# DERIVED CATEGORY OF MODULI OF POINTED CURVES - II 

ANA-MARIA CASTRAVET AND JENIA TEVELEV<br>In memory of Yuri Manin


#### Abstract

We prove a conjecture of Manin and Orlov: the moduli space of stable rational curves with $n$ marked points has a full exceptional collection invariant under the action of the symmetric group $S_{n}$, permuting the marked points. In particular, its K-group with integer coefficients is a permutation $S_{n}$-lattice.


## 1. INTRODUCTION

This is a final paper in the sequence devoted to the derived category of moduli spaces of stable rational curves with marked points. We will prove the following theorem conjectured by Manin and Orlov:
Theorem 1.1. The Grothendieck-Knudsen moduli space $\overline{\mathcal{M}}_{0, n}$, of stable rational curves with $n$ marked points, has a full exceptional collection invariant under the action of the symmetric group $S_{n}$ permuting the marked points. In particular, the $K$-group of $\overline{\mathcal{M}}_{0, n}$ with integer coefficients is a permutation $S_{n}$-lattice.

A sequence of objects $E_{1}, \ldots, E_{r}$ in the bounded derived category $D^{b}(X)$ of a smooth projective variety $X$ over $\mathbb{C}$ is exceptional if $R \operatorname{Hom}\left(E_{i}, E_{i}\right)=\mathbb{C}$ for all $i$ and $R \operatorname{Hom}\left(E_{i}, E_{j}\right)=0$ for all $i>j$. An exceptional collection is called full if $D^{b}(X)$ is the smallest full triangulated subcategory containing all $E_{i}$. If a full, exceptional collection exists, then the K-group of $X$ with integer coefficients is freely generated by the classes $\left[E_{1}\right], \ldots,\left[E_{r}\right]$.

The existence of full exceptional collections on $\overline{\mathcal{M}}_{0, n}$ is a straightforward consequence of Kapranov's description [Kap93] of $\overline{\mathcal{M}}_{0, n}$ as an iterated blow-up of $\mathbb{P}^{n}$, Orlov's theorem on semi-orthogonal decompositions (s.o.d.) on blow-ups [Or192] and Beilinson's theorem [Beĭ78]: the lines bundles $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ form a full exceptional collection in $D^{b}\left(\mathbb{P}^{n}\right)$. However, Kapranov's model and therefore these collections are not $S_{n^{-}}$ invariant. On the other hand, $\overline{\mathcal{M}}_{0, n}$ admits a presentation as an iterated $S_{n}$-equivariant blow-up of the symmetric GIT quotient $\mathrm{X}_{n}=\left(\mathbb{P}^{1}\right)^{n} / / \mathrm{PGL}_{2}$. We compute "equivariant minimal models" of all loci appearing in the process of blowing-up in Section 2, which gives an equivariant s.o.d. of derived category of $\overline{\mathcal{M}}_{0, n}$ into derived categories of much simpler moduli spaces, which we call $\overline{\mathrm{M}}_{p, q}$, which parametrize nodal curves with $p$ "heavy" and $q$ "light" points. The study of these moduli spaces, which are "equivariantly minimal" with respect to $S_{p} \times S_{q}$, is the main focus of this paper.

Understanding the derived category of $\overline{\mathcal{M}}_{0, n}$ was initiated in the work of Manin and Smirnov [MS13] and Ballard, Favero and Katzarkov [BFK19]. Part of the motivation in [MS13] was to study the relationship between
derived categories and quantum cohomology, by analogy with Dubrovin's conjecture for Fano varieties [Dub98, KS20].
The $S_{n}$-character of $\mathrm{H}^{*}\left(\overline{\mathcal{M}}_{0, n}, \mathbb{Q}\right)$ appears in the theory of modular operads [GK98, Get95]. In [Get95], Getzler gives recursive formulas for the $S_{n}$-character using mixed Hodge theory. Bergstrom and Minabe [BM13] gave another algorithm, using Hassett's moduli spaces of weighted stable curves [Has03], and computed the length of the $S_{n}$-module $\mathrm{H}^{*}\left(\overline{\mathcal{M}}_{0, n}, \mathbb{Q}\right)$, improving, for genus 0 , results of Faber and Pandharipande [FP13]. Other work on the $S_{n}$-representations given by (pieces of) the cohomology of special cases of Hassett spaces, such as symmetric GIT quotients of $\left(\mathbb{P}^{1}\right)^{n}$, appear in [HK98]. All mentioned recursive formulas are, however, not "effective" in the sense that the sums involve $\pm$ signs. The fact that the $S_{n^{-}}$ module $\mathrm{H}^{*}\left(\overline{\mathcal{M}}_{0, n}, \mathbb{Q}\right)$ is a permutation representation is new and allows for a straightforward decomposition of this module into a sum of irreducible $S_{n}$-representations. Permutation actions of $\operatorname{Aut}(X)$ on $K_{0}(X)$ seem rare, see Example 2.3 for an example of a variety with an involution such that the corresponding action of $S_{2}$ on $K_{0}$ is not a permutation representation. The insight of Yuri Manin was that $\overline{\mathcal{M}}_{0, n}$, as an $S_{n}$-variety, is "defined over the field with one element". Thus one can expect that the same is true for its total cohomology. From this perspective, it is also natural to wonder that if $X$ is a smooth toric variety admitting an action of a finite group $\Gamma$ normalizing the torus action, then $D^{b}(X)$ has a full $\Gamma$-equivariant exceptional collection and thus $K_{0}(X)$ is a permutation $\Gamma$-module. A similar question in the arithmetic setting was raised by Merkurjev and Panin [MP97]. We hope that techniques of this paper will be useful to handle this conjecture.

To prove Thm. [1.1 we follow a strategy outlined in [CT20a] and inspired by the work in [BM13]. The first step, accomplished in Section 2 is to show that $D^{b}\left(\overline{\mathcal{M}}_{0, n}\right)$ admits an invariant s.o.d. with blocks given by $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$ (for various $p$ and $q$ ), where $\overline{\mathrm{M}}_{p, q}$ is the Hassett space parametrizing $p$ "heavy" and $q$ "light" points on a nodal rational curve. Concretely, when $p=2 r+1$ is odd, at most $r$ of the heavy points may coincide and moreover, they may coincide with all the light points. When $p=2 r$ is even then $(r-1)$ of the heavy points may coincide with each other and with all the light points. Furthermore, if $q>0$ then $r$ heavy points may coincide, and they may further coincide with at most $\left\lfloor\frac{q-1}{2}\right\rfloor$ light points. We hope that these "blocks" of $\overline{\mathcal{M}}_{0, n}$ will find other uses in moduli theory. The group $S_{p} \times S_{q}$ acts on $\overline{\mathrm{M}}_{p, q}$ by permuting heavy and light points separately. The bulk of the paper is devoted to the equivariant description of $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$.

Exceptional collections on $\overline{\mathrm{M}}_{p, q}$ contain vector bundles $F_{l, E}$ of rank $l+1$ indexed by an integer $l \geq 0$ and a subset $E \subseteq\{1, \ldots, p+q\}$ such that $l+e$ is even, where $e=|E|$. For example, $F_{0, \emptyset}=\mathcal{O}$. At a point $[C]$ of $\overline{\mathrm{M}}_{p, q}$ which corresponds to an irreducible curve $C \cong \mathbb{P}^{1}$, the fiber of $F_{l, E}$ is equal to

$$
\left.F_{l, E}\right|_{[C]}=H^{0}\left(C, \omega_{C}^{\otimes \frac{e-l}{2}}(E)\right),
$$

where we identify $E$ with a subset of sections. The group $S_{p} \times S_{q}$ acts on the set of vector bundles $\left\{F_{l, E}\right\}$ via its action on the set of subsets $\{E\}$.

The result for $\overline{\mathrm{M}}_{p, q}$ is a combination of Theorems 1.2, 3.19, 1.5 and 1.8 . We start with the following basic case, where we denote $\overline{\mathrm{M}}_{p}:=\overline{\mathrm{M}}_{p, 0}$.
Theorem 1.2. Let $p=2 r+1$. The vector bundles $\left\{F_{l, E}\right\}$ form a full strong $S_{p}$-equivariant exceptional collection in $D^{b}\left(\overline{\mathrm{M}}_{p}\right)$ under the following condition: $l+\min (e, p-e) \leq r-1$ and $l+e$ is even. The vector bundles are ordered by increasing $e$, and for a given $e$, arbitrarily.

When $p$ is odd, $\overline{\mathrm{M}}_{p}$ is a symmetric GIT quotient $\left(\mathbb{P}^{1}\right)^{p} / / \mathrm{PGL}_{2}$ and we prove Theorem 1.2 using the theory of windows into derived categories of GIT quotients [Tel00, HL15, BFK19] . This is done in Section 3, where we also study derived category of the stack of $\mathbb{P}^{1}$-bundles with $n$ sections.
Example 1.3. $\overline{\mathrm{M}}_{3}$ is a point. The collection contains 1 object, $F_{0,0} \cong \mathcal{O}$. Here we abuse notations and write $F_{l, e}$ to indicate the objects $F_{l, E}$ with $|E|=e$.
The reduction map $\overline{\mathcal{M}}_{0,5} \rightarrow \overline{\mathrm{M}}_{5}$ is an isomorphism (although the universal families are different). The space $\overline{\mathrm{M}}_{5}$ is a del Pezzo surface of degree 5 . The collection contains 7 objects: $F_{0,0} \cong \mathcal{O}, 5$ line bundles $F_{0,4}$, which can be identified with $\pi_{i}^{*}(\mathcal{O}(1))$ for every conic bundle $\pi_{i}: \overline{\mathrm{M}}_{5} \rightarrow \mathbb{P}^{1}$, and a rank 2 vector bundle $F_{1,5}$. The map given by its global sections gives a well-known embedding of $\overline{\mathrm{M}}_{5}$ into the Grassmannian $G(2,5)$.

The reduction map $\overline{\mathcal{M}}_{0,7} \rightarrow \overline{\mathrm{M}}_{7}$ is the blow-up of 35 planes $\mathbb{P}^{2} \subset \overline{\mathrm{M}}_{7}$ intersecting transversally in 70 points. The exceptional collection on $\overline{\mathrm{M}}_{7}$ contains 38 objects: $F_{0,0}, F_{2,0}, F_{1,1}, F_{0,2}, F_{0,6}, F_{1,7}$. As $p$ grows, the reduction morphism $\overline{\mathcal{M}}_{0, p} \rightarrow \overline{\mathrm{M}}_{p}$ factors into more and more steps.

If $p$ is odd and $q>0$ then we have a morphism $\overline{\mathrm{M}}_{p, q} \rightarrow \overline{\mathrm{M}}_{p}$, which is an iterated ( $q$ times) universal $\mathbb{P}^{1}$-bundle of $\bar{M}_{p}$. By applying Orlov's theorem on the derived category of a projective bundle, we immediately construct an equivariant exceptional collection in this case (see Theorem 3.19)

The bulk of the paper, starting in Section 4 is focussed on the difficult case of even $p$. We introduce a new object, a torsion sheaf $\mathcal{T}_{l, E}$ on $\overline{\mathrm{M}}_{p, q}$ :
Notation 1.4. Let $P$ (resp., $Q$ ) be the set of heavy (resp., light) points. For every subset $E \subseteq P \cup Q$, we denote by $E_{p}$ (resp., $E_{q}$ ) its intersection with $P$ (resp., $Q$ ) and their cardinalities by $e_{p}$ and $e_{q}$. For $p=2 r \geq 4, q \geq 1$, $R \subseteq P,|R|=r$, let $i_{R}: Z_{R} \hookrightarrow \overline{\mathrm{M}}_{p, q}$ be the locus in $\overline{\mathrm{M}}_{p, q}$ where the points from $R$ come together. Let $\pi_{R}: \mathcal{U}_{R} \rightarrow Z_{R}$ be the universal family over of $\overline{\mathrm{M}}_{p, q}$ restricted to $Z_{R}$ and let $\sigma_{u}$ be the section of $\pi_{R}$ that corresponds to the combined points of $R$. For $l \geq 0, E \subseteq P \cup Q$ with $e=|E|$ such that $e+l$ is even and $E_{p}=R$, consider the following torsion sheaf on $\overline{\mathrm{M}}_{p, q}$ :

$$
\mathcal{T}_{l, E}=i_{R *} \sigma_{u}^{*}\left(\omega_{\pi_{R}}^{\frac{e-l}{2}}(E)\right)
$$

Theorem 1.5. Let $p=2 r \geq 4, q=2 s+1 \geq 1$. A full $S_{p} \times S_{q}$ invariant exceptional collection on $\overline{\mathrm{M}}_{p, q}$ consists of the following objects parametrized by integers $l \geq 0$ and subsets $E \subset P \cup Q$ such that $l+e$ is even:
(group 1A) The vector bundles $F_{l, E}$ for $l+\min \left(e_{p}, p+1-e_{p}\right) \leq r-1$;
(group 2) The torsion sheaves $\mathcal{T}_{l, E}$ for $e_{p}=r, l+\min \left(e_{q}, q-e_{q}\right) \leq s-1$.
These objects are arranged in blocks indexed by a subset $E_{q}$. Blocks are ordered by increasing $e_{q}$ and arbitrarily if $e_{q}$ is the same (but the set $E_{q}$ is different). Within
each block with the same $E_{q}$ we put the sheaves $\left\{\mathcal{T}_{l, E}\right\}$ first, in arbitrary order if $E_{p} \neq E_{p}^{\prime}$ and in order of decreasing $l$ when $E_{p}=E_{p}^{\prime}$. After the sheaves we put the bundles $\left\{F_{l, E}\right\}$ in order of increasing $e_{p}$ and, for a given $e_{p}$, arbitrarily.
Theorem 1.5 (and the next Theorem 1.8) remains true if $r=1$, i.e., if there are only $p=2$ heavy points. In this case the collection contains the vector bundles $F_{0, E}$ with $E \subseteq Q$ arbitrary and the objects $\mathcal{T}_{l, E}$ in group 2 (which are line bundles, since $Z_{R}$ is the whole moduli space). The dual of collection was constructed in [CT20b, Thm. 1.6, Rmk. 3.7].
Notation 1.6. For the case of both $p$ and $q$ even, we need yet another type of object. We always assume that $q=2 s+1$ is odd and work on $\overline{\mathrm{M}}_{p, q+1}$. In particular, $|Q|=q+1$ in the even case. The space $\overline{\mathrm{M}}_{p, q+1}$ is a resolution of singularities of the asymmetric GIT quotient $\mathrm{X}_{p, q+1}$ (see Notation 2.8) with $\frac{1}{2}\binom{p}{r}\binom{q+1}{s+1}$ exceptional divisors $\delta=\mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-1}$.

Let $\mathcal{A} \subset D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ be the triangulated subcategory generated by the torsion sheaves $\mathcal{O}_{\delta}(-a,-b)$ where either $1 \leq a, b \leq r+s-1$ or $a=$ $0,1 \leq b \leq \frac{r+s-1}{2}$ or $b=0,1 \leq a \leq \frac{r+s-1}{2}$. Let $\mathcal{B}={ }^{\perp} \mathcal{A}=\{T \in$ $D^{b}\left(\overline{\mathrm{M}}_{p, q+1} \mid \operatorname{Hom}(T, A)=0\right.$ for every $\left.A \in \mathcal{A}\right\}$.

We prove in Section 10 that $\mathcal{A}$ is an admissible ( $S_{p} \times S_{q+1}$ ) invariant subcategory and thus $\mathcal{B}$ is an $\left(S_{p} \times S_{q+1}\right)$ equivariant non-commutative resolution of singularities of the GIT quotient $\mathrm{X}_{p, q+1}$ in the sense of [KL14]. While $\mathrm{X}_{p, q+1}$ has many small resolutions related by flops obtained by contracting boundary divisors $\delta$ onto one of the factors $\mathbb{P}^{r+s-1}$, none of them is ( $S_{p} \times S_{q+1}$ ) equivariant unlike the category $\mathcal{B}$, which in some sense is the "minimal" equivariant resolution. The vector bundles $F_{l, E}$ belong to $\mathcal{B}$ (Prop.6.1) but the torsion objects have to be projected onto $\mathcal{B}$ :
Definition 1.7. We define the objects in $\mathcal{B} \subset D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ by $\tilde{\mathcal{T}}_{l, E}=\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}}$, where $T \rightarrow T_{\mathcal{B}}$ is a canonical functorial projection and the torsion sheaf $\mathcal{T}_{l, E}$ is defined as in Notation 1.4 for $E \subseteq P \cup Q, l \geq 0, e+l$ even.
Theorem 1.8. Let $p=2 r \geq 4, q=2 s+1 \geq 1$. The space $\overline{\mathrm{M}}_{p, q+1}$ has an $S_{p} \times S_{q+1}$ invariant full exceptional collection of torsion sheaves $\mathcal{O}(-a,-b)$ in subcategory $\mathcal{A}$ (Notation 1.6) followed by the following objects parametrized by integers $l \geq 0$ and subsets $E \subset P \cup Q$ such that $l+e$ is even:
(group 1A) The vector bundles $F_{l, E}$ for $l+\min \left(e_{p}, p+1-e_{p}\right) \leq r-1$;
(group 2B) The complexes $\tilde{\mathcal{T}}_{l, E}$ for $e_{p}=r, l+\min \left(e_{q}, q+2-e_{q}\right) \leq s$.
When combining group $2 B$ with $1 A$, the order is the same as in Theorem [1.5]
The same collection as in Thm. 1.8 works for the case of $s=-1$ :
Theorem 1.9. Let $p=2 r \geq 4$. Then $\overline{\mathrm{M}}_{p}$ has an $S_{p}$ invariant full exceptional collections of the torsion sheaves $\mathcal{O}_{\mathbb{P}^{r-2}} \times \mathbb{P}^{r-2}(-a,-b)$, where either $1 \leq a, b \leq$ $r-2$ or $a=0,1 \leq b \leq r / 2-1$ or $b=0,1 \leq a \leq r / 2-1$, followed by
(group 1A) The vector bundles $F_{l, E}$ with $l+e$ even, $l+\min (e, p+1-e) \leq r-1$. The order is first by increasing $e$, and for a given $e$, arbitrarily.
Example 1.10. For $\overline{\mathrm{M}}_{4} \cong \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^{1}, \mathcal{A}$ is empty. The collection contains 2 objects, $F_{0,0} \cong \mathcal{O}$ and $F_{0,4} \cong \mathcal{O}(1)$. In fact $F_{0, P}$ is always the pull-back of the GIT polarization from the symmetric GIT quotient $\mathrm{X}_{p}$ (see Corollary 6.3).

