) \cong \omega_{\mathbb{T}/M}(\mathbb{B}) \otimes \bigwedge^{n-r} \mathcal{Q}|_{\mathbb{S}}$. We explicitly compute this residue map on $\mathbb{T}/M$.

Let $\mathbb{T}^0 \subset \mathbb{T}$ denote the smooth locus of $\mathbb{T}/M$ and $\mathbb{B}_{I,T} = \bigcap_{i \in I} \mathbb{B}_{i,T}$. We have $\mathbb{B}_{I,T} = \mathbb{P}^{\bar{I}} \times M$ where $\mathbb{P}^{\bar{I}} = \mathbb{P}(k^n/\langle e_i \mid i \in I\rangle) \subset G(r,n)$; see the proof of Proposition 5.1. Let $\mathbb{B}^0_{I,T} \subset \mathbb{B}_{I,T}$ be the open (relative) toric stratum. Note $\mathbb{B}^0_{I,T} \subset \mathbb{T}^0$ by Proposition 5.1. Let $N_I = N/\langle e_{i_1}, \dots, e_{i_{r-1}}\rangle$ and $M_I = N_I^* \subset M$. Thus $N_I \otimes \mathbb{G}_m$ is the quotient torus acting faithfully on $\mathbb{B}_{I,T}$.

The adjunction $\omega_{\mathbb{T}^0/M}(\mathbb{B}) \to \omega_{\mathbb{B}^0_{I,T}/M}$ is identified, via the isomorphism of Lemma 4.10, with the map $\mathcal{O}_{\mathbb{T}^0} \otimes \bigwedge^{n-1} M \to \mathcal{O}_{\mathbb{B}^0_{I,T}} \otimes \bigwedge^{n-r} M_I$ induced by the map

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_{r-1}}, \cdot \rangle : \bigwedge^{n-1} M \to \bigwedge^{n-r} M_I.$$

Indeed, the facet $(x_i = 1)$ of P corresponding to $\mathbb{B}_{i,T}$ has outward normal $e_i \in N$, hence a torus invariant differential $d\chi^m/\chi^m$ has residue $\langle e_i, m\rangle$ along $\mathbb{B}^0_{i,T} := \mathbb{B}_{i,T} \cap \mathbb{T}^0$. So, the above map is the Poincaré residue map for $\mathbb{B}^0_{I,T} \subset \mathbb{T}^0$ (cf. [Oda88, p. 120], [Fulton93, p. 87]).

The section $\mathbb{B}_I \subset \mathbb{B}^0_{I,T}$ equals $[e] \times M \subset \mathbb{P}^{\bar{I}} \times M$, so $\mathcal{Q}|_{\mathbb{B}_I} = N_I \otimes \mathcal{O}_{\mathbb{B}_I}$, and $\omega_{\mathbb{B}^0_{I,T}/M} \cong \mathcal{O}_{\mathbb{B}^0_{I,T}} \otimes \bigwedge^{n-r} M_I$ by Lemma 4.10. The residue map $\omega_{\mathbb{B}^0_{I,T}/M} \otimes \bigwedge^{n-r} \mathcal{Q} \to \mathcal{O}_{\mathbb{B}_I}$ is induced by the pairing $\bigwedge^{n-r} M_I \otimes \bigwedge^{n-r} N_I \to \mathbb{Z}$. We obtain the residue map $\omega_{\mathbb{T}/M}(\mathbb{B}) \otimes \bigwedge^{n-r} \mathcal{Q} \to \mathcal{O}_{\mathbb{B}_I}$ as the composition

$$\omega_{\mathbb{T}/M}(\mathbb{B}) \otimes \bigwedge^{n-r} \mathcal{Q} \to \omega_{\mathbb{B}^0_T/M} \otimes \bigwedge^{n-r} \mathcal{Q} \to \mathcal{O}_{\mathbb{B}_I}.$$

