ARTIN’S APPROXIMATION THEOREM

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Basic question: Let $F$ be (some moduli) functor $(\text{Sch}/S)^0 \to \text{Sets}$. Can we assert representability of $F$ based on local properties of $F$, e.g. if its deformation theory is “good”?

Let $A$ be a Noetherian ring, $m \subset A$ an ideal (will be a maximal ideal, but does not matter). Let $\hat{A}$ be a completion. Let $F : (A - \text{Alg}) \to \text{Sets}$ be a functor. Let $c \in \mathbb{N}$.

Q1: can $\xi \in F(\hat{A})$ be approximated by some $\xi \in F(A)$ modulo $m^c$?

0.1. Definition. A functor $F : (A - \text{Alg}) \to \text{Sets}$ is called of finite presentation (FP) if it commutes with colimits:

$$\lim F(B_i) \simeq F(\lim B_i).$$

By Grothendieck, for representable functors this is equivalent to the usual definition of “finite presentation” (finitely many generators and relations).

Let $B$ be an $A$-algebra. Then $B = \lim_i B_i$, where all $B_i$’s are algebras of finite presentation. Let $B_i \simeq A[Y]/(f(Y))$, where $Y = \{Y_1, \ldots, Y_N\}$, $f = (f_1, \ldots, f_m)$.

Giving $\phi : B_i \to C$, for some $A$-algebra $C$ is equivalent to giving a solution to $f(Y) = 0$ over $C$.

Since $F$ is FP, for any $\xi \in F(B)$ there exists an $i$ such that $\xi$ is induced by $\xi_i$. Thus, given $\xi \in F(B)$, there exists a system $f(Y)$ such that to every solution $y_C \in C$ of $f(Y) = 0$, one has a functorial assignment of an object $\xi_{yc} \in F(C)$.

Thus Q1 can be answered affirmatively if the following question can answered:

Let $(f_1, \ldots, f_m) \in A[Y]$, let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in \hat{A}$ be a solution to $f(Y) = 0$ in $\hat{A}$. Then, given natural $c$, does there exist a solution $y = (y_1, \ldots, y_m)$ in $A$ such that $y_i = \bar{y}_i \mod m^c$?

0.2. Remark. Q2 admits an affirmative answer if $f(Y) = 0$ is a linear system and $A$ is local, $m \neq A$, by the faithful flatness of the inclusion $A \subset \hat{A}$.

Thus it is natural to study Q2 etale locally for Henselian rings.

0.3. Definition. $A$ is Henselian if given a solution $\bar{y}^0 \in A/m$ to a system with the Jacobian not equal to 0, there exists a solution $y$ in $A$ which reduces to $\bar{y}^0$.

Let $R$ be a field or an excellent DVR.

$A$ is a henselization of a finite type $R$-algebra at a prime ideal.

$m \subset A$ a proper ideal.

0.4. Theorem. Given a system $f(Y) = 0$ with coefficients in $A$, a solution $\bar{y}$ in $\hat{A}$, $c \in N$, there exists a solution to $f(Y) = 0$ over $A$ that reduces to $\bar{y}$ modulo $m^c$.

0.5. Corollary. With above assumptions, for any (FP) functor, given $\xi \in F(\hat{A})$, there exists $\xi \in F(A)$ such that $\xi = \xi \mod m^c$.

Application (R.A as above)

0.6. Theorem. Let $S = \text{Spec} A$, $f : X \to S$ proper morphism Then,

$$\theta : H^1(X, GL(N)) \to \lim H^1(X_n, GL(N))$$
is injective with dense image, where \( X_n = X \times_S \text{Spec}(A/m^{n+1}) \)

Proof. By Grothendieck’s existence theorem \( H^1(\hat{X}, GL(N)) \simeq \lim_{\leftarrow} H^1(X_n, GL(N)) \), where \( \hat{X} = X \times_S \text{Spec}(\hat{A}) \).

(EGA IV.8) implies that \( H^1(X \times_S \cdot, GL(N)) \) is FP.

By approximation theorem, the image of \( \theta \) is dense (just for the stupid direct limit topology).

For injectivity, if \( \theta(L) \) is free with trivializing sections \( \hat{s}_1, \ldots, \hat{s}_N \) then, by approximation theorem \( (H^0) \) is also FP) there exist sections \( s_1, \ldots, s_N \) of \( L \) that approximate \( \hat{s}_i \) modulo \( m \). So by Nakayama Lemma, \( s_1, \ldots, s_N \) trivialize \( L \). \qed

Now: algebraization theorem.

Let \( S \) be a scheme locally of FP over a field or an excellent Dedekind domain. Let \( F : (\text{Sch}/S)^0 \to \text{Sets} \) a functor.

Let \( X = \text{Spec} A \in \text{Sch}/S \).

0.7. Definition. A formal deformation \( \hat{A}, \xi_n \in F(A/m^{n+1}) \) is said to be effective if there exists a deformation \( \hat{A}, \xi \in F(\hat{A}) \) inducing \( \xi_n \).

0.8. Theorem. Assume \( F \) is FP and \( (\hat{A}, \xi) \) be an effective versal deformation. Let \( k' = A/m. \) Then there exists a scheme \( X \in (\text{Sch}/S) \), a closed point \( x \in X \) with residue field \( k' \), \( \xi \in F(X) \) such that \( \hat{O}_{X,x} \simeq \hat{A} \) inducing \( (\hat{A}, \xi_n) \). If \( (\hat{A}, \xi_n) \) is universal then \( X \) is unique.