The space $\mathrm{X}_{6}$ is the Segre cubic threefold in $\mathbb{P}^{4}$. The space $\overline{\mathrm{M}}_{6} \cong \overline{\mathcal{M}}_{0,6}$ is the blow-up of 10 singularities of $\mathrm{X}_{6}$ with exceptional divisors $\mathbb{P}^{1} \times \mathbb{P}^{1}$. The category $\mathcal{A}$ contains torsion sheaves $\mathcal{O}_{\mathbb{P}^{1} \times \mathbb{P}^{1}}(-1,-1)$. The category $\mathcal{B}$ is a weakly crepant resolution of $\mathrm{X}_{6}$ with the full exceptional collection of 24 vector bundles of type $F_{0,0}, F_{2,0}, F_{1,1}, F_{0,2}, F_{0,6}$.
Remark 1.11. In the setup of Theorems 1.5, 1.8 and 1.9, we will also show that there is another full invariant exceptional collection, obtained by swapping vector bundles from group 1A to vector bundles in
(group $1 B$ ) Vector bundles $F_{l, E}$ for $l+\min \left(e_{p}+1, p-e_{p}\right) \leq r-1$.
The order is the same. The groups $1 B$ and $1 A$ are related by the operation $E \mapsto E^{c}$. From this point of view, in Thm. 1.8, it is natural to consider also
(group 2A) Complexes $\left\{\tilde{\mathcal{T}}_{l, E}\right\}$ for $e_{p}=r, l+\min \left(e_{q}+1, q+1-e_{q}\right) \leq s$ and combine them with vector bundles from groups $1 A$ or $1 B$. The same proof as in Thm. 1.8 shows that this collection is exceptional and of the expected length, but we did not attempt to prove fullness.
Remark 1.12. Theorem 1.1 implies the existence of an s.o.d. in the derived category of the Deligne-Mumford stack $\left[\overline{\mathcal{M}}_{0, n} / S_{n}\right]$, see Lemma 2.2, which is known as the "symmetric $\overline{\mathrm{M}}_{0, n}$ " and naturally appears in moduli theory in the study of ample line bundles on $\overline{\mathcal{M}}_{g}$, see [GKM02]. Likewise, Theorems 1.2 and 1.9 provide categorification of the Deligne-Mumford stack [ $\mathrm{X}_{n} / S_{n}$ ], which was extensively studied in the 19th century by Silvester and others in the framework of invariant theory of binary forms [Dol03].
Connection with [CT20a]. In the first paper of this project [CT20a] we prove the existence of a full $S_{2} \times S_{n}$-invariant collection on the Losev-Manin space $\overline{\mathrm{LM}}_{n}$. The approach we take is different for the two types of spaces. In [CT20a] we proved that for both $\overline{\mathrm{LM}}_{n}$ and $\overline{\mathcal{M}}_{0, n}$ it suffices to find full invariant exceptional collections in the cuspidal block of the derived category (i.e., objects that push forward to 0 by all the forgetful maps). In [CT20a] we achieved this for the Losev-Manin spaces. An invariant description of $D_{\text {cusp }}^{b}\left(\overline{\mathcal{M}}_{0, n}\right)$ is an interesting open problem.
Structure of the sections focussing on $\overline{\mathrm{M}}_{p, q}$ for even $p$. In Section 4 we study numerical functions of pairs $(l, E)$, including the score $S(l, E)$. In Section 5 we extend the definition of the vector bundles $\overline{F_{l, E}}$ to more general Hassett spaces, while in Section 6 we give conditions for these vector bundles to be orthogonal to torsion sheaves supported on the boundary.

The exceptionality of the " $F_{l, E}$-part" of the collections in Thm. 1.5 and Thm. 1.8 is proven in Section 7 and Section 8 by induction on the number of light points and require different arguments (hence, the two sections). We always assume that $q$ is odd and introduce the alpha game to go from $\overline{\mathrm{M}}_{p, q-1}$ to $\overline{\mathrm{M}}_{p, q}$ and the beta game to go from $\overline{\mathrm{M}}_{p, q}$ to $\overline{\mathrm{M}}_{p, q+1}$.

We finish the proof of the exceptionality in Thm. 1.5 (i.e., including the " $\mathcal{T}_{l, E}$ part") in Section 9 by reducing it to a windows calculation on subvarieties $Z_{R} \subseteq \overline{\mathrm{M}}_{p, q}$ (the supports of the torsion sheaves $\mathcal{T}_{l, E}$ ) and their intersections. The exceptionality in Thm. 1.8 is finished in Section 10 where that we compare commutative (not equivariant) and non-commutative (equivariant) small resolutions of the singular GIT quotient $\mathrm{X}_{p, q+1}$.

Fullness of the exceptional collections on all the spaces $\overline{\mathrm{M}}_{p, q}$ is proved in the remaining sections: Section 11 for the collections in Thm. 1.5 and in Section 12 for the collections in Thm. 1.8.

Acknowledgements. We thank Ben Bakker, Matt Ballard, Arend Bayer, Daniel Halpern-Leistner, Alexander Kuznetsov, Chunyi Li, Emanuele Macrì, Georg Oberdieck, Alex Perry, Olivier Schiffmann and Maxim Smirnov for helpful conversations related to this work and Matt Ballard, Alex Duncan, Patrick McFaddin for sending us a preliminary version of their work related to Thm. 1.2 A.-M. C. was supported by the NSF grant DMS-1701752 and the ANR grant FanoHK. J. T. was supported by the NSF grants DMS1701704 and DMS-2101726, as well as Fulbright and Simons Fellowships.

## CONTENTS

1. Introduction ..... 1
2. Preliminaries on the Hassett spaces in genus 0 ..... 6
3. Derived categorv of the moduli stack $\mathcal{M}_{n}$ of $n$ points on $\mathbb{P}^{1}$ ..... 10
4. Functions of subsets, score and inequalities for pairs (l, $E$ ) ..... 17
5. Extending vector bundles $F_{1 E}$ to some Hassett spaces ..... 20
6. Perpendicularity of $F_{l-E}$ 's versus the sheaves $\mathcal{O}_{\delta}(-a,-b)$ ..... 29
7. $\left\{F_{l_{E}}\right\}$ exceptional on $\overline{\mathrm{M}}_{2 r}{ }_{2 s} \Rightarrow\left\{F_{l_{E}}\right\}$ exceptional on $\overline{\mathrm{M}}_{2 r 2 s+1}$ ..... 32
8. $\left\{F_{l-E}\right\}$ exceptional on $\overline{\mathrm{M}}_{2 r 2 s+1} \Rightarrow\left\{F_{l-E}\right\}$ exceptional on $\overline{\mathrm{M}}_{2 r, 2 s+2}$ ..... 37
9. Equivariant exceptional collection on $\overline{\mathrm{M}}_{2 r 2 s+1}$ ..... 43
10. Equivariant exceptional collection on $\overline{\mathrm{M}}_{2 r}{ }_{2 \varepsilon+}$ ] ..... 47
11. Fullness of the exceptional collection on $\overline{\mathrm{M}}_{2 r 2 s+1}$ ..... 57
12. Fullness of the exceptional collection on $\overline{\mathrm{M}}_{2 r 2 s+2}$ ..... 61
References ..... 69

## 2. Preliminaries on the Hassett spaces in genus 0

Definition 2.1. Let $\Gamma$ be a finite group acting on a smooth projective variety $X$. An exceptional collection $\left\{\mathcal{E}_{\alpha}^{\bullet}\right\}_{\alpha \in I}$ in $D^{b}(X)$. is called $\Gamma$-invariant if, for every $\gamma \in \Gamma$ and $\alpha \in I$, there exists $\beta \in I$ such that $\gamma^{*} \mathcal{E}_{\alpha}^{\bullet} \cong \mathcal{E}_{\beta}^{\bullet}$. A $\Gamma$-invariant exceptional collection is called $\Gamma$-equivariant if every complex $\mathcal{E}_{\alpha}^{\bullet}$ is quasi-isomorphic to a complex in $D^{b} \overline{\left(X / \Gamma_{\alpha}\right) \cong D_{\Gamma_{\alpha}}^{b}}(X)$, a bounded derived category of the category of $\Gamma_{\alpha}$-equivariant coherent sheaves, where $\Gamma_{\alpha} \subset \Gamma$ is the stabilizer of the isomorphism class of $\mathcal{E}_{\alpha}^{\bullet}$.

Lemma 2.2. If $D_{b}(X)$ admits a full $\Gamma$-invariant exceptional collection then $K_{0}(X)$ is a permutation $\Gamma$-lattice and $D_{\Gamma}^{b}(X)$ admits a s.o.d. with blocks isomorphic to representation categories (if the collection is equivariant) and twisted representation categories (if it is only invariant) of subgroups $\Gamma_{\alpha}$.

Proof. Since $K_{0}(X)$ is isomorphic to the Grothendieck $K$-group of $D^{b}(X)$, the first statement is clear (a s.o.d. of $D^{b}(X)$ induces a direct sum decomposition of its $K$-group). We refer to [Ela09, Theorem 2.3] for the precise formulation and proof of the second statement.

Example 2.3. Let us give an example of a variety with an involution such that the corresponding action of $S_{2}$ on its $K$-group is not a permutation representation. Consider a del Pezzo surface $S$ of degree 2, the double cover of $\mathbb{P}^{2}$ ramified along a smooth quartic curve, with the usual involution $\sigma$. Then $S$ has $56(-1)$-curves that come in pairs interchanged by $\sigma$. As $\sigma$ preserves $K$, it acts by -1 on the orthogonal complement $K^{\perp} \subset \operatorname{Pic}(S)$. Therefore the action of $S_{2}$ on both the total cohomology $H^{*}(S, \mathbb{Z})$ and the K-group $K_{0}(S)$ has type $(3,7)$, i.e. is diagonalizable with 3 eigenvalues 1 and 7 eigenvalues -1 . However, every diagonalizable permutation representation of $S_{2}$ of type ( $a, b$ ) obviously has $a \geq b$.

Our collections are not only $\Gamma$-invariant but in fact $\Gamma$-equivariant. To prove this we need a strengthened version of the equivariant Orlov blowup lemma [CT20a, Lem. 7.2]. Let $X$ be a smooth projective variety and let $Y_{1}, \ldots, Y_{n} \subset X$ be smooth transversal subvarieties of codimensions $l_{1}, \ldots, l_{n}$. Let $\Gamma$ be a finite group acting on $X$ permuting $Y_{1}, \ldots, Y_{n}$. For every subset $I \subset\{1, \ldots, n\}$, we denote by $Y_{I}$ the intersection $\cap_{i \in I} Y_{i}$. In particular, $Y_{\emptyset}=X$. Let $q: \tilde{X} \rightarrow X$ be an iterated blow-up of (proper transforms of) $Y_{1}, \ldots, Y_{n}$. Since the intersection is transversal, blow-ups can be done in any order. More canonically, the iterated blow-up is isomorphic to the blow-up of the ideal sheaf $\mathcal{I}_{Y_{1}} \cdot \ldots \cdot \mathcal{I}_{Y_{n}}$. Let $E_{i}$ be the exceptional divisor over $Y_{i}$ for every $i=1, \ldots, n$. For any subset $I \subset\{1, \ldots, n\}$, let $E_{I}=q^{-1}\left(Y_{I}\right)=\cap_{i \in I} E_{i}$, in particular, $E_{\emptyset}=\tilde{X}$. Let $i_{I}: E_{I} \hookrightarrow \tilde{X}$ be the inclusion and let $\Gamma_{I} \subset \Gamma$ be the normalizer of $Y_{I}$. The group $\Gamma$ acts on $\tilde{X}$ and the morphism $q$ is $\Gamma$-equivariant.
Lemma 2.4. Let $\left\{F_{I}^{\beta}\right\}$ be a (full) $\Gamma_{I}$-invariant (resp. equivariant) exceptional collection in $D^{b}\left(Y_{I}\right)$ for every subset $I \subset\{1, \ldots, n\}$. Choosing representative of $\Gamma$-orbits on the set $\left\{Y_{I}\right\}$, we can assume that if $Y_{I}=g Y_{I^{\prime}}$ for some $g \in \Gamma$ then $\left\{F_{I}^{\beta}\right\}=g\left\{F_{I^{\prime}}^{\beta}\right\}$. Then there exists a (full) $\Gamma$-invariant (resp. equivariant) exceptional collection in $D^{b}(\tilde{X})$ with blocks $B_{I, J}=\left(i_{I}\right)_{*}\left[\left(\left.L q\right|_{E_{I}}\right)^{*}\left(F_{I}^{\beta}\right)\left(\sum_{i=1}^{n} J_{i} E_{i}\right)\right]$ for every subset $I \subset\{1, \ldots, n\}$ (including the empty set) and for every $n$-tuple of integers $J$ such that $J_{i}=0$ if $i \notin I$ and $1 \leq J_{i} \leq l_{i}-1$ for $i \in I$.

The blocks are ordered in any linear order which respects the following partial order: $B_{I^{1}, J^{1}}$ precedes $B_{I^{2}, J^{2}}$ if $\sum_{i=1}^{n} J_{i}^{1} E_{i} \geq \sum_{i=1}^{n} J_{i}^{2} E_{i}$ (as effective divisors).
Proof. Exceptionality, fullness and invariance of the collection was proved in [CT20a, Lem. 7.2]. To prove equivariance, we use equivariant pull-back and push-forward and endow line bundles $\mathcal{O}\left(J_{1} E_{1}+\ldots+J_{n} E_{n}\right)$ with linearizations with respect to the stabilizer of the divisor $J_{1} E_{1}+\ldots+J_{n} E_{n}$. These linearizations are given by the canonical equivariant structure of the ideal sheaf of an invariant subscheme.

Notation 2.5. Fix a vector of positive rational weights $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with each $a_{i} \leq 1$ and $\sum a_{i}>2$. Let $\overline{\mathrm{M}}_{\mathrm{a}}$ be the Hassett space of weighted pointed stable rational curves, i.e., pairs ( $C, \sum a_{i} p_{i}$ ) such that $C$ is a nodal genus 0 curve, the points $p_{i}$ are smooth and the $\mathbb{Q}$-line bundle $\omega_{C}\left(\sum a_{i} p_{i}\right)$ is ample. The points $\left\{p_{i}: i \in I\right\}$ can become equal on $C$ if and only if $\sum_{i \in I} a_{i} \leq 1$. Note
that $\overline{\mathcal{M}}_{0, n}=\overline{\mathrm{M}}_{(1, \ldots, 1)}$. The weights have a chamber structure with walls

$$
\begin{equation*}
\sum_{i \in I} a_{i}=1 \quad \text { for every subset } I \subset\{1, \ldots, n\} \text {. } \tag{2.1}
\end{equation*}
$$

Moduli spaces within the interior of each chamber are isomorphic and carry the same universal family. There exist birational reduction morphisms $\overline{\mathrm{M}}_{\mathrm{a}} \rightarrow \overline{\mathrm{M}}_{\mathrm{a}^{\prime}}$ every time the weight vectors are such that $a_{i} \geq a_{i}^{\prime}$ for every $i$.
Notation 2.6. Fix a vector of positive rational weights $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with each $a_{i} \leq 1$ and $\sum a_{i}=2$. Let $\mathrm{X}_{\mathbf{a}}=\left(\mathbb{P}^{1}\right)^{n} / / \mathrm{PGL}_{2}$ be the GIT quotient of $\left(\mathbb{P}^{1}\right)^{n}$ by the diagonal action of $\mathrm{PGL}_{2}$ with respect to the fractional polarization $\mathcal{O}\left(a_{1}, \ldots, a_{n}\right)$. The polytope of GIT weights can be identified with the face of the polytope of Hassett weights and inherits its chamber structure with the walls (2.1), which encodes the variation of GIT (see [DH98]). Polarizations within the interior of each chamber have no strictly semistable points and carry a universal $\mathbb{P}^{1}$-bundle with $n$ sections. More generally, there always exists a morphism $\overline{\mathrm{M}}_{\mathrm{a}}:=\overline{\mathrm{M}}_{\left(a_{1}+\epsilon, \ldots, a_{n}+\epsilon\right)} \rightarrow \mathrm{X}_{\mathrm{a}}$, for some $0<\epsilon \ll 1$ that depends on a, which is an isomorphism if there are no strictly semistable points (i.e., no sum of $a_{i}{ }^{\prime}$ s equals 1) Has03, Thm. 8.2]. In the presence of strictly semistable points, $\mathrm{X}_{\mathrm{a}}$ can acquire isolated singularities (cones over $\mathbb{P}^{r} \times \mathbb{P}^{s}$ ) and $\overline{\mathrm{M}}_{\mathrm{a}}$ is the blow-up of these singularities ("Kirwan resolution"). In any case, the universal family over these varieties $\overline{\mathrm{M}}_{\mathrm{a}}$ "near the GIT face" has fibers with at most two irreducible components.
Lemma 2.7. A Hassett space $\overline{\mathrm{M}}_{\mathrm{a}}$ is a GIT quotient if and only if its universal family $\mathcal{U}$ is a $\mathbb{P}^{1}$-bundle. Furthermore, $\mathcal{U}$ in this case is also a GIT quotient.
Proof. One direction is clear. Now suppose that $\mathcal{U} \rightarrow \overline{\mathrm{M}}_{\mathrm{a}}$ is a $\mathbb{P}^{1}$-bundle. We claim that reducing all weights $a_{i}$ to $b_{i}$ such that $a_{i}>b_{i}$ for all $i$ and $\sum b_{i}=2$, does not change stability. Indeed, suppose that $\sum_{i \in I} a_{i}>1 \geq$ $\sum_{i \in I} b_{i}$ for some subset $I$. Then $\sum_{i \in I^{c}} a_{i}>\sum_{i \in I^{c}} b_{i}=2-\sum_{i \in I} b_{i} \geq 1$. So curves with two components with different points labeled by $I$ (resp., $I^{c}$ ) on these components are stable, which contradicts the assumption. The claim about $\mathcal{U}$ also follows.
Notation 2.8. For $p \geq 3, q \geq 0$, we let $\mathrm{X}_{p, q}$, resp., $\overline{\mathrm{M}}_{p, q}$, denote the spaces $\mathrm{X}_{\mathrm{a}}$, resp., $\overline{\mathrm{M}}_{\mathrm{a}}$, from Notation 2.6 with the weights $a=a_{1}=\ldots=a_{p}$, $b=a_{p+1}=\ldots=a_{p+q}$ satisfying the following conditions. If $q=0$ then $a=2 / p$. If $q>0$ then $a=\frac{2}{p}-\epsilon, b=\frac{p \epsilon}{q}$, where $0<\epsilon<\frac{1}{(2 r+1)(r+1)}$ if $p=2 r+1$ and $0<\epsilon<\frac{1}{r(r+1)}$ if $p=2 r$. We call the points with the weight $a$ heavy and the remaining points light.
We denote $\overline{\mathrm{M}}_{n}:=\overline{\mathrm{M}}_{n, 0}$. Note that $\mathrm{X}_{p, q} \cong \overline{\mathrm{M}}_{p, q}$ if and only if $p$ or $q$ is odd. If $p=2 r$ and $q=2 s$, then $\overline{\mathrm{M}}_{p, q}$ is a divisorial "Kirwan resolution" of singularities of $\mathrm{X}_{p, q}$. It has exceptional divisors $\mathbb{P}^{r+s-2} \times \mathbb{P}^{r+s-2}$ with normal bundle $\mathcal{O}(-1,-1)$ over each of the $\frac{1}{2}\binom{p}{r}\binom{q}{s}$ singular points of $\mathrm{X}_{p, q}$.

We will reduce the proof of Theorem 1.1 to the analysis of $\overline{\mathrm{M}}_{p, q}{ }^{\prime} \mathrm{s}$. We emphasize that all spaces $\overline{\mathrm{M}}_{p, q}$ are needed to prove Theorem 1.1 for $n \gg 0$. The same argument also proves the following more general result.

Theorem 2.9. Let a be a Hassett weight such that $a_{1} \geq \ldots \geq a_{n}$ and $\sum a_{i}>2$. Suppose further that either $a_{1}=1$ or $a_{j}>2 / j$ for some $j$. Then $\overline{\mathrm{M}}_{\mathbf{a}}$ has a full $\Gamma_{\mathbf{a}}$-invariant exceptional collection, where $\Gamma_{\mathbf{a}} \subseteq S_{n}$ is the stabilizer of the vector $\mathbf{a}$.

We note that full, exceptional (not $\Gamma_{\mathbf{a}}$-invariant) collections on some of the Hassett spaces $\overline{\mathrm{M}}_{\mathrm{a}}$ have been constructed in [BFK19]. More generally, full, exceptional collections on many GIT quotients (without any requirement of invariance) have been constructed in [HL15, BFK19].

In the remainder of this section we discuss the proof of Theorem 2.9, All along we assume that the spaces $\overline{\mathrm{M}}_{p, q}$ have a full $S_{p} \times S_{q}$-equivariant exceptional collection, which is proved in the subsequent sections.