We deduce that the composition

$$\bigwedge^{r-1} h^* \otimes \mathcal{O}_{\mathbb{S}} \to \mathcal{O}_{G_e}(1)|_{\mathbb{S}} \to \omega_p(\mathbb{B}) \to \mathcal{O}_{\mathbb{B}_I}$$

is induced by the map $e_{i_1} \wedge \cdots \wedge e_{i_{r-1}} : \bigwedge^{r-1} h^* \to k$. So, the composition

$$\bigwedge^{r-1} h^* \otimes \mathcal{O}_M \to p_*\omega_p(\mathbb{B}) \to \bigoplus_{|I|=r-1} p_*\mathcal{O}_{\mathbb{B}_I} = \bigwedge^{r-1} k^n \otimes \mathcal{O}_M$$

is induced by the inclusion $\bigwedge^{r-1} h^* \subset \bigwedge^{r-1} k^n$ as claimed. Finally, $p_*\omega_p(\mathbb{B})$ is locally free of rank $\binom{n-1}{r-1}$ by Proposition 5.4 below, so $\bigwedge^{r-1} h^* \otimes \mathcal{O}_M \to p_*\omega_p(\mathbb{B})$ is an isomorphism. $\square$

Lemma 5.3. *Let Y be a projective stable toric variety. Let Y^c denote the disjoint union of the strata of Y of codimension c which are not contained in the toric boundary and $p^c : Y^c \to Y$ the natural map. There is an exact sequence of $\mathcal{O}_Y$-modules,*

$$0 \to \mathcal{O}_Y \to p^0_*\mathcal{O}_{Y^0} \to p^1_*\mathcal{O}_{Y^1} \to \cdots . \tag{5.1}$$

Similarly, let B^c denote the disjoint union of the strata of the toric boundary B of codimension c and $q^c : B^c \to B$ the natural map. There is an exact sequence of $\mathcal{O}_B$-modules,

$$0 \to \mathcal{O}_B \to q^0_*\mathcal{O}_{B^0} \to q^1_*\mathcal{O}_{B^1} \to \cdots . \tag{5.2}$$

Proof. Let $\underline{P}$ be the subdivision of a lattice polytope $P \subset M_{\mathbb{R}}$ associated to Y, and write $d = \dim Y$. The sequences are defined as follows. Fix an orientation of each face $P' \in \underline{P}$. For $P'' \subset P'$ a facet and $Y'' \subset Y'$ the corresponding strata of Y, the map $\mathcal{O}_{Y'} \to \mathcal{O}_{Y''}$ is defined to be the restriction map with sign $+1$ if P' and P'' are oriented compatibly and -1 otherwise.

We assume that each maximal polytope and each boundary facet has the orientation induced by some fixed orientation of P, then the maps $\mathcal{O}_Y \to p^0_*\mathcal{O}_{Y^0}$ and $\mathcal{O}_B \to q^0_*\mathcal{O}_{B^0}$ are the restriction maps (no signs).

Let R be the homogeneous coordinate ring of Y. By the definition of stable toric varieties, R is the inverse limit of a system $(R_\sigma, p_{\sigma\tau})$. The sequence of homogeneous coordinate rings associated to the sequence (5.1) is the sequence

$$0 \to R \to R^0 \to R^1 \to \cdots \tag{5.3}$$

where R^c is the direct sum of the R_σ for $\sigma \in \Omega$ an interior cone of codimension c, and the maps $R_\sigma \to R_\tau$ for $\tau \subset \sigma$ a facet are $\pm p_{\sigma\tau}$, with the signs determined as above. Note that by definition the truncated sequence $0 \to R \to R^0 \to R^1$ is exact.

The sequence (5.3) is a direct sum of sequences of k-vector spaces

$$0 \to R_a \to R^0_a \to R^1_a \to \cdots$$

indexed by $a \in \omega \cap A$. Recall that $R_{\sigma,a} = k\cdot\chi^a$ if $a \in \sigma$ and $R_{\sigma,a} = 0$ otherwise. We identify the sequence R^i_a with the complex $C_{d-i}(K,L)$ computing the homology of the pair (K,L) of CW-complexes, where $K = \underline{P}$ and L is the subcomplex consisting of polytopes $P' \in \underline{P}$ such that $a \notin \mathrm{Cone}(P')$ or $P' \subset \partial P$. Let υ denote the cone of Ω containing a in its relative interior. The isomorphism $R^i_a \to C_{d-i}(K,L)$ is given by

$$R_{\sigma,a} \ni \chi^a \mapsto a(t_{\sigma\upsilon})[P'],$$

where $\sigma = \mathrm{Cone}(P')$ and $[P']$ denotes the generator of $C_{d-i}(K,L)$ corresponding to P' with its chosen orientation. (The coefficient $a(t_{\sigma\upsilon}) \in k^\times$ ensures that the isomorphism is compatible with the boundary maps.) For $a \neq 0$, the pair (K,L) is homotopy equivalent to the pair $(B^d, B^d - p)$, where B^d is a ball of dimension d and $p \in B^d$ an interior point. So $H_i(K,L) = k$ for $i = d$ and $H_i(K,L) = 0$ otherwise. Thus the graded piece of the sequence (5.3) of weight a is exact for $a \neq 0$. It follows that the sequence (5.1) of sheaves on Y associated to (5.3) is exact.