Proof of Theorem 2.9, Let $\Gamma_{\mathbf{a}}$ be the stabilizer of the vector a in the group $S_{n}$. Choose $p \leq n$ such that $a_{1}=\ldots=a_{p}>a_{p+1}$. We consider 4 cases
(i) $a_{1}=1, p \geq 2$; (ii) $a_{1}<1, p a_{1} \geq 2$; (iii) $a_{1}=1, p=1$; (iv) $a_{j} \geq \frac{2}{j}$ for some $j$;
and the following statement: (v) Consider weight vectors $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$, $\mathbf{a}^{\prime}=\left(a_{1}^{\prime}, \ldots, a_{n}^{\prime}\right)$ such that $a_{i} \geq a_{i}^{\prime}$ for every $i$. Suppose $\Gamma_{\mathbf{a}} \subset \Gamma_{\mathbf{a}^{\prime}}$. Then if $\overline{\mathrm{M}}_{\mathbf{a}^{\prime}}$ admits a $\Gamma_{\mathbf{a}^{-}}$-equivariant exceptional collection then so does $\overline{\mathrm{M}}_{\mathrm{a}}$.

We prove (i)-(v) simultaneously by induction on dimension of $\overline{\mathrm{M}}_{\mathrm{a}}$ using existence of full $S_{p} \times S_{q}$-equivariant exceptional collections on $\overline{\mathrm{M}}_{p, q}$. Note that all statement are clear in dimension 1 as in this case $\overline{\mathrm{M}}_{\mathrm{a}} \cong \mathbb{P}^{1}$.

Case (iv). We can assume without loss of generality that $j$ is the largest index with the property $a_{j} \geq 2 / j$. The statement follows from (i) if $a_{j}=1$. If $a_{j}<1$, we reduce to (ii) using (v) by taking the second weight vector $a_{1}^{\prime}=\ldots=a_{j}^{\prime}=a_{j}, a_{i}^{\prime}=a_{i}$ for $i>j$.

Case (iii). Let $A=a_{2}+\ldots+a_{n}>1$. Consider $a_{1}^{\prime}=1, a_{i}^{\prime}=\frac{a_{i}}{A-\epsilon}$ for $i \geq 2$ for some fixed $0<\epsilon<\min \left(a_{n}, A-1\right)$. By (v), it suffices to show that $\overline{\mathrm{M}}_{\mathbf{a}^{\prime}}$ admits a $\Gamma_{\mathbf{a}}$-invariant exceptional collection. Note that $\overline{\mathrm{M}}_{\mathbf{a}^{\prime}} \cong \mathbb{P}^{n-3}$ by [Has03, Section 6.2] since $\sum_{i \geq 2, i \neq j} a_{i}^{\prime} \leq \sum_{i=2}^{n-1} a_{i}^{\prime}=\frac{A-a_{n}}{A-\epsilon}<1$ for all $j \geq 2$. Note that the standard collection $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-3)$ on $\mathbb{P}^{n-3}$ is invariant under any group. For equivariance, we need a bit more. All appearing groups are contained in the symmetric group $S_{n-1}$ which acts on $\mathbb{P}^{n-3}$ by permuting $n-1$ fixed points in general linear position. The homomorphism $S_{n-1} \rightarrow \mathrm{PGL}_{n-2}$ can be factored through $\mathrm{GL}_{n-2}$, and therefore $\mathcal{O}(1)$ is a $S_{n-1}$-linearized line bundle. The corresponding action on $k^{n-2}$ is the irreducible $(n-2)$-dimensional representation of $S_{n-1}$, [KT09, Lemma 2.3].

Cases (i) and (ii). If $p=2$ (case (i)), we apply (v) to the weight vector $\mathbf{a}^{\prime}=(1,1, \epsilon, \ldots, \epsilon)$ of the Losev-Manin space and use the main result of [CT20a]. Let $p>2$. In both (i) and (ii) we apply (v) to the weight vector $\mathbf{a}^{\prime}=(a, \ldots, a, b \ldots, b)$ of $\overline{\mathrm{M}}_{p, q}$ (see Remark 2.8) with $q=n-p$. Concretely, if $q=0$ then $a=\frac{2}{p}+\epsilon$ and we choose $a<a_{1}$, while if $q>0$ then $a=\frac{2}{p}-\epsilon$, $b=\frac{p \epsilon}{q}(0<\epsilon \ll 1)$. Then $a<a_{1}$ and we choose $b<a_{n}$.

Case (v). We connect the weights by a $\Gamma_{\mathbf{a}}$-invariant homotopy $\mathbf{a}(t)=$ $t \mathbf{a}^{\prime}+(1-t) \mathbf{a}$. The Hassett chamber structure is semi-constant in the following sense: the reduction map $\overline{\mathrm{M}}_{b_{1}, \ldots, b_{n}} \rightarrow \overline{\mathrm{M}}_{b_{1}-\epsilon, \ldots, b_{n}-\epsilon}$ is an isomorphism for $0<\epsilon \ll 1$. It follows that our reduction map $\overline{\mathrm{M}}_{\mathbf{a}(0)}=\overline{\mathrm{M}}_{\mathbf{a}} \rightarrow \overline{\mathrm{M}}_{\mathbf{a}^{\prime}}=$
$\overline{\mathrm{M}}_{\mathbf{a}(1)}$ factors into the sequence of $\Gamma_{\mathrm{a}}$-equivariant reduction maps of the following form: $f_{t}: \overline{\mathrm{M}}_{\mathbf{a}(t-\epsilon)} \rightarrow \overline{\mathrm{M}}_{\mathbf{a}(t)}$ for $0<\epsilon \ll 1$ whenever there is a subset of indices $J \subset\{1, \ldots, n\}$ such that

$$
\begin{equation*}
\sum_{j \in J} a_{j}(0)>1=\sum_{j \in J} a_{j}(t) . \tag{2.2}
\end{equation*}
$$

Arguing by induction on the number of wall crossings, it suffices to analyze a single map $f_{t}$, descibed in [Has03], see also [BM13]. Let $J_{1}, \ldots, J_{s}$ be a full list of subsets satisfying (2.2). The group $\Gamma_{\mathrm{a}}$ permutes $J_{i}$ 's. The reduction map $f_{t}$ blows up the loci $\overline{\mathrm{M}}_{\mathbf{a}(t)}\left(J_{i}\right)$ in $\overline{\mathrm{M}}_{\mathbf{a}(t)}$ where the points in $J_{i}$ come together. Since $\sum_{j \in J_{i}} a_{j}(t)=1$, the loci $\overline{\mathrm{M}}_{\mathbf{a}(t)}\left(J_{i}\right)$ intersect transversely: $\overline{\mathrm{M}}_{\mathbf{a}(t)}\left(J_{i}\right) \cap \overline{\mathrm{M}}_{\mathbf{a}(t)}\left(J_{j}\right) \neq \emptyset$ if and only if $J_{i} \cap J_{j}=\emptyset$, in which case, the intersection is the locus where points in $J_{i}$, respectively $J_{j}$, coincide. Clearly, the loci $\overline{\mathrm{M}}_{\mathbf{a}(t)}\left(J_{i}\right)$ and their intersections are themselves Hassett spaces of the form $\bar{M}_{\mathbf{a}^{\prime \prime}}$, with the vector $\mathbf{a}^{\prime \prime}$ having at least one weight 1 (for marked points that correspond to combined points indexed by subsets $J_{1}, \ldots, J_{s}$ ). These spaces have invariant exceptional collections by cases (i), (iii) in smaller dimension. The theorem follows by Lemma 2.4 .

## 3. DERIVED CATEGORY OF THE MODULI STACK $\mathcal{M}_{n}$ OF $n$ POINTS ON $\mathbb{P}^{1}$

The goal of this section is to prove Theorem 1.2, which gives an equivariant full exceptional collection in $D^{b}\left(\overline{\mathrm{M}}_{n}\right)$ for $n$ odd. As a GIT quotient without strictly semistable points, $\overline{\mathrm{M}}_{n}$ is isomorphic to an open substack of the quotient stack $\mathcal{P}_{n}=\left[\left(\mathbb{P}^{1}\right)^{n} / \mathrm{PGL}_{2}\right]$ of $\mathbb{P}^{1}$-bundles with $n$ sections.

In what follows $n$ is arbitrary and $\mathcal{P}_{n}=\left[\left(\mathbb{P}^{1}\right)^{n} / \mathrm{PGL}_{2}\right]$. An associated $\mathbb{P}^{1}$ bundle of a $\mathrm{PGL}_{2}$-torsor is the "universal $\mathbb{P}^{1}$-bundle", i.e., a representable morphism $\pi=\pi_{n+1}: \mathcal{P}_{n+1} \rightarrow \mathcal{P}_{n}$ induced by the first projection $\left(\mathbb{P}^{1}\right)^{n+1}=$ $\left(\mathbb{P}^{1}\right)^{n} \times \mathbb{P}^{1} \rightarrow\left(\mathbb{P}^{1}\right)^{n}$. We first analyze the derived category of $\mathcal{P}_{n}$.
We denote $\sigma_{1}, \ldots, \sigma_{n}: \mathcal{P}_{n} \rightarrow \mathcal{P}_{n+1}$ the universal sections. They are representable morphisms induced by the big diagonals $\left(\mathbb{P}^{1}\right)^{n} \simeq \Delta_{i, n+1} \hookrightarrow$ $\left(\mathbb{P}^{1}\right)^{n+1}$ for $i=1, \ldots, n$. We identify the category of coherent sheaves on $\mathcal{P}_{n}$ with the category of $\mathrm{PGL}_{2}$-equivariant coherent sheaves on $\left(\mathbb{P}^{1}\right)^{n}$ and likewise for their bounded derived categories. For $\mathbf{i}=\left(i_{1}, \ldots, i_{n}\right) \in \mathbb{Z}^{n}$ with $\sum i_{k}$ even, the line bundle $\mathcal{O}_{\left(\mathbb{P}^{1}\right)^{n}}\left(i_{1}, \ldots, i_{n}\right)$ has a unique $\mathrm{PGL}_{2}$ - linearization and thus descends to $\mathcal{P}_{n}$. We call it $\mathcal{O}(\mathbf{i})$.

Definition 3.1. Let $\Sigma_{n}$ (or $\Sigma$ if $n$ is clear from the context) be the set $\{1, \ldots, n\}$. Fix a subset $E \subset \Sigma_{n}$ with $e=|E|$ and an integer $l \geq 0$ such that $e+l$ is even. For $j=1, \ldots, n$, let $i_{j}$ be 1 if $j \in E$ and 0 otherwise. Let $F_{l, E}=\pi_{*} N_{l, E}$, where $N_{l, E}=\mathcal{O}\left(i_{1}, \ldots, i_{n}, l\right)$ is a line bundle on $\mathcal{P}_{n+1}$.

Lemma 3.2. $F_{l, E}$ is a rank $l+1$ vector bundle on $\mathcal{P}_{n}$. There is an isomorphism of $\mathrm{SL}_{2}$-equivariant vector bundles on $\left(\mathbb{P}^{1}\right)^{n}, F_{l, E} \xlongequal{\underline{\mathrm{SL}_{2}}} \mathcal{O}\left(i_{1}, \ldots, i_{n}\right) \otimes V_{l}$, where $V_{l}$ is an $(l+1)$-dimensional irreducible $\mathrm{SL}_{2}$-module. In particular, $F_{0, E} \cong$ $\mathcal{O}\left(i_{1}, \ldots, i_{n}\right)$.

Proof. This follows from the $\mathrm{SL}_{2}$-equivariant projection formula applied to the projection morphism $\left(\mathbb{P}^{1}\right)^{n+1}=\left(\mathbb{P}^{1}\right)^{n} \times \mathbb{P}^{1} \rightarrow\left(\mathbb{P}^{1}\right)^{n}$.

Corollary 3.3. There are isomorphisms of $\mathrm{SL}_{2}$-equivariant vector bundles $F_{l, E}^{\vee} \cong$ $\pi_{*} \mathcal{O}\left(-i_{1}, \ldots,-i_{n}, l\right) \cong \mathcal{O}\left(-i_{1}, \ldots,-i_{n}\right) \otimes V_{l}$. If $n$ is even, then $F_{l, E} \cong F_{l, E^{c}}^{\vee} \otimes$ $F_{0, \Sigma}$, where $E^{c}=\Sigma_{n} \backslash E$.

Theorem 3.4. $D^{b}\left(\mathcal{P}_{n}\right)$ has a $S_{n}$-invariant s.o.d. with $2^{n}$ blocks $D^{b}\left(\mathcal{P}_{n}\right)_{E}$ indexed by subsets $E \subseteq \Sigma_{n}$ and ordered by increasing $e=|E|$ (different blocks with the same e are mutually orthogonal, i.e., $R$ Hom between blocks is 0 ). The block $D^{b}\left(\mathcal{P}_{n}\right)_{E}$ has a full exceptional collection $\left\{F_{l, E}: l+e\right.$ even $\}$ of infinitely many mutually orthogonal vector bundles. The combined infinite exceptional collection $\left\{F_{l, E}\right\}$ in $D^{b}\left(\mathcal{P}_{n}\right)$ is strong and $S_{n}$-equivariant.
The forgetful map $\pi_{i}: \mathcal{P}_{n+1} \rightarrow \mathcal{P}_{n}$ has the following properties:

$$
\left(L \pi_{i}\right)^{*} F_{l, E} \cong F_{l, E}, \quad\left(R \pi_{i}\right)_{*} F_{l, E}^{\vee} \cong\left\{\begin{array}{ll}
0 & i \in E \\
F_{l, E}^{\vee} & i \notin E
\end{array} .\right.
$$

Proof. The line bundles $\mathcal{O}\left(i_{1}, \ldots, i_{n}\right)$ form a strong $S_{n}$-equivariant exceptional collection in $D^{b}\left(\left(\mathbb{P}^{1}\right)^{n}\right)$, where each $j_{j}=0$ or 1 . We will denote them by $\left\{L_{k}\right\}, k=1, \ldots, 2^{n}$ (in some order). Schur's lemma and the fact that $R \Gamma\left(\mathbb{P}^{1}, \mathcal{O}(-1)\right)=0$ imply that $\left\{F_{l, E}\right\}$ with $l+e$ even is an exceptional collection in $D^{b}\left(\mathcal{P}_{n}\right)$ with required properties.

It remains to prove fullness, i.e., that any complex $F \in D^{b}\left(\mathcal{P}_{n}\right)$ can be obtained by finitely many extensions starting with objects in the exceptional collection $\left\{F_{l, E}\right\}$. This is a special case of [Ela09, Th. 2.10]. Alternatively, we give a simple ad hoc argument.

Viewing $F$ as a bounded complex of coherent sheaves on $\left(\mathbb{P}^{1}\right)^{n}$, let $k$ be the maximum index such that (in $D^{b}\left(\left(\mathbb{P}^{1}\right)^{n}\right)$ ) we have $\operatorname{RHom}\left(L_{k}, F\right) \neq 0$ or $k=0$ if $\operatorname{RHom}\left(L_{k}, F\right)=0$ for every $k$. We argue by induction on $k$. If $k=0$ then $F \cong 0$ because $\mathcal{O}\left(i_{1}, \ldots, i_{n}\right)$ form a full exceptional collection in $D^{b}\left(\left(\mathbb{P}^{1}\right)^{n}\right)$. Otherwise, consider the left mutation triangle $F^{\prime} \rightarrow$ $R H o m\left(L_{k}, F\right) \otimes L_{k} \rightarrow F \rightarrow$ in $D^{b}\left(\left(\mathbb{P}^{1}\right)^{n}\right)$. We first prove that this is an exact triangle in $D^{b}\left(\mathcal{P}_{n}\right)$. Since $F$ is an $\mathrm{SL}_{2}$-equivariant complex and $L_{k}$ is an $\mathrm{SL}_{2}{ }^{-}$ equivariant line bundle, it suffices to prove that $-\mathrm{Id} \in \mathrm{SL}_{2}$ acts trivially on RHom $\left(L_{k}, F\right) \otimes L_{k}$. This is clear since $-\mathrm{Id} \in \mathrm{SL}_{2}$ acts either trivially or by multiplication by -1 on both terms of the tensor product. It follows that this triangle is $\mathrm{PGL}_{2}$-equivariant. To finish the proof, we need to prove that $F^{\prime}$ and $\operatorname{RHom}\left(L_{k}, F\right) \otimes L_{k}$ are generated by $F_{l, E}$ 's. For $F^{\prime}$ this follows by induction since $\operatorname{RHom}\left(L_{k}, F^{\prime}\right)=0$. Furthermore, $\operatorname{RHom}\left(L_{k}, F\right) \otimes L_{k}$ is isomorphic to a direct sum of the vector bundles $F_{l, E}$ by Lemma 3.2
Proposition 3.5. On $\mathcal{P}_{n}$, we have exact sequences

$$
\begin{gathered}
0 \rightarrow F_{l-1, E \backslash k} \rightarrow F_{l, E} \rightarrow Q_{l, E}^{k} \rightarrow 0 \\
0 \rightarrow F_{l-1, E \cup\{k\}}^{\vee} \rightarrow F_{l, E}^{\vee} \rightarrow S_{l, E}^{k} \rightarrow 0
\end{gathered} \quad \text { for } k \in E, ~ \text { for } k \notin E, ~ \$
$$

where $Q_{l, E}^{k}=\mathcal{O}\left(i_{1}, \ldots, i_{k}+l, \ldots, i_{n}\right)$ and $S_{l, E}^{k}=\mathcal{O}\left(-i_{1}, \ldots,-i_{k}+l, \ldots,-i_{n}\right)$.
Proof. Apply $R \pi_{*}$ to an exact sequence $0 \rightarrow \mathcal{O}\left(-\Delta_{k, n+1}\right) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\Delta_{k, n+1}} \rightarrow 0$ on $\mathcal{P}_{n+1}$ tensored with $\mathcal{O}\left(i_{1}, \ldots, i_{n}, l\right)$ (resp., $\mathcal{O}\left(-i_{1}, \ldots,-i_{n}, l\right)$ ).
Proposition 3.6. $F_{l, E} \otimes F_{l^{\prime}, E^{\prime}} \cong F_{l+l^{\prime}, E \cup E^{\prime}} \oplus F_{l+l^{\prime}-2, E \cup E^{\prime}} \oplus \ldots \oplus F_{\mid l-l^{\prime}, E \cup E^{\prime}}$ if $E \cap E^{\prime}=\emptyset$. In particular, $F_{l, E} \otimes F_{0, E^{\prime}} \cong F_{l, E \cup E^{\prime}}$. Here we assume that all these bundles are defined, i.e., all parity conditions are satisfied.

Proof. This follows from the Clebsch-Gordan formula (Lemma 3.7).
Lemma 3.7. (Clebsch-Gordan) $\left(V_{l} \otimes V_{l^{\prime}}\right)=V_{l+l^{\prime}} \oplus V_{l+l^{\prime}-2} \oplus \ldots \oplus V_{\left|l-l^{\prime}\right|}$. In particular, if $l>l_{1}+\ldots+l_{r}$, then $\left(V_{l} \otimes V_{l_{1}} \otimes \ldots \otimes V_{l_{r}}\right)^{\mathrm{SL}_{2}}=0$.

Remark 3.8. In the next sections we will consider Hassett moduli spaces of pointed curves which do not always admit a quotient stack interpretation. Therefore, it is useful to relate the vector bundles $F_{l, E}$ to tautological line bundles on the stack of $\mathbb{P}^{1}$-bundles with $n$ sections.

The tautological line bundle $\psi_{i}=\sigma_{i}^{*} \omega_{\pi}$ on $\mathcal{P}_{n}$ is identified with the $\mathrm{PGL}_{2}$-equivariant line bundle $\mathcal{O}(0, \ldots, 0,-2,0, \ldots, 0)$, where -2 is in position $i$. The line bundle $\delta_{i j}=\sigma_{i}^{*} \mathcal{O}\left(\sigma_{j}\right)$ is identified with the $\mathrm{PGL}_{2}$-equivariant line bundle $\mathcal{O}(0, \ldots, 0,1,0, \ldots, 0,1,0, \ldots, 0)$, where 1 is in position $i$ and $j$. Furthermore, on $\mathcal{P}_{n+1}$,

$$
N_{l, E} \cong \omega_{\pi}^{\otimes\left(\frac{e-l}{2}\right)}(E)
$$

where, by abuse of notation, we let $\mathcal{O}(E)=\mathcal{O}\left(\sum_{j \in E} \sigma_{j}\right)$.
Proposition 3.9. More generally, we let $N_{l, E}:=\omega_{\pi}^{\frac{e-l}{2}}(E)$ for any $l$, positive or negative, and $F_{l, E}:=R \pi_{*} N_{l, E}$. When $l=-1$, we have $F_{l, E}=0$. When $l \leq-2$, we have $F_{l, E} \cong F_{-l-2, E}[-1]$.

Proof. By Grothendieck-Verdier duality (see Remark 3.10), we have

$$
\begin{equation*}
R \pi_{*} \mathcal{O}\left(-i_{1}, \ldots,-i_{n},-l-2\right) \cong F_{l, E}^{\vee}[-1] \tag{3.1}
\end{equation*}
$$

where $i_{j}=1$ if $j \in E$ and $i_{j}=0$ otherwise. But by Cor. 3.3 when $-l-2 \geq 0$, $F_{-l-2, E}^{\vee} \cong R \pi_{*} \mathcal{O}\left(-i_{1}, \ldots,-i_{n},-l-2\right)$.
Remark 3.10. We recall the Grothendieck-Verdier duality [Huy06, Thm. 3.34] in a particular case that we will use often: for a morphism $\alpha: X \rightarrow Y$ of relative dimension 1, with $X, Y$ smooth varieties, if $N$ is a line bundle on $X$, then $R \alpha_{*}\left(N^{\vee} \otimes \omega_{\alpha}[1]\right)=\left(R \alpha_{*} N\right)^{\vee}$.

In the remainder of this section we study derived categories of open substacks of $\mathcal{P}_{n}$, with an eye towards proving Theorem 1.2
Proposition 3.11. Let $\mathcal{U} \subseteq \mathcal{P}_{n}$ be an open substack. We abuse notations and denote $F_{l, E} \mid \mathcal{U}$ by $F_{l, E}$. Then $D^{b}(\mathcal{U})$ is generated by the $F_{l, E}$ 's (equivalently, by $F_{l, E}^{\vee}$ 's). All results about $\mathcal{P}_{n}$ proved previously, except Theorem 3.4 are valid on $\mathcal{U}$.
Proof. We have $\mathcal{U}=\left[U / \mathrm{PGL}_{2}\right]$ for an open equivariant subset $U \subset\left(\mathbb{P}^{1}\right)^{n}$. It is enough to show that any equivariant coherent sheaf $F$ on $U$ can be obtained by a finite number of equivariant extensions starting with restrictions of the vector bundles $F_{l, E}$. This follows from Theorem3.4 since $F$ is a restriction of an equivariant coherent sheaf on $\left(\mathbb{P}^{1}\right)^{n}$ (see e.g. [Tho87]). For the rest of the proposition, we restrict isomorphisms on $\mathcal{P}_{n}$ to $\mathcal{U}$.

Open substacks $\mathcal{U} \subseteq \mathcal{P}_{n}$ include all Hassett spaces when the universal family is a $\mathbb{P}^{1}$-bundle. More precisely, fix weights $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with $\sum a_{i}=2$. Recall from the introduction that the GIT quotient $X_{\mathbf{a}}$ is a quotient of the semi-stable locus $\left(\mathbb{P}^{1}\right)_{s s}^{n}$ for the fractional $\mathrm{PGL}_{2}$-polarization given by $\mathbf{a}$. We denote by $\mathcal{M}_{\mathbf{a}}$ the stack quotient $\left[\left(\mathbb{P}^{1}\right)_{s s}^{n} / \mathrm{PGL}_{2}\right]$, which is
an open substack of $\mathcal{P}_{n}$. If there are no strictly semistable points then $\mathrm{PGL}_{2}$ acts on $\left(\mathbb{P}^{1}\right)_{s s}^{n}$ freely and $\mathcal{M}_{\mathrm{a}} \cong \overline{\mathrm{M}}_{\mathrm{a}} \cong \mathrm{X}_{\mathrm{a}}$. In particular, for every partition $P \amalg Q=\{1, \ldots, n\}$ with $p=|P| \geq 2, q=|Q|$, we define an open substack $\mathcal{M}_{p, q} \subset \mathcal{P}_{n}$ of $\mathbb{P}^{1}$ bundles such that at most $p / 2$ sections indexed by $P$ (heavy points) are allowed to coincide and if $p=2 r$ then $r$ heavy points are allowed to coincide with at most $q / 2$ points indexed by $Q$ (light points). The corresponding Hassett space is $\overline{\mathrm{M}}_{p, q}$ and the GIT quotient is $\mathrm{X}_{p, q}$ (see Notation (2.8). We have $\mathcal{M}_{p, q}=\overline{\mathrm{M}}_{p, q}=\mathrm{X}_{p, q}$ unless both $p$ and $q$ are even. We use notation $\mathcal{M}_{n}, \overline{\mathrm{M}}_{n}$ and $\mathrm{X}_{n}$ if $q=0$.

Theorem 3.12. Consider a collection $\left\{F_{l, E}\right\}$ on $\mathcal{M}_{\mathbf{a}}$ for some set of pairs $(E, l)$. Order them by increasing $e=|E|$ (and arbitrarily when $e$ is the same). Choose an integer $w_{K}$ for every subset $K \subseteq \Sigma_{n}$ such that $\sum_{i \in K} a_{i}>1$ and suppose that $w_{K} \leq-l+\left(e_{0}-e_{\infty}\right)$ and $l+\left(e_{0}-e_{\infty}\right)<w_{K}+2|K|-2$ for all bundles in the collection and all subsets as above, where $e_{\infty}=|E \cap K|$ and $e_{0}=\left|E \cap K^{c}\right|$, for $K^{c}=\Sigma_{n} \backslash K$. Then $\left\{F_{l, E}\right\}$ is a strong exceptional (but not necessarily full) collection in $D^{b}\left(\mathcal{M}_{\mathbf{a}}\right)$.

The proof relies on the following theorem of Halpern-Leistner:
Theorem 3.13. [HL15] Let $[X / G]$ be the stack quotient of a smooth projective variety by a reductive group $G$ and let $\left[X^{\text {ss }} / G\right]$ the open substack corresponding to the semistable locus $X^{s s}$ with respect to a choice of polarization and linearization. For a choice of a Kempf-Ness (KN) stratification of the unstable locus $X^{u s}$ with data $Z_{i}, S_{i}, \lambda_{i}, \sigma_{i}: Z_{i} \hookrightarrow S_{i}$, define the integers $\eta_{i}=$ weight $_{\lambda_{i}} \operatorname{det}\left(N_{S_{i} \mid X}^{\vee}\right)_{\mid Z_{i}}>$ 0 . For each KN stratum $S_{i}$, choose an integer $w_{i}$. Define the full subcategory $\mathcal{G}_{w}$ of all objects $F \in D^{b}[X / G]$ such that the cohomology sheaves of $\sigma_{i}^{*} F$ have weights in $\left[w_{i}, w_{i}+\eta_{i}\right)$ for all $i$. Then the restriction functor $i^{*}: \mathcal{G}_{w} \rightarrow D^{b}\left[X^{s s} / G\right]$ is an equivalence of categories.

Remark 3.14. There are two sign conventions for weights used in the literature. Above we follow [HL15] where the ample polarization of the GIT quotient has negative weights on the unstable locus, see (3.2). However, starting with $\S 9$ we take the opposite weight as in [Tel00].
Proof of Theorem [3.12] Here $X=\left(\mathbb{P}^{1}\right)^{n}$ and $G=\mathrm{PGL}_{2}$. Up to conjugation, a 1- parameter subgroup $\lambda$ has the form $\lambda(t)=\left[\begin{array}{cc}t & 0 \\ 0 & t^{-1}\end{array}\right]$. For every subset $K \subseteq \Sigma_{n}$, consider the $\lambda$-invariant point $z_{K}=\left(p_{1}, \ldots, p_{n}\right)$, with $p_{i}=0=[0$ : 1] if $i \notin K$ and $p_{i}=\infty=[1: 0]$ otherwise. Then weight ${ }_{\lambda} \mathcal{O}(1)_{\mid x}=1$ if $x=0$ and weight ${ }_{\lambda} \mathcal{O}(1)_{\mid x}=-1$ if $x=\infty$. If $\mathcal{L}=\mathcal{O}\left(a_{1}, \ldots, a_{n}\right)$, it follows that

$$
\begin{equation*}
\text { weight }_{\lambda} \mathcal{L}_{\mid z_{K}}=\sum_{i \in K^{c}} a_{i}-\sum_{i \in K} a_{i} . \tag{3.2}
\end{equation*}
$$

Let $\Delta_{K}$ be a diagonal in $\left(\mathbb{P}^{1}\right)^{n}$ consisting of points $\left(p_{1}, \ldots, p_{n}\right)$ such that for all $i \in K$ the points $p_{i}$ are equal. Let $S_{K} \subset \Delta_{K}$ be the complement to the union of smaller diagonals. The KN strata of $X^{u s}$ are given by $Z_{K}=\left\{z_{K}\right\}$, $S_{K}$ and the canonical inclusions $\sigma_{K}: Z_{K} \hookrightarrow S_{K}$, for all $K \subseteq \Sigma_{n}$ such that $\sum_{i \in K} a_{i}>1$. The destabilizing 1-PS is $\lambda$. Next we compute $\operatorname{det}\left(N_{\Delta_{K} \mid\left(\mathbb{P}^{1}\right)^{n}}^{\vee}\right)$.

We may assume that $n \in K$. As $\Delta_{K}$ is a complete intersection in $X$ of big diagonals $\Delta_{j n}$, for all $j \in K \backslash\{n\}$, it follows that
$N_{\Delta_{K} \mid\left(\mathbb{P}^{1}\right)^{n}}^{\vee} \cong \bigoplus_{j \in K \backslash\{n\}} \mathcal{O}\left(-\Delta_{j n}\right)_{\mid \Delta_{K}}, \operatorname{det}\left(N_{\Delta_{K} \mid\left(\mathbb{P}^{1}\right)^{n}}^{\vee}\right) \cong \mathcal{O}\left(-\sum_{j \in K \backslash\{n\}} \Delta_{j n}\right)_{\mid \Delta_{K}}$.
It follows from (3.2) that $\eta_{K}:=$ weight $\left._{\lambda} \operatorname{det}\left(N_{\Delta_{K} \mid\left(\mathbb{P}^{1}\right)^{n}}^{\vee}\right)\right|_{z_{K}}=2(|K|-1)$. Theorem 3.12 then follows from Theorem 3.4, Theorem 3.13, and the following Lemma 3.15
Lemma 3.15. $F_{l, E_{\mid z_{K}}}$ has weights $m+\left(e_{0}-e_{\infty}\right)$ for $m=l, l-2, \ldots, 2-l,-l$.
Proof. For this calculation we can view $F_{l, E}$ as an $\mathrm{SL}_{2}$ - (rather than $\mathrm{PGL}_{2^{-}}$) equivariant bundle. By Lemma 3.2 and in the notations of Definition 3.1, we have that $F_{l, E}=\mathcal{O}\left(i_{1}, \ldots, i_{n}\right) \otimes V_{l}$. Note that $V_{l}=\operatorname{Sym}^{l} V_{1}$, where $V_{1}$ is a trivial rank 2 vector bundle with the standard $\mathrm{SL}_{2}$-action. The weights of $V_{l}$ (at any $\lambda$-fixed point) are $l, l-2, l-4, \ldots,-l$ and the formula follows.
Proof of Theorem [1.2] We consider the following score function,

$$
S(l, E)=l+\min (e, p-e) .
$$

We need to prove that the vector bundles $F_{l, E}$ with $S(l, E) \leq r-1$ form a full exceptional collection. First we prove exceptionality. We prove that condition $S(l, E) \leq r-1$ implies that the weights of the bundles $F_{l, E}$ fit in a window as in Theorem 3.13, Lemma 3.15 implies that the maximum weight of $F_{l, E}$ over all subsets $K$ with fixed $|K|=k$ is $\min \{l+e, l+2 p-$ $2 k-e\}$. Similarly, the minimum weight of $F_{l, E}$ is $-\min \{l+e, l+2 k-e\}$. The conditions in Theorem 3.12 for existence of a window are equivalent to requiring that for any pairs $(l, E),\left(l^{\prime}, E^{\prime}\right)$, if we let $e=|E|$ and $e^{\prime}=\left|E^{\prime}\right|$

$$
\min \{l+e, l+2 k-e\}+\min \left\{l^{\prime}+e^{\prime}, l^{\prime}+2(p-k)-e^{\prime}\right\}<\eta_{k}=2(k-1),
$$

for all $k \geq r+1$. We now prove that this is the case for the list of pairs $(l, E)$ in Theorem 1.2. We consider three cases.

Case $\mathbf{I}: e, e^{\prime} \leq r$. By assumption $l+e, l^{\prime}+e^{\prime} \leq r-1$. Then $\min \{l+e, l+$ $2 k-e\}+\min \left\{l^{\prime}+e^{\prime}, l^{\prime}+2(p-k)-e^{\prime}\right\} \leq(l+e)+\left(l^{\prime}+e^{\prime}\right) \leq 2 r-2<2(k-1)$ for all $k \geq r+1$.

Case II: $e, e^{\prime} \geq r+1$. By assumption, $l+p-e, l^{\prime}+p-e^{\prime} \leq r-1$. We have: $\min \{l+e, l+2 k-e\}+\min \left\{l^{\prime}+e^{\prime}, l^{\prime}+2(p-k)-e^{\prime}\right\} \leq(l+2 k-e)+$ $\left(l^{\prime}+2(p-k)-e^{\prime}\right) \leq 2(r-1)<2(k-1)$ for all $k \geq r+1$.

Case III: $e \leq r$ and $e^{\prime} \geq r+1$ (or the opposite). By assumption $l+e \leq$ $r-1, l^{\prime}+p-e^{\prime} \leq r-1$. It follows that $\min \{l+e, l+2 k-e\}+\min \left\{l^{\prime}+\right.$ $\left.e^{\prime}, l^{\prime}+2(p-k)-e^{\prime}\right\} \leq(l+e)+\left(l^{\prime}+2(p-k)-e^{\prime}\right) \leq 2(r-1)<2(k-1)$ for all $k \geq r+1$. We finish by applying Theorem 3.4.

Next we prove fullness. By Proposition 3.11, it suffices to prove the following claim.
Claim 3.16. Let $p=2 r+1$. Every vector bundle $F_{l, E}$ (with $l+e$ is even) on $\overline{\mathrm{M}}_{p}$ is generated by the vector bundles in the collection in Theorem 1.2

For simplicity, let $\mathcal{C}$ be the collection of vector bundles $F_{l, E}$ in Theorem 1.2, i.e., those in the range $S(l, E) \leq r-1$. We will prove the equivalent dual statement that every vector bundle $F_{l, E}^{\vee}$ on $\overline{\mathrm{M}}_{p}$ is generated by the dual
collection $\mathcal{C}^{\vee}$. We argue by induction on the score and for a fixed score, by induction on $l$. Clearly, the statement holds when $S(l, E) \leq r-1$. Let $a \geq r$ and assume that all the bundles $F_{l, E}^{\vee}$ with $S(l, E)<a$ are generated by $\mathcal{C}^{\vee}$. Let $F_{l, E}^{\vee}$ be a bundle such that $S(l, E)=a$. We consider two cases: $e=|E| \leq r$ and $e=|E| \geq r+1$.

Assume $e \leq r$. Let $I \subseteq\{1, \ldots, p\}$ with $I \cap E=\emptyset$ and $|I|=r+1$. We consider the Koszul complex for the diagonal $\Delta=\Delta_{I \cup\{x\}} \subseteq\left(\mathbb{P}^{1}\right)^{p} \times \mathbb{P}_{x}^{1}$,

$$
\begin{aligned}
& 0 \leftarrow \mathcal{O}_{\Delta} \leftarrow \mathcal{O} \leftarrow \bigoplus_{i \in I} \mathcal{O}\left(-\mathbf{e}_{\mathbf{i}}-\mathbf{e}_{\mathbf{x}}\right) \leftarrow \bigoplus_{i, k \in I} \mathcal{O}\left(-\mathbf{e}_{\mathbf{i}}-\mathbf{e}_{\mathbf{k}}-2 \mathbf{e}_{\mathbf{x}}\right) \leftarrow \ldots \\
& \leftarrow \mathcal{O}\left(-\sum_{i \in I} \mathbf{e}_{\mathbf{i}}-(r+1) \mathbf{e}_{\mathbf{x}}\right) \leftarrow 0 .
\end{aligned}
$$

Here $\left\{\mathbf{e}_{\mathbf{i}}\right\}$ for $i \in P$ is the standard basis of $\operatorname{Pic}\left(\mathbb{P}^{1}\right)^{p} \cong \mathbb{Z}^{p}$.
We tensor this resolution with $\mathcal{O}\left(-\sum_{j \in E} \mathbf{e}_{\mathbf{j}}+l \mathbf{e}_{\mathbf{x}}\right)$ and take derived pushforwards of its terms via the projection map $\pi:\left(\mathbb{P}^{1}\right)^{p+1} \rightarrow\left(\mathbb{P}^{1}\right)^{p}$, which we then restrict to the semistable locus in $\left(\mathbb{P}^{1}\right)^{p}$. Since $\pi(\Delta)$ is in the unstable locus, we obtain the following objects:

where the terms $F_{l-j, E \cup J}^{\vee}$ appear as $R^{0} \pi_{*}$ as long as $0 \leq j \leq l$, the 0 term appears when $j=l+1$ (which happens if $l \leq r$; if $l>r$ then the last term that appears is $\left.F_{l-r-1, E \cup I}^{\vee}\right)$, while the terms $F_{j-l-2, E \cup J}^{\vee}$ appear as $R^{1} \pi_{*}$ in the range $l+2 \leq j \leq r+1$ by Grothendieck-Verdier duality (see 3.1).
We claim that the first object $F_{l, E}^{\vee}$ is generated by the remaining objects, which are all generated by $\mathcal{C}^{\vee}$ by induction. It will follow that $F_{l, E}^{\vee}$ is generated by $\mathcal{C}^{\vee}$. The first claim is a special case of the following lemma, which will be used repeatedly in the remainder of the paper:
Lemma 3.17. Let $F: D(\mathcal{A}) \rightarrow \mathcal{T}$ be an exact functor of triangulated categories and let $0 \rightarrow A_{1} \rightarrow \ldots \rightarrow A_{n} \rightarrow 0$ be an exact sequence in an abelian category $\mathcal{A}$. Then any of the objects $F\left(A_{1}\right), \ldots, F\left(A_{n}\right)$ belong to the triangulated subcategory of $\mathcal{T}$ generated by the remaining objects.
Proof. It suffices to prove that each $A_{i}$ is generated by the remaining ones in $D(\mathcal{A})$. This is clear by converting the short exact sequences arising from $0 \rightarrow A_{1} \rightarrow \ldots \rightarrow A_{n} \rightarrow 0$ into exact triangles in $D(\mathcal{A})$.

We now prove that the remaining terms are generated by $\mathcal{C}^{\vee}$ by induction. Consider the two types of terms:
(a) The terms $F_{\tilde{l}, \tilde{E}}^{\vee}=F_{l-j, E \cup J}^{\vee}$ have scores $S(\tilde{l}, \tilde{E})=(l-j)+\min \{e+$ $j, p-e-j\} \leq(l-j)+(e+j)=l+e=S(l, E)=a$, and we are done by induction, since $\tilde{l} \leq l$, with equality if and only if $(\tilde{l}, \tilde{E})=(l, E)$.
(b) The terms $F_{\tilde{l}, \tilde{E}}^{\vee}=F_{j-l-2, E \cup J}^{\vee}(l+2 \leq j \leq r+1)$ have scores $S(\tilde{l}, \tilde{E})=$ $(j-l-2)+\min \{e+j, p-e-j\} \leq(j-l-2)+(p-e-j)=p-e-l-2=p-a-2$
since $S(l, E)=l+e=a$. But $p-a-2<a$ since since $a \geq r$. It follows that these terms are generated by $\mathcal{C}^{\vee}$ by induction.