A similar argument shows that the sequence (5.2) is exact. Let

$$0 \to S \to S^0 \to S^1 \to \cdots \tag{5.4}$$

be the associated sequence of homogeneous coordinate rings. The sequence S^i_a is identified with $C_{d-1-i}(K,L)$, where K is the subcomplex of $\underline{P}$ with support ∂P and $L \subset K$ is the subcomplex of faces P' such that $a \notin \mathrm{Cone}(P')$. For $a \neq 0$, the pair (K,L) is homotopy equivalent to $(S^{d-1}, S^{d-1} - p)$, where S^{d-1} is a sphere of dimension $(d-1)$ and $p \in S^{d-1}$ a point. We deduce that the graded piece of the sequence (5.4) of weight a is exact for $a \neq 0$, and the sequence (5.2) of sheaves on Y associated to (5.4) is exact, as required. □

Proposition 5.4. *For each fibre (S,B) of $(\mathbb{S},\mathbb{B})/M$, $\dim_k H^0(\omega_S(B)) = \binom{n-1}{r-1}$ and $H^i(\omega_S(B)) = 0$ for $i > 0$. Thus $p_*\omega_p(\mathbb{B})$ is locally free of rank $\binom{n-1}{r-1}$ and commutes with base change.*

Proof. The variety S is Cohen-Macaulay by [Alexeev02, 2.3.29] and Corollary 4.6. By Serre duality,

$$H^i(\omega_S(B)) = \operatorname{Ext}^{r-1-i}(\omega_S(B), \omega_S)^* = H^{r-1-i}(\mathcal{O}_S(-B))^*,$$

using S Cohen-Macaulay and $\omega_S(B)$ invertible. We calculate the cohomology groups $H^i(\mathcal{O}_S(-B))$ using the exact sequence

$$0 \to \mathcal{O}_S(-B) \to \mathcal{O}_S \to \mathcal{O}_B \to 0.$$

We compute below that $H^i(\mathcal{O}_S) = 0$ for $i > 0$, $H^i(\mathcal{O}_B) = 0$ for $0 < i < r-2$ and $\dim_k H^{r-2}(\mathcal{O}_B) = \binom{n-1}{r-1}$, thus $H^i(\mathcal{O}_S(-B)) = 0$ for $i < r-1$ and $\dim_k H^{r-1}(\mathcal{O}_S(-B)) = \binom{n-1}{r-1}$, as required.

Let (T, B_T) be the fibre of $(\mathbb{T}, \mathbb{B}_T)/M$ associated to (S,B). Let T^c denote the disjoint union of the strata of T of codimension c which are not contained in the boundary B_T and $p^c : T^c \to T$ the natural map. By Lemma 5.3, there is an exact sequence

$$0 \to \mathcal{O}_T \to p^0_*\mathcal{O}_{T^0} \to p^1_*\mathcal{O}_{T^1} \to \cdots .$$

Defining $p^c : S^c \to S$ analogously, we obtain an exact sequence

$$0 \to \mathcal{O}_S \to p^0_*\mathcal{O}_{S^0} \to p^1_*\mathcal{O}_{S^1} \to \cdots$$

by restriction, using smoothness of $H \times S \to T$. For each stratum S' of S we have $H^i(\mathcal{O}_{S'}) = 0$ for $i > 0$ by Lemma 5.5. So $H^i(\mathcal{O}_S)$ is the ith cohomology of the complex

$$0 \to H^0(\mathcal{O}_{S^0}) \to H^0(\mathcal{O}_{S^1}) \to \cdots .$$

By Theorem 4.5, the nonboundary strata of S are in bijection with the nonboundary strata of T. Let $K = \underline{P}$, the subdivision of P associated to T, and let $L \subset K$ be the subcomplex with support ∂P. Then the complex $H^0(\mathcal{O}_{S^i})$ is identified with the complex $C_{n-1-i}(K, L)$ computing the homology of the pair (K, L) of CW-complexes (cf. Proof of Lemma 5.3). We deduce that $H^i(\mathcal{O}_S) = 0$ for $i > 0$.

Similarly, we obtain an exact sequence

$$0 \to \mathcal{O}_B \to q^0_*\mathcal{O}_{B^0} \to q^1_*\mathcal{O}_{B^1} \to \cdots$$

where $q^c : B^c \to B$ are the strata of B of codimension c, and $H^i(\mathcal{O}_B)$ is the ith cohomology of the complex

$$0 \to H^0(\mathcal{O}_{B^0}) \to H^0(\mathcal{O}_{B^1}) \to \cdots .$$

The strata of B are in bijection with the strata of B_T which are not contained in $\bigcup B^-_{i,T}$. Here $B^-_{i,T}$ is the component of B corresponding to the facet $(x_i = 0)$ of P. Let K denote the subcomplex of $\underline{P}$ with support ∂P and let $L \subset K$ be the subcomplex with support $\bigcup(x_i = 0)$. Then the complex $H^0(\mathcal{O}_{B^i})$ is identified with the complex $C_{n-2-i}(K, L)$. To compute the homology, we may replace $\underline{P}$ by the trivial subdivision. There is then an isomorphism of chain complexes $C.(K, L) \to C.(\Delta^{(n-2)}_{[n]}, \Delta^{(n-r)}_{[n]})$, where $\Delta_{[n]}$ denotes the simplex with vertices labelled by $[n]$ and $\Delta^{(m)}_{[n]}$ its m-skeleton, which sends the facet $(x_i = 1)$ of P to $\Delta_{[n]\setminus\{i\}}$. We find $\dim_k H^{n-r}(K, L) = \binom{n-1}{r-1}$ and $H^i(K, L) = 0$ for $i \neq n-2, n-r$. Explicitly, $H^{n-r}(K, L)$ is the cokernel of the boundary map $C_{n-r+1}(\Delta_{[n]}) \to C_{n-r}(\Delta_{[n]})$, which may be identified with the map

$$\bigwedge^{r-2} k^n \to \bigwedge^{r-1} k^n, \ v \mapsto e \wedge v.$$

Then $H^{n-r}(K, L)$ is identified with $\bigwedge^{r-1} h$ where $h = k^n/k \cdot e$. We deduce that $\dim_k H^{r-2}(\mathcal{O}_B) = \binom{n-1}{r-1}$ and $H^i(\mathcal{O}_B) = 0$ for $0 < i < r-2$. □

Lemma 5.5. *Let S' be a closed stratum of a fibre S of the visible contour family $\mathbb{S} \to M$. S' is rational with rational singularities.*

Proof. By Theorem 4.5, S' has singularities no worse than those of the corresponding stratum of T (the corresponding fibre of $\mathbb{T} \to M$), which is a normal toric variety (and, in particular, has at worst rational singularities) by Corollary 3.11. By [Kapranov93, 3.1.9], S' is rational—it compactifies the complement to a hyperplane arrangement. □

6. Very stable pairs

Definition 6.1. A very stable pair over a k-scheme Z is a family $q\colon (\mathcal{S}, \mathcal{B}) \to Z$ of pairs with stable toric singularities, where $\mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_n$, satisfying the following conditions:

(1) $\mathcal{S}, \mathcal{B}_1, \ldots, \mathcal{B}_n$ are flat over Z and the sheaf $\omega_q(\mathcal{B})$ is a line bundle.
(2) For each subset $I \subset [n]$ with $|I| = r-1$, $\mathcal{B}_I := \bigcap_{i \in I} \mathcal{B}_i \subset \mathcal{S}$ is a section of q. For each fibre (S, B) of q, S is smooth and B has normal crossings at B_I.
(3) The residue map $q_*\omega_q(\mathcal{B}) \to \bigoplus_I q_*\mathcal{O}_{\mathcal{B}_I} = \bigwedge^{r-1} k^n \otimes \mathcal{O}_Z$ is an isomorphism onto $\bigwedge^{r-1} h^* \otimes \mathcal{O}_Z \subset \bigwedge^{r-1} k^n \otimes \mathcal{O}_Z$. Let $c : \bigwedge^{r-1} h^* \otimes \mathcal{O}_Z \to q_*\omega_q(\mathcal{B})$ denote its inverse.
(4) The line bundle $\omega_q(\mathcal{B})$ and the isomorphism c define an embedding $\mathcal{S} \subset \mathbb{P}(\bigwedge^{r-1} h) \times Z$ which factors through $G(r-1, h) \times Z$.