Assume $e \geq r+1$. Let $I \subseteq E,|I|=r+1$ and let $E^{\prime}=E \backslash I$. As before, we tensor the above Koszul resolution of $\Delta_{I \cup\{x\}} \subseteq\left(\mathbb{P}^{1}\right)^{p} \times \mathbb{P}^{1}$ with $\mathcal{O}\left(-\sum_{k \in E^{\prime}} \mathbf{e}_{\mathbf{k}}+(r-1-l) \mathbf{e}_{\mathbf{x}}\right)$, take derived push-forwards via the projection map $\pi$ and apply Lemma 3.17.

$$
\begin{array}{rlllllllll}
\begin{array}{lllllll}
0 & F_{r-1-l, E^{\prime}}^{\vee} & \cdots & \bigoplus_{J \subseteq I,|J|=j}^{\vee} & F_{r-1-l-j, E^{\prime} \cup J}^{\vee} & \cdots & \bigoplus_{J \subseteq I,|J|=r-1-l}^{\vee} \\
& \bigoplus_{0, E^{\prime} \cup J}^{\vee} & 0 \\
J \subseteq I,|J|=r+1-l
\end{array} & F_{0, E^{\prime} \cup J}^{\vee} & \cdots & \bigoplus_{J \subseteq I,|J|=j}^{\vee} & F_{l+j-r-1, E^{\prime} \cup J}^{\vee} & \ldots & F_{l, E}^{\vee},
\end{array}
$$

where the terms $F_{r-1-l-j, E^{\prime} \cup J}^{\vee}$ appear as $R^{0} \pi_{*}$ for $0 \leq j<r-l$, the 0 term appears when $j=r-l$, and the terms $F_{l+j-r-1, E^{\prime} \cup J}^{\vee}$ appear as $R^{1} \pi_{*}$ for $r+1-l \leq j<r+1$. Consider the two types of terms and we compare theirs scores with $a=S(l, E)=l+p-e$.
(a) The terms $F_{\tilde{l}, \tilde{E}}^{\vee}=F_{r-1-l-j, E^{\prime} \cup J}^{\vee}$ have scores $S(\tilde{l}, \tilde{E})=(r-1-l-$ $j)+\min \left\{e^{\prime}+j, p-e^{\prime}-j\right\} \leq(r-1-l-j)+\left(e^{\prime}+j\right)=e-l-2$ (where $e^{\prime}=\left|E^{\prime}\right|=e-r-1$ ). But since we assume $a=l+p-e \geq r$, we have $e-l \leq r+1$, hence, $S(\tilde{l}, \tilde{E}) \leq r-1$, i.e., $F_{\tilde{l}, \tilde{E}}^{\vee}$ is in $\mathcal{C}^{\vee}$.
(b) The terms $F_{\tilde{l}, \tilde{E}}^{\vee}=F_{l+j-r-1, E^{\prime} \cup J}^{\vee}$ have scores $S(\tilde{l}, \tilde{E})=(l+j-r-1)+$ $\min \left\{e^{\prime}+j, p-e^{\prime}-j\right\} \leq(l+j-r-1)+p-e^{\prime}-j=l+p-e=S(l, E)$, and we are done by induction, since $\tilde{l}=l+j-r-1 \leq l($ as $j \leq r+1)$ with equality if and only if $j=r+1$, i.e., $J=I$ (that is $(\tilde{l}, \overline{\tilde{E}})=(l, E)$ ).
Corollary 3.18. Let $p=2 r$. A strong $S_{p}$-equivariant full exceptional collection in $D^{b}\left(\overline{\mathrm{M}}_{p, 1}\right)$ is given by $\left\{F_{l, E}\right\}$ with $l+\min \left\{e_{p}, p-e_{p}\right\} \leq r-1$.
Proof. We have $\overline{\mathrm{M}}_{p, 1} \simeq \overline{\mathrm{M}}_{\left(\frac{1}{a}, \ldots, \frac{1}{2}, \epsilon\right)} \simeq \overline{\mathrm{M}}_{p+1}$. Indeed, the stability condition is the same: no $r+1$ points are allowed to collide. The statement now follows from Theorem 1.2.
Theorem 3.19. Let $p=2 r+1, q>0$. The vector bundles $F_{l, E}$ form a full strong $\left(S_{p} \times S_{q}\right)$-equivariant exceptional collection on $\overline{\mathrm{M}}_{p, q}$ for subsets $E_{p} \subseteq P, E_{q} \subseteq Q$ such that $l+e$ is even and $l+\min \left(e_{p}, p-e_{p}\right) \leq r-1$. The order is first by increasing $e_{q}$, arbitrarily if $e_{q}$ is the same, but the set $E_{q}$ is different. If $E_{q}=E_{q}^{\prime}$, the order is by increasing $e_{p}$, and for a given $e_{p}$, arbitrarily.
Proof. We have a morphism $f: \overline{\mathrm{M}}_{p, q} \rightarrow \overline{\mathrm{M}}_{p}$, an iterated ( $q$ times) universal $\mathbb{P}^{1}$-bundle of $\overline{\mathrm{M}}_{p}$. It is induced by the projection $\left(\mathbb{P}^{1}\right)^{p+q} \rightarrow\left(\mathbb{P}^{1}\right)^{p}$ which maps $\left(\mathbb{P}^{1}\right)_{s s}^{p+q}$ to $\left(\mathbb{P}^{1}\right)_{s s}^{p}$. Using Theorem 3.4, we can identify the pull-back $f^{*}\left\{F_{l, E_{p}}\right\}$ of the collection of Theorem 1.2 with a collection $\left\{F_{l, E_{p}}\right\}$ on $\overline{\mathrm{M}}_{p, q}$ when $E_{p}$ is a subset of $P$. For every $j \in Q$, let $L_{j}=F_{0,\{1, \ldots, p, j\}}$. For every subset $J \subset Q$, let $L_{J}=\bigotimes_{j \in J} L_{j}$. From Orlov's theorem on the derived category of a projective bundle [Orl92], we have a s.o.d. of $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$ into $2^{q}$ blocks equivalent to $D^{b}\left(\overline{\mathrm{M}}_{p}\right)$ via functors $D^{b}\left(\overline{\mathrm{M}}_{p}\right) \rightarrow D^{b}\left(\overline{\mathrm{M}}_{p, q}\right), E \mapsto$ $L f^{*}(E) \otimes L_{J}$ for every subset $J \subset Q$. In blocks with $|J|=2 s$ even, we introduce an exceptional collection $\left\{F_{l, E_{p}}\right\} \otimes F_{0,\{1, \ldots, p\}}^{\otimes-2 s} \otimes L_{J}=\left\{F_{l, E_{p} \cup J}\right\}$
(use Proposition 3.6). In blocks with $|J|=2 s+1$ odd, we use an exceptional collection $\left\{F_{l, E_{p}^{c}}^{\vee}\right\} \otimes F_{0,\{1, \ldots, p\}}^{\otimes-2 s} \otimes L_{J} \simeq\left\{F_{l, E_{p} \cup J}\right\}$, where $E_{p}^{c}=P \backslash E_{p}$.

## 4. Functions of subsets, sCore and inequalities for pairs $(l, E)$

In this section we collect numerical functions of subsets and inequalities used throughout the paper in order to relate exceptional collections on different Hassett spaces. The reader may skip this section at a first reading and refer to it as needed.

Let $l \in \mathbb{Z}$ and let $E, T$ be subsets of the set $N=\{1, \ldots, n\}$.
Notation 4.1. When $e+l$ is even, we denote

$$
\begin{equation*}
f_{T, E, l}=|E \cap T|-\frac{e-l}{2}=\frac{|E \cap T|-\left|E \cap T^{c}\right|+l}{2}, \quad T^{c}=N \backslash T . \tag{4.1}
\end{equation*}
$$

We use notation $f_{T}$ when $E$ and $l$ are clear from the context. Note that $f_{T}+f_{T^{c}}=l$, so if $l \geq-1$ then at least one of $f_{T}, f_{T^{c}}$ is non-negative. Typically $l \geq 0$, in which case if $A \sqcup B=N \backslash\{z\}$ then $f_{A}$ or $f_{B}$ is $\geq 0$. Let

$$
\begin{gather*}
\alpha_{T, E, l}=\max \left\{0,-f_{T, E, l}\right\} .  \tag{4.2}\\
m_{T, E, l}=\max \left\{0, f_{T, E, l}\right\} . \tag{4.3}
\end{gather*}
$$

We use notation $\alpha_{T}, m_{T}$ when $E$ and $l$ are clear from the context.
Notation 4.2. Let $P$ be the set of heavy points, $|P|=p=2 r \geq 4$. Let $Q$ (resp., $\tilde{Q}$ ) be the set of light points, $|Q|=q=2 s+1 \geq 1$ (resp., $|\tilde{Q}|=q+1=$ $2 s+2 \geq 0$ ). For $l \in \mathbb{Z}, E \subseteq P \cup Q$ (resp., $E \subseteq P \cup \tilde{Q}$ ) we define the score

$$
\begin{gathered}
S(l, E)=l+\min \left\{e_{p}, p-e_{p}\right\}+\min \left\{e_{q}, q-e_{q}\right\} \\
\text { (resp., } \left.S^{\prime}(l, E)=l+\min \left\{e_{p}, p-e_{p}\right\}+\min \left\{e_{q}, q+1-e_{q}\right\}\right) .
\end{gathered}
$$

We use different notation for $S(l, E)$ and $S^{\prime}(l, E)$ because we often view $Q$ as subset of $\tilde{Q}$ (in which case we denote the extra light index by $y$ or $z$ ).

Lemma 4.3. The score functions satisfy the following:
(i) For $l \geq 0, E \subseteq P \cup Q$ and $(l, E)$ in any of the groups $1 A, 1 B$ or 2 of Theorem [1.5 and Remark 1.11] we have $S(l, E) \leq r+s-1$.
(ii) For $l \geq 0, E \subseteq P \cup \tilde{Q}$ and $(l, E)$ in any of the groups $1 A, 1 B, 2 A$ or $2 B$ of Theorem 1.8 and Remark 1.11] we have $S^{\prime}(l, E) \leq r+s$.
Moreover equality in (i) holds if and only if either $l+e_{p}=r-1, e_{q}=s$ (group $1 A)$, or $l+\left(p-e_{p}\right)=r-1, e_{q}=s($ group $1 B)$, or $e_{p}=r, l+e_{q}=s-1$ (group $2)$, or $e_{p}=r, l+\left(q-e_{q}\right)=s-1$ (group 2).

Similarly, equality in (ii) holds if and only if either $l+e_{p}=r-1, e_{q}=s+1$ (group $1 A)$, or $l+\left(p-e_{p}\right)=r-1, e_{q}=s+1($ group $1 B)$, or $e_{p}=r, l+q+1-e_{q}=$ $s($ group $2 A)$, or $e_{p}=r, l+e_{q}=s($ group $2 B) e_{p}=r$.
Proof. For groups $1 A, 1 B$ (on $P \cup Q$ and $P \cup \tilde{Q}$ ) it follows from the definitions that $l+\min \left\{e_{p}, p-e_{p}\right\} \leq r-1$. In this case (i), resp., (ii), follow from $\min \left\{e_{q}, q-e_{q}\right\} \leq s$, resp., $\min \left\{e_{q}, q+1-e_{q}\right\} \leq s+1$. For group $2($ on $P \cup Q$ ) it follows from the definitions that $l+\min \left\{e_{q}, q-e_{q}\right\} \leq s-1$, while for groups $2 A, 2 B$ (on $P \cup \tilde{Q}$ ) we have that $l+\min \left\{e_{q}, q+1-e_{q}\right\} \leq s$, and (i), (ii) follow from $\min \left\{e_{p}, p-e_{p}\right\} \leq r$.

Lemma 4.4. Let $l \in \mathbb{Z}$ (positive or negative), $E \subseteq P \cup \tilde{Q}$. Then

$$
\begin{gather*}
\max _{P \cup \bar{Q}=T \sqcup T^{c}} f_{T, E, l}=S^{\prime}(l, E) / 2,  \tag{4.4}\\
\max _{P \cup \tilde{Q}=T \sqcup T^{c}}-f_{T, E, l}=S^{\prime}(l, E) / 2-l . \tag{4.5}
\end{gather*}
$$

Here we assume that $T=T_{p} \amalg T_{q}$ is a subset of markings split into heary and light markings such that $\left|T_{p}\right|=r$ and $\left|T_{q}\right|=s+1$. Equality in (4.4) is attained if and only if ( $T_{p} \subseteq E_{p}$ or $E_{p} \subseteq T_{p}$ ) and ( $T_{q} \subseteq E_{q}$ or $E_{q} \subseteq T_{q}$ ), while equality in (4.5) is attained if and only if ( $T_{p}^{c} \subseteq E_{p}$ or $E_{p} \subseteq T_{p}^{c}$ ) and ( $T_{q}^{c} \subseteq E_{q}$ or $E_{q} \subseteq T_{q}^{c}$ ).

Proof. $|E \cap T|$ is maximized when $2|E \cap T|=2\left(\min \left(r, e_{p}\right)+\min \left(s+1, e_{q}\right)\right)=$ $\min \left(p, 2 e_{p}\right)+\min \left(q+1,2 e_{q}\right)$ and so $2 f_{T, E, l}=2|E \cap T|-(e-l)=\min (p-$ $\left.e_{p}, e_{p}\right)+\min \left(q+1-e_{q}, e_{q}\right)+l=S^{\prime}(l, E)$. Similarly, $-|E \cap T|$ is maximized when $2\left|E \cap T^{c}\right|-(e-l)=S^{\prime}(l, E)$ and so $(e-l)-2|E \cap T|=$ $(e-l)+2\left|E \cap T^{c}\right|-2 e=S^{\prime}(l, E)-2 l$, which proves the lemma.
Lemma 4.5. Let $E \subseteq P \cup \tilde{Q}$. Suppose ( $l, E)$ is a pair from the groups $1 A, 1 B$, $2 A$ or $2 B$ of Theorem 1.8 and Remark 1.11 Then

$$
\begin{equation*}
m_{T, E, l}=\max \left\{0, f_{T, E, l}\right\} \leq(r+s) / 2, \tag{4.6}
\end{equation*}
$$

for every subset of markings $T=T_{p} \amalg T_{q}$ split into heavy and light markings such that $\left|T_{p}\right|=r,\left|T_{q}\right|=s+1$. We call $T$ critical for ( $l, E$ ) if equality holds in (4.6). Then $T$ is critical if and only if $r+s$ is even and we have one of the cases listed in the following table, where we also list critical subsets:

$$
\begin{array}{ccl}
\text { (group 1A) } & l+e_{p}=r-1, e_{q}=s+1 & E_{p} \subseteq T_{p}, E_{q}=T_{q} ; \\
\text { (group 1B) } & l+\left(p-e_{p}\right)=r-1, e_{q}=s+1 & T_{p} \subseteq E_{p}, E_{q}=T_{q} ; \\
\text { (group 2A) } & e_{p}=r, l+q+1-e_{q}=s & E_{p}=T_{p}, T_{q} \subseteq E_{q} ; \\
\text { (group 2B) } & e_{p}=r, l+e_{q}=s & E_{p}=T_{p}, E_{q} \subseteq T_{q} .
\end{array}
$$

In addition, we have:

$$
\begin{equation*}
f_{T^{c}, E, l} \geq-(r+s) / 2 \tag{4.7}
\end{equation*}
$$

The inequality (4.7) is strict unless $r+s$ is even, $l=0$ and $T$ is critical, in particular appears in the table above.
Proof. The first statement is a direct consequence of Lemmas 4.3 and 4.4 The second claim follows from the first, since $l \geq 0, f_{T^{c}, E, l} \geq-f_{T, E, l} \geq$ $-(r+s) / 2$ with the first inequality becoming an equality iff $l=0$.
Corollary 4.6. If $(l, E)$ is in group $2 A$ or $2 B$ on $\overline{\mathrm{M}}_{p, q+1}$, then for $I \subseteq \tilde{Q},|I|=$ $s+1$, we have (i) $f_{I, E_{q}, l} \leq s / 2$, with equality iff either $e_{q}=l+s+2, I \subseteq E_{q}$ (group $2 A$ ) or $l+e_{q}=s, E_{q} \subseteq I($ group $2 B)$; (ii) $f_{I^{c}, E_{q}, l} \geq-s / 2$.
Proof. Since $\left|E_{p}\right|=r$, both inequalities follow from Lemma 4.5 for a set $T$ with heavy indices $T_{p}=E_{p}$ and light indices $T_{q}=I$.

We use analogues of Lemmas 4.4, 4.5 for $P \cup Q$ as follows:
Lemma 4.7. Let $l \in \mathbb{Z}$ (positive or negative), $E \subseteq P \cup Q, y \in Q$. Then

$$
\begin{equation*}
\max _{P \cup(Q \backslash\{y\})=T \sqcup T^{c}} f_{T, E, l}=S^{\prime}(l, E \backslash\{y\}) / 2 . \tag{4.8}
\end{equation*}
$$

Here we assume that $T=T_{p} \amalg T_{q}$ is a subset of markings split into heavy and light markings such that $\left|T_{p}\right|=r$ and $\left|T_{q}\right|=s$ and the score function $S^{\prime}(l, E \backslash\{y\})$ is
considered on $P \cup(Q \backslash\{y\})$ (i.e., $\tilde{Q}=(Q \backslash\{y\})$ in Notation 4.2). Equality in (4.8) is attained if and only if $\left(T_{p} \subseteq E_{p}\right.$ or $\left.E_{p} \subseteq T_{p}\right)$ and $\left(T_{q} \subseteq E_{q}\right.$ or $\left.E_{q} \subseteq T_{q}\right)$.

Proof. This is the same proof as for Lemma4.5, as $E \cap T=(E \backslash\{y\}) \cap T$.
Lemma 4.8. Let $E \subseteq P \cup Q$. Suppose $(l, E)$ is a pair from the groups $1 A, 1 B$ or 2 of Theorem 1.5 and Remark1.11 Let $y \in Q$ and $T=T_{p} \amalg T_{q}=P \cup(Q \backslash\{y\})$ be split into heavy and light markings such that $\left|T_{p}\right|=r$ and $\left|T_{q}\right|=s$. Then

$$
\begin{equation*}
m_{T, E, l}=\max \left\{0, f_{T, E, l}\right\} \leq(r+s-1) / 2 \tag{4.9}
\end{equation*}
$$

We call $T$ critical if equality holds in (4.9), which happens if and only if $r+s$ is odd, $y \notin E$ and we are in one of the following cases:

$$
\begin{array}{ccl}
(\text { group } 1 A) & l+e_{p}=r-1, e_{q}=s, & E_{p} \subseteq T_{p}, E_{q}=T_{q} ; \\
(\text { group } 1 B) & l+\left(p-e_{p}\right)=r-1, e_{q}=s, & T_{p} \subseteq E_{p}, E_{q}=T_{q} ; \\
(\text { group } 2) & e_{p}=r, l+e_{q}=s-1, & E_{p}=T_{p}, E_{q} \subseteq T_{q} .
\end{array}
$$

In particular, $m_{T, E, l} \leq \frac{r+s}{2}-|E \cap\{y\}|$.
Proof. By Lemma 4.4 we have that $2 f_{T, E, l} \leq S^{\prime}(l, E \backslash\{y\})$. The proof of Lemma 4.4 shows that equality holds iff $S^{\prime}(l, E \backslash\{y\})=l+\min \left\{e_{p}, p-\right.$ $\left.e_{p}\right\}+\min \left\{e_{q}, q-1-e_{q}\right\}$. By Lemma 4.3 we have $S^{\prime}(l, E \backslash\{y\}) \leq r+s-1$ and equality holds in the cases listed there. Note that in the case when $e_{p}=r, l+\left(q-e_{q}\right)=s-1$, equality in (4.9) is not possible, as the equality $l+\min \left\{e_{p}, p-e_{p}\right\}+\min \left\{e_{q}, q-1-e_{q}\right\}=r+s-1$ is not possible.