(5) Let $\mathcal{T}$ denote the sweep closure $\overline{H\mathcal{S}}$ of

$$\mathcal{S} \subset G(r-1,h) \times Z = G(r-1,n-1)_e \times Z \subset G(r,n) \times Z$$

and similarly let $\mathcal{B}^+_{i,T} = \overline{H\mathcal{B}_i}$ for each i. Then the affine cone over $\mathcal{T}/Z$ is a Z-valued point of the toric Hilbert scheme H^h_S, and $\mathcal{B}^+_{i,T}$ is the component of the relative toric boundary of $\mathcal{T}/Z$ corresponding to the facet $(x_i = 1)$ of $\Delta(r,n)$.

Remark 6.2. For $H \curvearrowright X$ a group acting on a scheme X and $Y \subset X$ a subscheme of X, the sweep closure $\overline{HY}$ is by definition the scheme-theoretic image of the multiplication map $H \times Y \to X$. For $f : Z \to X$ a map of schemes, the scheme-theoretic image of f is the closed subscheme of X defined by the ideal sheaf $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Z)$.

We stress (at the referee's suggestion) that all the properties above are *conditions*: we make no claims, only requirements.

Theorem 6.3. *M is a fine moduli space of very stable pairs, with universal family the family of visible contours $p : (\mathbb{S}, \mathbb{B}) \to M$.*

Proof. An arbitrary pullback of the visible contour family $(\mathbb{S}, \mathbb{B})/M$ is a family of very stable pairs by Theorem 4.5, Theorem 4.9, Proposition 5.1, Theorem 5.2, Proposition 5.4, and Lemma 6.4 below. It remains to check that $(\mathbb{S}, \mathbb{B})/M$ is universal. Let $(\mathcal{S}, \mathcal{B})/Z$ be a family of very stable pairs, and consider the associated visible contour family

$$(\mathcal{S}', \mathcal{B}') = (\overline{H\mathcal{S}}, \overline{H\mathcal{B}}) \cap G_e \times Z$$

which is obtained by pullback from $(\mathbb{S}, \mathbb{B})/M$. Consider the closed embedding $\mathcal{S} \subset \mathcal{S}'$. Let $S \subset S'$ be the restriction to a general fibre; we claim $S = S'$. Since S and S' are reduced and have pure dimension $r-1$, S is a union of irreducible components of S'. Each component S'_j of S' is of the form $T_j \cap G_e$, where T_j is a component of the stable toric variety $T = \overline{HS}$. Let x_j be a point of S in the interior of the toric variety T_j. Then S'_j is the only irreducible component of S' containing x_j, so $S'_j \subset S$. Hence $S = S'$ as claimed. We deduce $\mathcal{S} = \mathcal{S}'$ by flatness. The same argument shows $\mathcal{B}_i = \mathcal{B}'_i$. □

Lemma 6.4. *Let $Z \to M$ be a morphism and let $\mathcal{S}, \mathcal{B}_i, \mathcal{T}, \mathcal{B}^+_{i,T}$ denote the pullbacks of $\mathbb{S}, \mathbb{B}_i, \mathbb{T}, \mathbb{B}^+_{i,T}$. The sweep closures $\overline{H\mathcal{S}}, \overline{H\mathcal{B}_i}$ are equal to $\mathcal{T}, \mathcal{B}^+_{i,T}$.*

Proof. The map $H \times \mathcal{S} \to \mathcal{T}$ is smooth, with image $\mathcal{T}^0 := \mathcal{T} - \bigcup \mathcal{B}^-_{i,T}$, where $\mathcal{B}^-_{i,T} = \mathbb{B}^-_{i,T}|_{\mathcal{T}}$. Hence $\overline{H\mathcal{S}} = \overline{\mathcal{T}^0}$. Since $\mathcal{T}/Z$ is flat with reduced fibres, any embedded component of $\mathcal{T}$ contains a fibre by [Matsumura89, 23.2]. In particular, there are no embedded components contained in $\mathcal{T} - \mathcal{T}^0$, so $\overline{\mathcal{T}^0} = \mathcal{T}$. The same argument proves $\overline{H\mathcal{B}_i} = \mathcal{B}^+_{i,T}$. □

7. Example

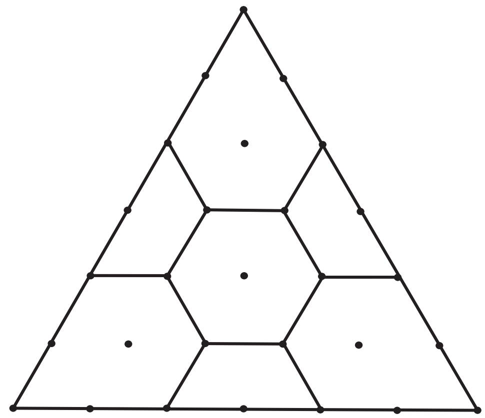
We show that, for $(r, n) = (3, 9)$, M has an irreducible component besides the closure of M^0. Moreover, this component is not contained in the image of the Lafforgue space $\overline{\Omega}$ (see Section 3.4). The example is a version of Alexeev's example [Alexeev02, 2.16].

We describe a stable pair (S, B) which is a limit of generic arrangements of 9 lines in $\mathbb{P}^2$ such that the deformation space $\mathrm{Def}(S, B)$ is reducible. More precisely, $\mathrm{Def}(S, B)$ has two smooth components D_1 and D_2 such that D_1 parametrises locally trivial deformations and D_2 contains the smoothings of (S, B). Let $P = [(S, B)]$ denote the corresponding point of M. We show that the map of germs $(P \in M) \to \mathrm{Def}(S, B)$ is an isomorphism, and the image of Lafforgue's space $\overline{\Omega}$ in M maps isomorphically onto the smoothing component D_2.

Let $\bar{S} = \mathbb{P}^2$ and let $\bar{B} = \bar{B}_1 + \cdots + \bar{B}_9$ be an arrangement of 9 lines in $\mathbb{P}^2$ as follows: for $i = 1, 2, 3$, the lines $\bar{B}_i$ are in general position, $\bar{B}_{i+3} = \bar{B}_i$, and $\bar{B}_{i+6}$ is a generic line through $\bar{B}_i \cap \bar{B}_{i+1 \bmod 3}$. Let $(\bar{\mathcal{S}}, \bar{\mathcal{B}})/T$ be a generic one-parameter smoothing of the pair $(\bar{S}, \bar{B})$. Let $\mathcal{S} \to \bar{\mathcal{S}}$ be the birational morphism given by first blowing up the points $B_i \cap B_{i+1 \bmod 3}$, $i = 1, 2, 3$, then blowing up the strict transforms of the lines B_i, $i = 1, 2, 3$. Let $\mathcal{B}$ denote the strict transform of $\bar{\mathcal{B}}$ and (S, B) the special fibre of $(\mathcal{S}, \mathcal{B})/T$. Then $\mathcal{S}$ is smooth and $S + \mathcal{B}$ is a simple normal crossing divisor. One checks that the line bundle $\omega_S(B) = \omega_{\mathcal{S}/T}(\mathcal{B})|_S$ is ample. Thus (S, B) is a stable pair.

The deformation space of the surface S may be computed using the results of [Friedman83]. We find that $\mathrm{Def}\, S$ is the union of two smooth curve germs V_1 and V_2 which intersect transversely. Here V_1 parametrises locally trivial deformations of S, and V_2 gives the (essentially unique) one-parameter smoothing. The forgetful map $F : \mathrm{Def}(S, B) \to \mathrm{Def}(S)$ is smooth since B_i is Cartier and $H^1(\mathcal{N}_{B_i/S}) = 0$ for each i (here $\mathcal{N}_{B_i/S}$ denotes the normal bundle of B_i in S). Thus $\mathrm{Def}(S, B)$ is a union of two smooth components $D_i = F^{-1}(V_i)$, $i = 1, 2$, as claimed.