Corollary 4.9. Assume $p=2 r, q=2 s+1$. For $(l, E)$ in group $1 A$ or $1 B$ on $\overline{\mathrm{M}}_{p, q+1}$ or $\overline{\mathrm{M}}_{p, q}$, with $P=R \sqcup R^{\prime}$ a partition of $P$ with $|R|=\left|R^{\prime}\right|=r$, then

$$
\begin{equation*}
f_{R, E_{p}, l} \leq(r-1) / 2 \tag{4.10}
\end{equation*}
$$

Proof. Note that if $(l, E)$ is in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q+1}$, then $\left(l, E_{p}\right)$ is in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p}$. Similarly if $(l, E)$ is in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q}$, then $\left(l, E_{p}\right)$ is in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, 1}$. The result follows by applying Lemma4.5in the case when $p=2 r, s=-1$, and Lemma 4.8 in the case when $p=2 r, s=0$ for the partition $T=R, T^{c}=R^{\prime}$.

We will also need the following:
Lemma 4.10. Let $q=|Q|=2 s+1, l \geq 0, E \subseteq Q$ a set with $e=|E|$ satisfying $l+\min \{e, q-e\} \leq s-1$. Let $I \subseteq Q$ be any subset. Then

$$
l+\left|E \cap I^{c}\right|-|E \cap I| \leq \begin{cases}s-1 & \text { if } \quad e \leq s  \tag{4.11}\\ 2\left|I^{c}\right|-s-2 & \text { if } \quad e \geq s+1\end{cases}
$$

Proof. If $e \leq s$ then the inequality follows from $l+e \leq s-1$. If $e \geq s+1$, then the inequality follows from $l-e \leq-s-2$.

The same way we obtain the following:
Lemma 4.11. Let $q=|Q|=2 s+2, l \geq 0, E \subseteq Q$ a set with $e=|E|$, and let $I \subseteq Q$ be any subset. If $l+\min \{e+1, q+1-e\} \leq s$, then

$$
l+\left|E \cap I^{c}\right|-|E \cap I| \leq \begin{cases}s-1 & \text { if } \quad e \leq s  \tag{4.12}\\ 2\left|I^{c}\right|-s-2 & \text { if } \quad e \geq s+1\end{cases}
$$

$$
\begin{align*}
& \text { If } l+\min \{e, q+2-e\} \leq s, \text { then } \\
& \qquad l+\left|E \cap I^{c}\right|-|E \cap I| \leq \begin{cases}s & \text { if } \\
2\left|I^{c}\right|-s-3 & \text { if } \\
e \geq s+1\end{cases} \tag{4.13}
\end{align*}
$$

## 5. Extending vector bundles $F_{l, E}$ TO some Hassett spaces

In this section we will extend the construction of vector bundles $F_{l, E}$, defined in Section 3 for open substacks of the stack $\left.\left[\mathbb{P}^{1}\right)^{n} / \mathrm{PGL}_{2}\right]$, to certain Hassett spaces. They will include all spaces $\overline{\mathrm{M}}_{p, q}$ when $p$ and $q$ are both even as well as the universal families over them.

Note 5.1. Throughout this section we consider Hassett spaces $\overline{\mathrm{M}}:=\overline{\mathrm{M}}_{\mathcal{A}}$ ( $\sum a_{i}>2$ ) such that $\sum_{i \in I} a_{i} \neq 1$ for all $|I| \geq 2$ (i.e., the weight data is contained in a fine open chamber) and we will denote $\alpha: W \rightarrow \overline{\mathrm{M}}$ the universal family, with $\sigma_{1}, \ldots, \sigma_{n}$ the corresponding sections.

Recall that on $\overline{\mathrm{M}}$ we have tautological classes $\psi_{i}=\sigma_{i}^{*}\left(\omega_{\alpha}\right)=-\sigma_{i}^{*}\left(\sigma_{i}\right)$, $\delta_{i j}=\sigma_{i}^{*}\left(\sigma_{j}\right)$. We denote the extra marking on $W$ by $y$. We will further consider the universal family $\pi: \mathcal{U} \rightarrow W$, denoting the extra marking on $\mathcal{U}$ with $x$. We recall several facts about Hassett spaces which will be used extensively in what follows:
(1) $W$ is the Hassett space $\overline{\mathrm{M}}_{\left(a_{1}, \ldots, a_{n}, \epsilon\right)}$ for $\epsilon \ll 1$ [Has03, Prop. 5.4]. The space $W$ is a Hassett space whose weight data is also contained in a fine open chamber. In particular, W is smooth and so is its universal family $\mathcal{U}$.
(2) If $\alpha: W \rightarrow \overline{\mathrm{M}}$ is a $\mathbb{P}^{1}$-bundle, i.e., all $\mathcal{A}$-stable curves are irreducible, then $\psi_{i}+\psi_{j}=-2 \delta_{i j}$ in $\operatorname{Pic}(\overline{\mathrm{M}})$ [CT20b, Lemma 2.1]. Note that $\delta_{i j}=0$ when $a_{i}+a_{j}>1$ (the sections $\sigma_{i}, \sigma_{j}$ are disjoint).
(3) If $a_{1}=1$ and $\sum_{k \neq 1, j} a_{k} \leq 1$ for all $j \geq 2$, then $\overline{\mathrm{M}}_{\mathcal{A}} \cong \mathbb{P}^{n-3}$ [Has03, §6.2] and we can assume without loss of generality that $a_{j}=\frac{1}{n-1}+\epsilon$ for $j \geq 2$. Furthermore, $\delta_{i j}=\mathcal{O}(1)(i \neq j, i, j \neq 1), \psi_{1}=\mathcal{O}(1), \psi_{j}=\mathcal{O}(-1)(j \neq 1)$.
(4) The boundary divisors of a Hassett space $\overline{\mathrm{M}}=\overline{\mathrm{M}}_{\left(a_{i}\right)}$ have two types:

Type I: divisors $\delta_{T, T^{c}}$ with $|T|,\left|T^{c}\right| \geq 2$, whose generic element parametrizes curves with two components, marked by $T$ and $T^{c}$ respectively $\left(\sum_{i \in T} a_{i}>\right.$ 1, $\sum_{i \in T^{c}} a_{i}>1$ ). We can identify $\delta_{T, T^{c}}=\overline{\mathrm{M}}_{T \cup\{u\}} \times \overline{\mathrm{M}}_{T^{c} \cup\{u\}}$. Here $u$ denotes the attaching point and comes with weight 1.

Type II: divisors parametrizing curves where markings $i, j$ coincide (so $a_{i}+a_{j} \leq 1$ ). When there is no risk of confusion, we abuse notations and denote this by $\delta_{i j}$ (this is consistent with the definition of the tautological class $\delta_{i j}=\sigma_{i}^{*} \sigma_{j}$ ). We identify $\delta_{i j}=\overline{\mathrm{M}}_{\{i, j\}^{c} \cup\{u\}}$, with the marking $u$ having weight $a_{i}+a_{j}$. The image of the section $\sigma_{i}: \overline{\mathrm{M}} \rightarrow W$ may be identified with $\delta_{i y} \subset W$ when $a_{i}<1$.
(5) Let $\overline{\mathrm{M}}=\overline{\mathrm{M}}_{\left(a_{i}\right)}, \overline{\mathrm{M}}^{\prime}=\overline{\mathrm{M}}_{\left(b_{i}\right)}$ be Hassett spaces related by a reduction $\operatorname{map} p: \overline{\mathrm{M}} \rightarrow \overline{\mathrm{M}}^{\prime}$ (i.e., $a_{i} \geq b_{i}$ for all $i$ ). Let $\pi: \mathcal{U} \rightarrow \overline{\mathrm{M}}, \pi^{\prime}: \mathcal{U}^{\prime} \rightarrow \overline{\mathrm{M}}^{\prime}$ be the corresponding universal families. There is an induced map $v: \mathcal{U} \rightarrow \mathcal{U}^{\prime}$. By [CT20b, Lemma 2.3], we have:

$$
\begin{equation*}
p^{*} \psi_{i}=\psi_{i}-\sum_{i \in I,|I| \geq 2, \sum_{i \in I} a_{i}>1, \sum_{i \in I} b_{i} \leq 1} \delta_{I, I^{c}} \tag{5.1}
\end{equation*}
$$

$$
\begin{gather*}
p^{*} \delta_{i j}=\delta_{i j}+\sum_{i, j \in I,|I| \geq 3, \sum_{i \in I} a_{i}>1, \sum_{i \in I} b_{i} \leq 1} \delta_{I, I^{c}}  \tag{5.2}\\
v^{*} \omega_{\rho}=\omega_{\pi}-\sum_{T} \delta_{T \cup\{y\}, T^{c}}, \quad v^{*} \delta_{i y}=\delta_{i y}+\sum_{i \in T} \delta_{T \cup\{y\}, T^{c}} . \tag{5.3}
\end{gather*}
$$

Here and everywhere we identify line bundles with divisor classes (hence, we use both additive and multiplicative notation). We often write $\delta_{I}$ for the boundary divisor $\delta_{I, I^{c}} \subset \overline{\mathrm{M}}$, when the set of indices $I$ and its complement $I^{c}$ is clear from the context.

Throughout the remaining part of this section, we assume (in addition to $\sum_{i \in I} a_{i} \neq 1$ for $|I| \geq 2$ ) that the Hassett spaces $\overline{\mathrm{M}}:=\overline{\mathrm{M}}_{\mathcal{A}}$ are such that all $\mathcal{A}$-stable curves have at most two components. Let $N=\{1, \ldots, n\}$. We construct vector bundles $F_{l, E}$ on both $\overline{\mathrm{M}}$ and $W$ as follows. Recall that $\alpha: W \rightarrow \overline{\mathrm{M}}, \pi: \mathcal{U} \rightarrow W$ denote the universal families.

Definition 5.2 ( $F_{l, E}$ on $\overline{\mathrm{M}}$ ). For a subset $E \subseteq N$ with $e=|E|$ and integer $l \geq-1$ such that $e+l$ is even, we let $F_{l, E}=R \alpha_{*}\left(N_{l, E}\right)$, where

$$
N_{l, E}=\omega_{\alpha}^{\frac{e-l}{2}}(E) \otimes \mathcal{O}\left(-\sum_{T} \alpha_{T, E, l} \delta_{T \cup\{y\}, T^{c}}\right),
$$

(see (4.2) for the definition of $\alpha_{T, E, l}$ ), and the sum is over all $T \subset N$ such that $\delta_{T, T^{c}} \subseteq \overline{\mathrm{M}}$ is a boundary component (here $T^{c}:=N \backslash T$ ).

Definition 5.3 ( $F_{l, E}$ on $W$ ). For $E \subseteq N \cup\{y\}$ with $e=|E|$ and integer $l \geq 0$ such that $e+l$ is even, on $W$ we let $F_{l, E}=R \pi_{*}\left(N_{l, E}\right)$, where

$$
N_{l, E}=\omega_{\pi}^{\frac{e-l}{2}}(E) \otimes \mathcal{O}\left(-\sum_{T} \alpha_{T} \delta_{T \cup\{x\}, T^{c} \cup\{y\}}-\sum_{T} \alpha_{T \cup\{y\}} \delta_{T \cup\{y, x\}, T^{c}}\right)
$$

where for $S=T$ or $S=T \cup\{y\}$, we define $\alpha_{S}:=\alpha_{S, E, l}$ (see (4.2); hence, $\alpha_{T}=\alpha_{T \cup\{y\}}=0$ or $\left.\alpha_{T \cup\{y\}}=\alpha_{T}-|E \cap\{y\}| \geq 0\right)$. The sum is over all $T \subset N$ such that $\delta_{T, T^{c}} \subseteq \overline{\mathrm{M}}$ is a boundary component ( $T^{c}:=N \backslash T$ ).

Before we prove that these are indeed vector bundles (Lemmas 5.11, 5.16), we first discuss the geometry of the spaces $\overline{\mathrm{M}}, W$ and $\mathcal{U}$.

Lemma 5.4. Let $\delta \subseteq \overline{\mathrm{M}}$ be a boundary divisor in $\overline{\mathrm{M}}$. Using the identification $\delta=\overline{\mathrm{M}}_{T \cup\{u\}} \times \overline{\mathrm{M}}_{T^{c} \cup\{u\}}$, we have $\delta_{\mid \delta} \cong\left(-\psi_{u}\right) \boxtimes\left(-\psi_{u}\right)$ in $\operatorname{Pic}(\delta)$ (where we identify $\delta=\overline{\mathrm{M}}_{T^{c} \cup\{u\}}$ when $\overline{\mathrm{M}}_{T \cup\{u\}}$ is a point).
Proof. We first check that the identity holds on $\overline{\mathrm{M}}_{0, n}$. Using the forgetful map $\pi: \overline{\mathrm{M}}_{0, n} \rightarrow \overline{\mathrm{M}}_{0,\{1,2,3,4\}}$, we have that $\pi^{*} \delta_{12,34}=\sum_{1,2 \in T^{\prime}, 3,4 \in T^{\prime \prime}} \delta_{T^{\prime}, T^{\prime c}}$ and hence we have in $\operatorname{Pic}\left(\overline{\mathrm{M}}_{0, n}\right)$ that

$$
\begin{equation*}
\sum_{1,2 \in T, 3,4 \in T^{c}} \delta_{T, T^{c}}=\sum_{1,3 \in T, 2,4 \in T^{c}} \delta_{T, T^{c}} . \tag{5.4}
\end{equation*}
$$

Fix $\delta:=\delta_{T_{0}, T_{0}^{c}}$ a boundary divisor on $\overline{\mathrm{M}}_{0, n}$. We may assume without loss of generality that $1,2 \in T_{0}, 3,4 \in T_{0}^{c}$. Restricting (5.4) to $\delta$ we obtain
that $\delta_{\mid \delta}+\sum_{\left(T_{0} \subseteq T, 3,4 \in T^{c}\right) \text { or }\left(1,2 \in T \subseteq T_{0}\right)} \delta_{T, T^{c} \mid \delta}=0$. Identifying $\delta=\overline{\mathrm{M}}_{T 0} \cup\{u\} \times$ $\overline{\mathrm{M}}_{T_{0}^{c} \cup\{u\}}$, it follows that

$$
-\delta_{\mid \delta}=\left(\sum_{1,2 \in S \subseteq T_{0}} \delta_{S,\left(T_{0} \backslash S\right) \cup\{u\}}\right) \boxtimes\left(\sum_{3,4 \in S \subseteq T_{0}^{c}} \delta_{S,\left(T_{0}^{c} \backslash S\right) \cup\{0\}}\right)
$$

in $\operatorname{Pic}(\delta)$. Using the Kapranov model given by the line bundle $\psi_{u}$, we have that $\psi_{u}=\sum_{1,2 \in S \subseteq T_{0}} \delta_{S}$ in $\operatorname{Pic}\left(\overline{\mathrm{M}}_{T_{0} \cup\{u\}}\right)$. By symmetry, we have that $\psi_{u}=$ $\sum_{3,4 \in S \subseteq T_{0}^{c}} \delta_{S,\left(T_{0} \backslash S\right) \cup\{u\}}$ in $\operatorname{Pic}\left(\overline{\mathrm{M}}_{T_{0}^{c} \cup\{u\}}\right)$ and the identity follows.

Consider now an arbitrary Hassett space $\overline{\mathrm{M}}=\overline{\mathrm{M}}_{\mathcal{A}}$ and let $p: \overline{\mathrm{M}}_{0, n} \rightarrow \overline{\mathrm{M}}$ be the reduction map. Fix $\delta$ a boundary divisor on $\overline{\mathrm{M}}$, corresponding to a partition $N=T_{0} \sqcup T_{0}^{c}$ (when $\delta=\delta_{i j}$ is of type II (see Note 5.1(4)), we let $\left.T_{0}=\{i, j\}\right)$. We let $\delta^{\prime}$ be the boundary divisor $\delta_{T_{0}, T_{0}^{c}}$ on $\overline{\mathrm{M}}_{0, n}$ and let $p^{\prime}=p_{\mid \delta^{\prime}}: \delta^{\prime} \rightarrow \delta$ be the restriction map, which, after identifying $\delta=$ $\overline{\mathrm{M}}_{1} \times \overline{\mathrm{M}}_{2}, \delta^{\prime}=\overline{\mathrm{M}}_{0, T_{0} \cup\{u\}} \times \overline{\mathrm{M}}_{0, T_{0}^{c} \cup\{u\}}$, is the product of reduction maps $p_{1}$ : $\overline{\mathrm{M}}_{0, T_{0} \cup\{u\}} \rightarrow \overline{\mathrm{M}}_{1}, p_{2}: \overline{\mathrm{M}}_{0, T_{0}^{c} \cup\{u\}} \rightarrow \overline{\mathrm{M}}_{2}$. It suffices to check that $\left(p^{*} \mathcal{O}(\delta)\right)_{\mid \delta^{\prime}}=$ $p_{1}^{*}\left(-\psi_{u}\right) \boxtimes p_{2}^{*}\left(-\psi_{u}\right)$.

If $\delta$ is of type I, then $p^{*} \mathcal{O}(\delta)=\mathcal{O}\left(\delta^{\prime}\right)$. By (5.1) we have that $p_{1}^{*} \psi_{u}=\psi_{u}$, $p_{2}^{*} \psi_{u}=\psi_{u}$, since the marking $u$ has weight 1 on $\overline{\mathrm{M}}_{1}$ and $\overline{\mathrm{M}}_{2}$. The statement now follows, since $\mathcal{O}\left(\delta^{\prime}\right)_{\mid \delta^{\prime}}=\left(-\psi_{u}\right) \boxtimes\left(-\psi_{u}\right)$ on $\overline{\mathrm{M}}_{0, n}$.

If $\delta$ is of type II, we may assume that $T_{0}=\{1,2\}$ (so $a_{1}+a_{2} \leq 1$ ). By (5.2) we have $p^{*} \delta_{12}=\delta_{12}+\sum_{1,2 \in T,|T| \geq 3, \sum_{i \in T} a_{i} \leq 1} \delta_{T, T^{c}}$. Identifying $\delta^{\prime}=$ $\overline{\mathrm{M}}_{0, T_{0}^{c} \cup\{u\}}$, we have in $\operatorname{Pic}\left(\delta^{\prime}\right)$ that

$$
p^{*} \mathcal{O}(\delta)_{\mid \delta^{\prime}}=\mathcal{O}\left(\delta^{\prime}\right)_{\mid \delta^{\prime}}+\sum_{\emptyset \neq S \subset T_{0}^{c}, a_{1}+a_{2}+\sum_{i \in S} a_{i} \leq 1} \delta_{S \cup\{u\}, T_{0}^{c} \backslash S}
$$

and we already proved that $\mathcal{O}\left(\delta^{\prime}\right)_{\mid \delta^{\prime}}=-\psi_{u}$. We identify $\delta=\overline{\mathrm{M}}_{2}$ (as $\overline{\mathrm{M}}_{1}$ is a point), and note that the map $p^{\prime}: \delta^{\prime} \rightarrow \delta$ is a reduction map. It follows by (5.1) that $p^{*}\left(-\psi_{u}\right)=-\psi_{u}+\sum_{\emptyset \neq S \subset T_{0}^{c}, a_{1}+a_{2}+\sum_{i \in S} a_{i} \leq 1} \delta_{S \cup\{u\}, T_{0}^{c} \backslash S}$.

Lemma 5.5. Assume $a_{1}=1$ and assume that the universal family $\alpha: W \rightarrow \overline{\mathrm{M}}$ is a $\mathbb{P}^{1}$-bundle, i.e., all $\mathcal{A}$-stable curves are irreducible. Then:
(i) $\overline{\mathrm{M}} \cong \mathbb{P}^{n-3}$ and $W \cong \mathrm{Bl}_{p} \mathbb{P}^{n-2}$. If we denote $H$ the hyperplane class and by $\Delta$ the exceptional divisor on $\mathrm{Bl}_{p} \mathbb{P}^{n-2}$, then we have $\psi_{1}=\mathcal{O}(H), \delta_{N \backslash\{1\},\{1, y\}}=\Delta$ in $\operatorname{Pic}(W)$.
(ii) The map $\alpha: W=\mathrm{Bl}_{p} \mathbb{P}^{n-2} \rightarrow \overline{\mathrm{M}}=\mathbb{P}^{n-3}$ is induced by the linear system $|H-\Delta|$. Moreover, if $f$ is a fiber of $\alpha$, then $\psi_{1} \cdot f=H \cdot f=1$.
(iii) In $\operatorname{Pic}(W)$ we have that $\delta_{j y}=H$, for all $j$ in $N \backslash\{1\}$.

Proof. Since there are no reducible $\mathcal{A}$-curves, it follows that for all $j \neq 1$, we have $\sum_{k \neq 1, j} a_{k} \leq 1$ and therefore $\bar{M} \cong \mathbb{P}^{n-3}$ (Note5.1(3)). The only reducible curves parametrized by the Hassett space $W$ are those given by the boundary divisor $\delta:=\delta_{1 y}$. Let $\overline{\mathrm{M}}^{\prime}$ be the Hassett space with the weights $\left(1, \frac{1}{n-1}, \ldots, \frac{1}{n-1}, \epsilon\right)$ (where $\frac{1}{n-1}$ appears $(n-1)$ times). Then $\overline{\mathrm{M}}^{\prime} \cong \mathbb{P}^{n-2}$ by Note 5.1(2). The reduction map $\phi: W \rightarrow \overline{\mathrm{M}}^{\prime}$ contracts $\delta$ to a point $p \in \overline{\mathrm{M}}^{\prime}$. Hence, $W=\mathrm{Bl}_{p} \overline{\mathrm{M}}^{\prime}$ with exceptional divisor $\Delta=\delta$. By (5.1), we have $\phi^{*} \psi_{1}=\psi_{1}$. This proves (i).

Clearly, the map $\alpha$ restricted to $\Delta=\delta$ is an isomorphism. The only morphism $\pi: \mathrm{Bl}_{p} \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-3}$ which is an isomorphism on the exceptional divisor $\Delta$ is the one given by the linear system $|H-\Delta|$. We have then from (i) that $H \cdot f=1$ for any fiber $f$ of $\pi$. This proves (ii).

The curve $f$ is obtained by moving the marking $y$ along a $\mathbb{P}^{1}$ with fixed markings from $N$. Hence, $\delta_{j y} \cdot f=1$, for all $j \in N$. As $\delta_{1 y}=\Delta$ and $\delta_{j y} \cdot \Delta=0$, it follows that $\delta_{j y}=H$ in $\operatorname{Pic}(W)$ if $j \neq 1$. This proves (iii).

Lemma 5.6. Every boundary divisor $\delta_{T}:=\delta_{T, T^{c}} \subseteq \overline{\mathrm{M}}$ of type I is isomorphic to $\mathbb{P}^{m-2} \times \mathbb{P}^{n-m-2}$ for some $m$. Its preimage in $W$ is the union of $\delta_{T \cup\{y\}}$ and $\delta_{T^{c} \cup\{y\}}$, where $\delta_{T \cup\{y\}} \cong \mathrm{Bl}_{p} \mathbb{P}^{m-1} \times \mathbb{P}^{n-m-2}$. The restriction map $\alpha_{\mid \delta_{T \cup\{y\}}}: \delta_{T \cup\{y\}} \rightarrow \delta_{T}$ is the product map $\pi \times I d: \mathrm{Bl}_{p} \mathbb{P}^{m-1} \times \mathbb{P}^{n-m-2} \rightarrow \mathbb{P}^{m-2} \times \mathbb{P}^{n-m-2}$ where $\pi: \mathrm{Bl}_{p} \mathbb{P}^{m-2} \rightarrow \mathbb{P}^{m-3}$ is a $\mathbb{P}^{1}$-bundle. The description for $\delta_{T^{c} \cup\{y\}}$ is similar.

Proof. The boundary divisor $\delta_{T} \subseteq \overline{\mathrm{M}}$ corresponds to $\mathcal{A}$-stable curves with two components and markings from $T, T^{c}$, respectively. Since $\delta_{T}=\overline{\mathrm{M}}^{\prime} \times$ $\overline{\mathrm{M}}^{\prime \prime}$, where $\overline{\mathrm{M}}^{\prime}$ and $\overline{\mathrm{M}}^{\prime \prime}$ are Hassett spaces parametrizing only irreducible curves, we have for all $j \in T$ that $\sum_{i \in T \backslash\{j\}} a_{i} \leq 1<\sum_{i \in T} a_{i}$ (recall that $\overline{\mathrm{M}}^{\prime}$ and $\overline{\mathrm{M}}^{\prime \prime}$ have an extra marking corresponding to the attaching point). By Lemma 5.5, $\overline{\mathrm{M}}^{\prime}=\mathbb{P}^{m-2}$ and similarly, $\overline{\mathrm{M}}^{\prime \prime}=\mathbb{P}^{n-m-2}$, with $m=|T|$. Consider now $\delta_{T \cup\{y\}} \subseteq W$, the corresponding boundary component in $W$. We have $\delta_{T \cup\{y\}}=\mathcal{U}^{\prime} \times \overline{\mathrm{M}}^{\prime \prime}$, where $\pi: \mathcal{U}^{\prime} \rightarrow \overline{\mathrm{M}}^{\prime}$ is the universal family over $\overline{\mathrm{M}}^{\prime}$ and $\mathcal{U}^{\prime}$ can be identified with the Hassett space with weights $\left\{a_{i}\right\}_{i \in T} \cup$ $\{\epsilon, 1\}$, with the weight $\epsilon$, resp., 1 , corresponding to the marking $y$, resp., the attaching point. It follows from Lemma 5.5 that $\delta_{T \cup\{y\}}=\mathrm{Bl}_{p} \mathbb{P}^{m-2} \times \mathbb{P}^{n-m-2}$ and the restriction map $\alpha_{\mid \delta_{T \cup\{y\}}}$ is as claimed.

Lemma 5.7. Let $\delta_{T}:=\delta_{T, T^{c}} \subseteq \overline{\mathrm{M}}$ be a boundary divisor of type I. Using the identification $\delta_{T}=\mathbb{P}^{m-2} \times \mathbb{P}^{n-m-2}$, we have in $\operatorname{Pic}\left(\delta_{T}\right)$ :
(i) $\delta_{i j \mid \delta_{T}} \cong \begin{cases}\mathcal{O}(1,0) & \text { if } \quad i, j \in T \\ \mathcal{O}(0,1) & \text { if } \quad i, j \in T^{c} \\ \mathcal{O} & \text { otherwise }\end{cases}$
(ii) $\psi_{j \mid \delta_{T}} \cong\left\{\begin{array}{lll}\mathcal{O}(-1,0) & \text { if } & j \in T \\ \mathcal{O}(0,-1) & \text { if } & j \in T^{c}\end{array}\right.$
(iii) $\delta_{T \mid \delta_{T}}=\mathcal{O}(-1,-1)$.

Proof. Clearly, if $i \in T, j \in T^{c}, \delta_{i j}$ is disjoint from $\delta_{T}$, while if $i, j \in T$ then $\delta_{i j \mid \delta_{T}} \cong \delta_{i j} \boxtimes \mathcal{O}=\mathcal{O}(1,0)$ (see Note5.1(3)). This proves (i). To prove (ii), let $j \in T$. Recall that the universal family over $\delta_{T}$ is the union $\mathcal{U}^{\prime} \cup \mathcal{U}^{\prime \prime}$, where $\alpha^{\prime}: \mathcal{U}^{\prime} \rightarrow \overline{\mathrm{M}}^{\prime}, \alpha^{\prime \prime}: \mathcal{U}^{\prime \prime} \rightarrow \overline{\mathrm{M}}^{\prime \prime}$ are the corresponding universal families. As $\mathcal{U}^{\prime}, \mathcal{U}^{\prime \prime}$ are attached along the section corresponding to the attaching point (disjoint from the section $\sigma_{j}$ ), we have that $\psi_{j \mid \delta_{T}}=\sigma_{j}^{*} \omega_{\alpha_{l \delta_{T}}}=\sigma_{j}^{*} \omega_{\alpha^{\prime}} \boxtimes \mathcal{O}=$ $\psi_{j} \boxtimes \mathcal{O}$, which is $\mathcal{O}(-1,0)$ (see Note $\left.5.1(3)\right)$. This proves (ii). Part (iii) is a particular case of Lemma 5.4 (using again Note5.1(3))

Lemma 5.8. In the set-up and notations of Lemma 5.6, we use the identification $\delta_{T \cup\{y\}} \cong \mathrm{Bl}_{p} \mathbb{P}^{m-1} \times \mathbb{P}^{n-m-2}$, and denote by $H$ the hyperplane class and by $\Delta$ the exceptional divisor on $\mathrm{Bl}_{p} \mathbb{P}^{m-1}$. We have:
(i) $\left(\omega_{\alpha}\right)_{\mid \delta_{T \cup\{y\}}} \cong \mathcal{O}(-H) \boxtimes \mathcal{O}$.
(ii) $\mathcal{O}\left(\sigma_{j}\right)_{\mid \delta_{T \cup\{y\}}} \cong \mathcal{O}$ if $j \notin T$ and $\mathcal{O}(H) \boxtimes \mathcal{O}$ otherwise.
(iii) $\quad\left(\delta_{T^{\prime} \cup\{y\}}\right)_{\mid \delta_{T \cup\{y\}}} \cong \begin{cases}\mathcal{O}(\Delta) \boxtimes \mathcal{O} & \text { if } \quad T^{\prime}=T^{c} \\ \mathcal{O}(-H) \boxtimes \mathcal{O}(-1) & \text { if } \quad T^{\prime}=T \\ \mathcal{O} & \text { if } \quad T^{\prime} \neq T, T^{c} .\end{cases}$

Also, $(\pi \times I d)^{*} \mathcal{O}(-a,-b)=\pi^{*} \mathcal{O}(-a) \boxtimes \mathcal{O}(-b)=\mathcal{O}(-a H+a \Delta) \boxtimes \mathcal{O}(-b)$.
Proof. Denote for simplicity $\delta=\delta_{T \cup\{y\}} \subseteq W, \bar{\delta}=\delta_{T} \subseteq \overline{\mathrm{M}}$. Since $\alpha$ is a family of nodal curves, we have $\omega_{\alpha} \cong K_{\mathcal{U}}-\alpha^{*} K_{\overline{\mathrm{M}}}$. We have by Lemmas 5.4, 5.5(i) and adjunction that $K_{\mathcal{U} \mid \delta}=K_{\delta}-\delta_{\mid \delta}=\mathcal{O}((-m+1) H+(m-2) \Delta) \boxtimes$ $\mathcal{O}(-n+m+2)$. Similarly, we have $K_{\overline{\mathrm{M}} \mid \bar{\delta}}=K_{\bar{\delta}}-\bar{\delta}_{\mid \bar{\delta}}=\mathcal{O}(-m+2,-n+m+2)$. It follows from Lemma5.5(ii) that $\omega_{\alpha \mid \delta}=\mathcal{O}(-H) \boxtimes \mathcal{O}$. This proves (i). Part (ii) follows from $\sigma_{j}=\delta_{j y}$ and Lemma 5.5(iii).

To prove (iii), note that two boundary divisors $\delta_{T \cup\{y\}}, \delta_{T^{\prime} \cup\{y\}}$ intersect if and only if $T^{\prime}=T$ or $T^{c}$. Furthermore, $\delta_{T^{c} \cup\{y\}_{\mid \delta}}$ has class $\Delta \boxtimes \mathcal{O}$ by Lemma 5.5(i). By Lemmas 5.4, 5.5(iii) and Note 5.1(3), we have $\delta_{\mid \delta}=\left(-\psi_{u}\right) \boxtimes$ $\left(-\psi_{u}\right)=\mathcal{O}(-H) \boxtimes \mathcal{O}(-1)$. The last statement follows from Lemma 5.5(ii).

Lemma 5.9. Let $m \geq 2$. Consider the $\mathbb{P}^{1}$-bundle $\pi: \mathrm{Bl}_{p} \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-2}$. Then

$$
R \pi_{*}(\mathcal{O}(a \Delta))=\mathcal{O}(-a) \oplus \ldots \oplus \mathcal{O}(-1) \oplus \mathcal{O} \quad \text { if } \quad a \geq 0
$$

while $R \pi_{*}(\mathcal{O}(-\Delta))=0$ and, when $a \geq 2, R \pi_{*}(\mathcal{O}(-a \Delta))$ is generated by $\mathcal{O}(1), \ldots, \mathcal{O}(a-1)$. More generally, $R \pi_{*}(\mathcal{O}(a H-b \Delta))$ is either 0 if $a=b-1$, or it is generated by: (i) $\mathcal{O}(b), \mathcal{O}(b+1), \ldots, \mathcal{O}(a)$ if $a \geq b$; (ii) $\mathcal{O}(a+1), \mathcal{O}(a+$ $2), \ldots, \mathcal{O}(b-1)$ if $a \leq b-2$. In particular, we have that $R \pi_{*}(\mathcal{O}(a H-b \Delta))$ is generated by $\mathcal{O}(u)$ with $\min \{b, a+1\} \leq u \leq \max \{a, b-1\}$.
Proof. The lemma follows by applying $R \pi_{*}(-)$ to the exact sequences

$$
0 \rightarrow \mathcal{O}((i-1) \Delta) \rightarrow \mathcal{O}(i \Delta) \rightarrow \mathcal{O}_{\Delta}(-i) \rightarrow 0
$$

induction on $a$, projection formula and the fact $\pi^{*} \mathcal{O}(1)=\mathcal{O}(H-\Delta)$.
Lemma 5.10. Let $m \geq 2$. Then $\mathcal{O}_{\mathrm{Bl}_{p} \mathbb{P}^{m-1}}(a H-b \Delta)$ is acyclic if $0<-a \leq m-1$, $0 \leq-b \leq m-2$.
Proof. Indeed, $R \Gamma\left(R \pi_{*}(\mathcal{O}(a H-b \Delta))\right)=0$ by Lemma 5.9
Lemma 5.11. Let $\overline{\mathrm{M}}$ be a Hassett space such that all $\mathcal{A}$-stable curves have at most two components. Let $a,\left\{\alpha_{T}\right\}$ be integers. Consider the line bundle $L:=\omega_{\alpha}^{a}(E) \otimes$ $\mathcal{O}\left(-\sum_{T} \alpha_{T} \delta_{T \cup\{y\}, T^{c}}\right)$ on $W$ (where the sum is over partitions $N=T \sqcup T^{c}$ giving rise to boundary divisors $\delta_{T, T^{c}}$ of type I on $\left.\overline{\mathrm{M}}\right)$. The complex $R \alpha_{*}(L)$ is a vector bundle of rank $e-2 a+1$ (and $R \alpha_{*} L=\alpha_{*} L$ ) if

$$
\begin{equation*}
\left|E \cap T^{c}\right|-a \geq \alpha_{T}-\alpha_{T^{c}} \geq a-|E \cap T| \tag{5.5}
\end{equation*}
$$

Furthermore, we can weaken (5.5) by either adding 1 to $\left|E \cap T^{c}\right|-a$ or subtracting 1 from a $-|E \cap T|$ (but not both at once).

In particular, the complex $F_{l, E}$ in Def. 55.2 is a vector bundle of rank $l+1$.

Proof. Note that (5.5) implies that $e \geq 2 a-1$. If $C$ is an irreducible fiber of $\alpha$ then $\operatorname{deg}\left(L_{\mid C}\right)=\operatorname{deg}\left(\omega_{\alpha}^{a}(E)\right)_{\mid C}=e-2 a \geq-1$. Let now $C$ be a reducible fiber of $\alpha$ over a point of $\delta_{T} \subseteq \overline{\mathrm{M}}$, for a partition $T \sqcup T^{c}$ of $N$. The curve $C$ has components $C_{1}$ and $C_{2}$, with $C_{1}$ (resp., $C_{2}$ ) having fixed markings from $T$ (resp., $T^{c}$ ). The curve $C_{i} \subseteq W$ is given by the point $y$ moving along $C_{i}$. Therefore, we have $C_{1} \subseteq \delta_{T \cup\{y\}}=\delta_{T^{c}}, C_{2} \subseteq \delta_{T^{c} \cup\{y\}}=\delta_{T}, C_{1} \cdot \delta_{T}=1$, $C_{2} \cdot \delta_{T^{c}}=1$ and $C_{i}$ intersects all boundary other than $\delta_{T}, \delta_{T^{c}}$ trivially. Furthermore, one has $\delta_{T^{c}} \cdot C_{1}=-1$ from $\left(\delta_{T}+\delta_{T^{c}}\right) \cdot C_{1}=\alpha^{-1}\left(\delta_{T}\right) \cdot C_{1}=$ 0 . Similarly $\delta_{T} \cdot C_{2}=-1$. Furthermore, $\left(\omega_{\alpha}\right)_{\mid C_{i}}=\omega_{C_{i}}(1)=\mathcal{O}(-1)$ and $\operatorname{deg}\left(\mathcal{O}\left(\sigma_{j}\right)\right)_{\mid C_{1}}=1$ if $j \in T$ and 0 otherwise. It follows that
$\operatorname{deg}\left(L_{\mid C_{1}}\right)=-a+|E \cap T|+\alpha_{T}-\alpha_{T^{c}}, \operatorname{deg}\left(L_{\mid C_{2}}\right)=-a+\left|E \cap T^{c}\right|+\alpha_{T^{c}}-\alpha_{T}$.
Similarly, $\operatorname{deg}\left(L_{\mid C_{2}}\right)=-a+|E \cap T|+\alpha_{T}-\alpha_{T^{c}}$.
There is an exact sequence $0 \rightarrow L \rightarrow L_{\mid C_{1}} \oplus L_{\mid C_{2}} \rightarrow \mathcal{O}_{k(u)} \rightarrow 0$. We have $\mathrm{H}^{1}\left(L_{\mid C_{i}}\right)=0$ for $i=1,2$ since $\operatorname{deg}\left(L_{\mid C_{i}}\right) \geq-1$ for $i=1,2$. Since at least one of the inequalities is strict, the induced map $\mathrm{H}^{0}\left(L_{\mid C_{1}} \oplus L_{\mid C_{2}}\right) \rightarrow k(u)$ is surjective. Therefore, $\mathrm{H}^{1}(L)=0, h^{0}(L)=e-2 a+1$ and $R \alpha_{*} L=\alpha_{*} L$ is a vector bundle of rank $e-2 a+1$.

Assume now that $a=\frac{e-l}{2}$ and $\alpha_{T}$ is as in (4.2). If $f_{T}, f_{T^{c}} \geq 0$ (see 4.2) for the definition of $f_{T}$ ) then $\alpha_{T}=\alpha_{T^{c}}=0$ and $\operatorname{deg}\left(L_{\mid C_{1}}\right)=f_{T} \geq 0$, $\operatorname{deg}\left(L_{\mid C_{2}}\right)=f_{T^{c}} \geq 0$. If $f_{T} \geq 0, f_{T^{c}}<0$ then $\alpha_{T}=0, \alpha_{T^{c}}=-f_{T^{c}}$, $\operatorname{deg}\left(L_{\mid C_{1}}\right)=l \geq-1, \operatorname{deg}\left(L_{\mid C_{2}}\right)=0$. The case $f_{T^{c}} \geq 0, f_{T}<0$ is similar. Since $f_{T}+f_{T^{c}}=l \geq-1$, it is not possible to have $f_{T}, f_{T^{c}}<0$.