We briefly explain the existence of locally trivial deformations of S. If S is a reducible surface with simple normal crossing singularities, there is a canonically defined line bundle $\mathcal{O}_D(-S)$ on the double curve D of S given by $\mathcal{I}_{S_1}|_D \otimes \cdots \otimes \mathcal{I}_{S_l}|_D$, where $S_1, \cdots, S_l$ are the irreducible components of S, and $\mathcal{I}_{S_i}$ denotes the ideal sheaf of $S_i \subset S$. If S admits a one-parameter smoothing $\mathcal{S}/T$ such that the total space is smooth, then $\mathcal{O}_D(-S)$ is isomorphic to $\mathcal{O}_D$ (because $\mathcal{O}_D(-S) = \mathcal{O}_{\mathcal{S}}(-S)|_D$ and $\mathcal{O}_{\mathcal{S}}(-S) \cong \mathcal{O}_{\mathcal{S}}$). If S' is a locally trivial deformation of S, the line bundle $\mathcal{O}_{D'}(-S')$ lies in $\mathrm{Pic}^0(D')$ but is



nontrivial in general. In our example, $\mathrm{Pic}^0(D) \cong \mathbb{G}_m$ (because D is a union of rational components and contains a unique cycle), and there are locally trivial deformations S' of S given by changing the gluing of the components of S such that $\mathcal{O}_{D'}(-S')$ is a nontrivial line bundle on D'.

We show that the map $(P \in M) \to \mathrm{Def}(S, B)$ is an isomorphism. By Theorem 6.3, it is a closed embedding, and its image contains the smoothing component D_2. It remains to prove that a general fibre over the component D_1 of $\mathrm{Def}(S, B)$ is a fibre of the visible contour family $(\mathbb{S}, \mathbb{B})/M$. Let (S, B) be an arbitrary fibre over D_1. The surface S may be identified with the stable toric variety defined by a subdivision of the standard triangle of side length 6 (see the figure) and some gluing data. The torus action determines a locally free sheaf $\Omega_S(\log)$ on S obtained by gluing the locally free sheaves $\Omega_{S_i}(\log \Delta_i)$ on the components S_i at the double locus (here Δ_i denotes the double locus on S_i). There is a natural map $\Omega_S \to \Omega_S(\log)$. Let $\Omega_S(\log B)$ be the $\mathcal{O}_S$-module generated by $\Omega_S(\log)$ and $\{\frac{df}{f} \mid f \in \mathcal{O}_U^\times\}$, where $U = S \setminus B$. Then $\Omega_S(\log B)$ is also locally free, and there is an exact sequence

$$0 \to \Omega_S(\log) \to \Omega_S(\log B) \to \bigoplus \mathcal{O}_{B_i} \to 0$$

where the last map is given by taking residues along the B_i. The residue map induces an isomorphism $H^0(\Omega_S(\log B)) \to h^* = (\sum x_i = 0) \subset k^n$. This defines an embedding $(S, B) \subset G(r-1, h) = G(r-1, n-1)_e$. For S' a component of S, let B' denote the divisor on S' given by the restriction of B and the double locus. Then $U' = S' \backslash B'$ is the complement of a hyperplane arrangement, (S', B') is the log canonical model of U', and $\Omega_{S'}(\log B') = \Omega_S(\log B)|_{S'}$. One checks that the induced map $h^* \to H^0(\Omega_{S'}(\log B'))$ coincides with the map of Theorem 2.2. Thus the locus $\overline{HS'}$ in $G(r, n)$ is the closure of a single H-orbit. The weight polytopes $P' \subset P = \Delta(r, n)$ of the orbit closures $\overline{HS'}$

define a subdivision of P (because this only depends on the combinatorial type of (S, B), and holds for the fibre over $0 \in D_2$). Hence $\overline{HS}$ defines a point of the toric Hilbert scheme H_S^h, and (S, B) is its visible contour. Thus (S, B) is a fibre of the visible contour family over M, as required.

The Lafforgue space $\overline{\Omega}$ is a moduli space of varieties with log structures. We refer to [Kato89] for background on log structures. Given a pair $[(S, B)] \in M$ which lies in the image of $\overline{\Omega}$, a point of $\overline{\Omega}$ over $[(S, B)]$ corresponds to a log structure on S/k which (in particular) determines the divisors $B_i \subset S$. In our example, the log structure on S/k is the restriction of the log structure on the smoothing $\mathcal{S}/T$ defined by the divisors $S + \mathcal{B} \subset \mathcal{S}$ and $0 \in T$. By [KN94] the log deformations of S/k are parametrised by the component $D_2 \subset \mathrm{Def}(S, B)$, thus the germ of $\overline{\Omega}$ at S/k maps isomorphically onto D_2.

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