Lemma 5.12. Assume that $n$ is even. Let $\overline{\mathrm{M}}$ be a Hassett space such that all $\mathcal{A}$ stable curves have at most two components and any boundary component $\delta_{T, T^{c}}$ of type $I$ is such that $|T|=\left|T^{c}\right|=\frac{n}{2}$ (for example the spaces $\overline{\mathrm{M}}_{p, q}$ when $p$ and $q$ both even). Let $\Sigma$ be the set of all indices. When $l \geq 0$ is even, the vector bundles $F_{l, \Sigma}$ satisfy the property $F_{l, \Sigma}=F_{l, \emptyset} \otimes F_{0, \Sigma}$.
Proof. By Def. 5.2, $F_{0, \Sigma}$ is the line bundle $\alpha_{*}\left(N_{0, \Sigma}\right)$, where $N_{0, \Sigma}=\omega_{\alpha}^{n / 2}(\Sigma)$ is a line bundle on the universal family $\alpha: W \rightarrow \overline{\mathrm{M}}$, with degree 0 on the generic fiber. Furthermore, the assumption $|T|=\frac{n}{2}$ implies that $N_{0, \Sigma}$ has trivial restriction to each component of every reducible fiber of $\alpha$. Hence, $N_{0, \Sigma}=\alpha^{*}\left(F_{0, \Sigma}\right)$. By definition $F_{l, \Sigma}=\alpha_{*}\left(\omega_{\alpha}^{(n-l) / 2}(\Sigma), F_{l, \emptyset}=\alpha_{*}\left(\omega_{\alpha}^{-\frac{l}{2}}\right)\right.$ and thus $F_{l, \Sigma}=\alpha_{*}\left(\omega_{\alpha}^{-\frac{l}{2}} \otimes \alpha^{*} F_{0, \Sigma}\right) \cong R \alpha_{*}\left(\omega_{\alpha}^{-\frac{l}{2}}\right) \otimes F_{0, \Sigma} \cong F_{l, \emptyset} \otimes F_{0, \Sigma}$.
Notation 5.13. Let $N=\{1, \ldots n\}, n=2 m \geq 4$ even. Assume $\overline{\mathrm{M}}$ is a Hassett space such that all $\mathcal{A}$-stable curves have at most two components and any boundary component $\delta_{T, T^{c}} \subset \overline{\mathrm{M}}$ of type I is such that $|T|=m$. Let $\alpha$ : $W \rightarrow \overline{\mathrm{M}}$ be the universal family. Recall from Lemma 5.6 that $W$ is a Hassett space with boundary of type I of the form $\delta_{T \cup\{y\}, T^{c}} \cong \mathrm{Bl}_{p} \mathbb{P}^{m-1} \times \mathbb{P}^{m-2}$ for all partitions $T \sqcup T^{c}=N$. We let $\pi: \mathcal{U} \rightarrow W$ be the universal family, with $x$ the new marking on $\mathcal{U}$.

Lemma 5.14. There are two types of boundary divisors of type I on $\mathcal{U}$ :
(1) Let $\delta=\delta_{T \cup\{x\}, T^{c} \cup\{y\}}=\overline{\mathrm{M}}_{T \cup\{x, u\}} \times \overline{\mathrm{M}}_{T \cup\{y, u\}}$ (with $u$ the attaching point). Then $\delta \cong \mathrm{Bl}_{p} \mathbb{P}^{m-1} \times \mathrm{Bl}_{p} \mathbb{P}^{m-1}$. If on $\mathrm{Bl}_{p} \mathbb{P}^{m-1}$ we denote by $H$ the hyperplane
class and by $\Delta$ the exceptional divisor, then the restriction $\pi_{\mid \delta}$ is the pair ( $q, I d$ ), where $q: \mathrm{Bl}_{p} \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-2}, q^{*} \mathcal{O}(1)=\mathcal{O}(H-\Delta), \Delta=\delta_{u x}$. Moreover, $\omega_{\pi \mid \delta}=\mathcal{O}(-H) \boxtimes \mathcal{O}, \delta_{y x \mid \delta}=0, \delta_{j x \mid \delta}$ is $\mathcal{O}(H) \boxtimes \mathcal{O}$ if $j \in T$ and $\mathcal{O}$ otherwise.
(2) Let $\delta=\delta_{T \cup\{y, x\}, T^{c}}=\overline{\mathrm{M}}_{T \cup\{y, x, u\}} \times \overline{\mathrm{M}}_{T \cup\{u\}}$. Then $\delta \cong \mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m} \times \mathbb{P}^{m-2}$, where $\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m}$ denotes the blow-up of $\mathbb{P}^{m}$ at two distinct points $p_{1}, p_{2}$ and the proper transform of the line through them. On $\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m}$ we denote by $H$ the hyperplane class and by $E_{1}, E_{2}, E_{12}$ the corresponding exceptional divisors. We denote pullbacks of these divisors to $\delta$ by the same letters. We have: $E_{1}=\delta_{T \cup\{x\} \mid \delta}$, $E_{2}=\delta_{\left.T \cup\{y\}\right|_{\delta}}, E_{12}=\delta_{\left.T\right|_{\delta}}$. The restriction $\pi_{\mid \delta}$ is the pair ( $\left.\tilde{q}, I d\right)$, where $\tilde{q}$ : $\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m} \rightarrow \mathrm{Bl}_{p_{1}} \mathbb{P}^{m-1}, \tilde{q}^{*} \mathcal{O}(H)=\mathcal{O}\left(H-E_{2}\right), \tilde{q}^{*} \Delta=E_{1}+E_{12}$, and $\omega_{\pi \mid \delta}=$ $\mathcal{O}\left(-H+E_{1}\right) \boxtimes \mathcal{O}, \delta_{j x \mid \delta}=\mathcal{O}\left(H-E_{1}\right) \boxtimes \mathcal{O}$ for $j \in T, \delta_{j y \mid \delta}=\mathcal{O}\left(H-E_{2}\right) \boxtimes \mathcal{O}$ for $j \in T, \delta_{y x \mid \delta}=\mathcal{O}(H) \boxtimes \mathcal{O}$.
Proof. For (1), the statements about $q$ and $\delta_{j x \mid \delta}$ follow from Lemma 5.5] By adjunction and Lemma $\left.5.4, K_{W \mid \pi(\delta)}=\mathcal{O}(-m+2)\right) \boxtimes(-(m-1) H+(m-2) \Delta)$, $K_{\mathcal{U} \mid \delta}=(-(m-1) H+(m-2) \Delta) \boxtimes(-(m-1) H+(m-2) \Delta)$. Using $\omega_{\pi}=$ $K_{\mathcal{U}}-\pi^{*} K_{W}$, we have that $\omega_{\pi \mid \delta}=\mathcal{O}(-H) \boxtimes \mathcal{O}$.

We now prove (2). Denote $\overline{\mathrm{M}}:=\overline{\mathrm{M}}_{T \cup\{y, x, u\}}, \overline{\mathrm{M}}_{W}:=\overline{\mathrm{M}}_{T \cup\{y, u\}}$ and let $\overline{\mathrm{N}}^{\prime}$ (resp., $\overline{\mathrm{N}}^{\prime \prime}$ ) be the Hassett space with markings $T \cup\{y, x, u\}$ such that the points in $T$ (resp., $T, T \cup\{x\}, T \cup\{y\}$, but not $T \cup\{x, y\}$ ) are allowed to coincide. This is possible since $\overline{\mathrm{M}}_{T \cup\{u\}} \cong \mathbb{P}^{m-2}$, we may assume that the weights of the points in $T$ are $\epsilon=\frac{1}{m-1}$ and those of $y, x$ are $\eta \ll 1$. Then for $\overline{\mathrm{N}}^{\prime}$, resp. $\overline{\mathrm{N}}^{\prime \prime}$, we can lower the weights of the points in $T$ to $\epsilon^{\prime}=\frac{2-\eta}{2 m}$, resp., $\epsilon^{\prime \prime}=\frac{2-3 \eta}{2 m}$. Note that $\overline{\mathrm{N}}^{\prime \prime} \cong \mathbb{P}^{m}$, since all but one of the markings $T \cup\{y, x\}$ may coincide.

There are reduction maps $\pi^{\prime}: \overline{\mathrm{M}} \rightarrow \overline{\mathrm{N}}^{\prime}, \pi^{\prime \prime}: \overline{\mathrm{N}}^{\prime} \rightarrow \overline{\mathrm{N}}^{\prime \prime}$. The map $\pi^{\prime \prime}$ contracts $\delta_{T \cup\{x\}}$ and $\delta_{T \cup\{y\}}$ to points, which we denote $p_{1}:=\pi^{\prime \prime}\left(\delta_{T \cup\{x\}}\right)$, $p_{2}:=\pi^{\prime \prime}\left(\delta_{T \cup\{y\}}\right)$, while the composition $\pi^{\prime \prime} \circ \pi^{\prime}$ contracts $\delta_{T}$ to the line through $p_{1}$ and $p_{2}$. Hence, $\overline{\mathrm{M}}$ is isomorphic to the blow-up $\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m}$ and we have $E_{1}=\delta_{T \cup\{x\}}, E_{2}=\delta_{T \cup\{y\}}, E_{12}=\delta_{T}$.

By Lemma 5.6, we have $\overline{\mathrm{M}}_{W} \cong \mathrm{Bl}_{p} \mathbb{P}^{m-1}$. The map $\tilde{q}: \overline{\mathrm{M}} \rightarrow \overline{\mathrm{M}}_{W}$ forgets the $x$ marking; hence, $\tilde{q}$ is an isomorphism on $E_{2}=\delta_{T \cup\{y\}}$. Since the only fibrations $\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m} \rightarrow \mathrm{Bl}_{p} \mathbb{P}^{m-1}$ are given by the linear systems $\left|H-E_{1}\right|$, $\left|H-E_{2}\right|$, it follows that $\tilde{q}^{*} \mathcal{O}(H)=\mathcal{O}\left(H-E_{2}\right)$ (the projection from $p_{2}$ ). As $\Delta=\delta_{T} \subseteq \overline{\mathrm{M}}_{W}$, it follows that $\tilde{q}^{*} \Delta=\delta_{T}+\delta_{T \cup\{x\}}=E_{12}+E_{1}$. The restriction $\delta_{j x \mid \delta}$ is $\delta_{j x} \boxtimes \mathcal{O}$ if $j \in T$ and trivial otherwise. If $j \in T$, the pull-backs of $\delta_{j x}$ via the reduction maps $\pi^{\prime}, \pi^{\prime \prime}$ are given by $\pi^{\prime \prime *} \delta_{j x}=\delta_{j x}+$ $\delta_{T \cup\{x\}},\left(\pi^{\prime} \circ \pi^{\prime \prime}\right)^{*} \delta_{j x}=\delta_{j x}+\delta_{T \cup\{x\}}$. As $\delta_{j x}=\mathcal{O}(1)$ on $\overline{\mathrm{N}}^{\prime \prime}=\mathbb{P}^{m}$ (Note 5.1(3)), it follows that we have $\delta_{j x}=\mathcal{O}\left(H-E_{1}\right)$ in $\operatorname{Pic}(\overline{\mathrm{M}})$. By symmetry, when $j \in T$ we also have $\delta_{j y}=\mathcal{O}\left(H-E_{2}\right)$ in $\operatorname{Pic}(\overline{\mathrm{M}})$. Similarly, since $\left(\pi^{\prime} \circ \pi^{\prime \prime}\right)^{*} \delta_{y x}=\delta_{y x}$ and $\delta_{y x}=\mathcal{O}(1)$ on $\overline{\mathrm{N}}^{\prime \prime}=\mathbb{P}^{m}$, we have $\delta_{y x \mid \delta}=\mathcal{O}(H) \boxtimes$ $\mathcal{O}$. By adjunction, $K_{W \mid \pi(\delta)}=(-(m-1) H+(m-2) \Delta) \boxtimes \mathcal{O}(-(m-2))$, $K_{\mathcal{U} \mid \delta}=\left(-m H+(m-1)\left(E_{1}+E_{2}\right)+(m-2) E_{12}\right) \boxtimes \mathcal{O}(-(m-2))$. Using $\omega_{\pi}=K_{\mathcal{U}}-\pi^{*} K_{W}$, it follows that $\omega_{\pi \mid \delta}=\mathcal{O}\left(-H+E_{1}\right) \boxtimes \mathcal{O}$.

Remark 5.15. In the notations of Lemma 5.14, the same proof as in Lemma 5.9 shows that if $a \geq 0$ we have

$$
R \tilde{q}_{*}\left(\mathcal{O}\left(a E_{2}\right)\right)=\mathcal{O}(-a H) \oplus \ldots \oplus \mathcal{O}(-H) \oplus \mathcal{O}
$$

Lemma 5.16. The complex $F_{l, E}$ from Def. 5.3 is a vector bundle of rank $l+1$.
Proof. Denote $N:=N_{l, E}$ for simplicity. If $C$ is an irreducible fiber of $\pi$, we have $\operatorname{deg}\left(N_{\mid C}\right)=l$. Consider now a reducible fiber $C$ with two components $C_{1}$ and $C_{2}$, with markings from $T \cup\{y\}$ (resp., $T^{c}$ ) on $C_{1}$ (resp. $C_{2}$ ). Recall that $E \subseteq N \cup\{y\}$ and $T^{c}=N \backslash T$. Since $f_{T}+f_{T^{c}}+|E \cap\{y\}|=l \geq 0$, it is not possible to have $f_{T}, f_{T^{c}}<0$. We have $C_{1} \cdot \delta_{T \cup\{y, x\}}=-1, C_{1} \cdot \delta_{T \cup\{y\}}=$ $1, C_{2} \cdot \delta_{T \cup\{y, x\}}=1, C_{2} \cdot \delta_{T \cup\{y\}}=-1$, while all other intersections with boundary are 0 . We have:

$$
\begin{gathered}
\operatorname{deg}\left(N_{\mid C_{1}}\right)=|E \cap\{y\}|+f_{T}+\alpha_{T \cup\{y\}}-\alpha_{T^{c}}, \\
\operatorname{deg}\left(N_{\mid C_{2}}\right)=f_{T^{c}}-\alpha_{T \cup\{y\}}+\alpha_{T^{c}} .
\end{gathered}
$$

If $f_{T}, f_{T^{c}} \geq 0$, then $\alpha_{T \cup\{y\}}=\alpha_{T^{c}}=0$ and $\operatorname{deg}\left(N_{\mid C_{1}}\right) \geq 0, \operatorname{deg}\left(N_{\mid C_{2}}\right) \geq 0$. If $f_{T} \geq 0, f_{T^{c}}<0$, then $\alpha_{T \cup\{y\}}=0, \alpha_{T^{c}}=-f_{T^{c}}, \operatorname{deg}\left(N_{\mid C_{1}}\right)=l \geq 0$, $\operatorname{deg}\left(N_{\mid C_{2}}\right)=0$. If $f_{T}<0, f_{T^{c}} \geq 0$, then $\alpha_{T \cup\{y\}}=-f_{T}-|E \cap\{y\}|, \alpha_{T^{c}}=0$, $\operatorname{deg}\left(N_{\mid C_{1}}\right)=0, \operatorname{deg}\left(N_{\mid C_{2}}\right)=l$. In all cases, $\mathrm{h}^{1}\left(N_{\mid C}\right)=0, \mathrm{~h}^{0}\left(N_{\mid C}\right)=l+1$.

Consider now a reducible fiber $C$ with 3 components $C_{1}, C_{2}, C_{3}$, with markings from $T,\{y\}, T^{c}$ respectively. Then $C_{1} \cdot \delta_{T \cup\{x\}}=-1, C_{1} \cdot \delta_{T c \cup\{y, x\}}=$ $1, C_{2} \cdot \delta_{T \cup\{y, x\}}=C_{2} \cdot \delta_{T^{c} \cup\{y, x\}}=-1, C_{2} \cdot \delta_{T \cup\{x\}}=C_{2} \cdot \delta_{T^{c} \cup\{x\}}=1$, $C_{3} \cdot \delta_{T^{c} \cup\{x\}}=-1, C_{3} \cdot \delta_{T \cup\{y, x\}}=1$, while intersections with other boundary are 0. We have: $\operatorname{deg}\left(N_{\mid C_{1}}\right)=f_{T}+\alpha_{T}-\alpha_{T^{c} \cup\{y\}}, \operatorname{deg}\left(N_{\mid C_{2}}\right)=\alpha_{T \cup\{y\}}+$ $\alpha_{T^{c} \cup\{y\}}-\alpha_{T}-\alpha_{T^{c}}+|E \cap\{y\}|, \operatorname{deg}\left(N_{\mid C_{3}}\right)=f_{T^{c}}+\alpha_{T^{c}}-\alpha_{T \cup\{y\}}$.

If $f_{T}, f_{T^{c}} \geq 0$, then $\alpha_{T}=\alpha_{T \cup\{y\}}=\alpha_{T^{c}}=\alpha_{T^{c} \cup\{y\}}=0, \operatorname{deg}\left(N_{\mid C_{1}}\right) \geq 0$, $\operatorname{deg}\left(N_{\mid C_{3}}\right) \geq 0, \operatorname{deg}\left(N_{\mid C_{2}}\right) \geq 0$. If $f_{T} \geq 0, f_{T^{c}}<0$, then $\alpha_{T}=\alpha_{T \cup\{y\}}=0$, $\alpha_{T^{c}}=-f_{T^{c}}, \alpha_{T^{c} \cup\{y\}}=\alpha_{T^{c}}-|E \cap\{y\}|, \operatorname{deg}\left(N_{\mid C_{1}}\right)=l \geq 0, \operatorname{deg}\left(N_{\mid C_{3}}\right)=0$, $\operatorname{deg}\left(N_{\mid C_{2}}\right)=0$. If $f_{T}<0, f_{T^{c}} \geq 0$ then $\alpha_{T^{c}}=\alpha_{T^{c} \cup\{y\}}=0, \alpha_{T}=-f_{T}$, $\alpha_{T \cup\{y\}}=\alpha_{T}-|E \cap\{y\}|, \operatorname{deg}\left(N_{\mid C_{1}}\right)=0, \operatorname{deg}\left(N_{\mid C_{3}}\right)=l, \operatorname{deg}\left(N_{\mid C_{2}}\right)=$ 0 . Hence, in all cases $\mathrm{h}^{1}\left(N_{\mid C}\right)=0, \mathrm{~h}^{0}\left(N_{\mid C}\right)=l+1$ and we are done by cohomology and base change

Lemma 5.17. Consider the set-up of Notation 5.13] Let $l \geq 0, E \subseteq N \cup\{y\}$, with $e+l$ even. (i) If $y \notin E$, then $F_{l, E}=\alpha^{*} F_{l, E}$ on $W$. (ii) If $y \in E$, then there is an exact sequence

$$
\begin{equation*}
0 \rightarrow F_{l-1, E \backslash\{y\}} \rightarrow F_{l, E} \rightarrow Q_{l, E}^{y} \rightarrow 0, \tag{5.6}
\end{equation*}
$$

of vector bundles on $W$, with $Q_{l, E}^{y}:=\sigma_{y}^{*} N_{l, E}$, where $\sigma_{y}$ is the section of $\pi: \mathcal{U} \rightarrow$ $W$ corresponding to the $y$ marking. Furthermore, $R \alpha_{*}\left(F_{l, E}^{\vee}\right)=0$ in this case.
Proof of Lemma 5.17 There is a commutative diagram

where $\alpha: W \rightarrow \overline{\mathrm{M}}$ is the universal family and the notation $\alpha=\alpha_{x}$ indicates the marking that is forgotten. The right square is Cartesian. Let $g=\phi \circ v$. The map $v$ is small and contracts the codimension 2 loci

$$
\pi^{-1}\left(\delta_{T} \cap \delta_{T^{c}}\right)=\mathbb{P}^{m-2} \times \mathbb{P}^{1} \times \mathbb{P}^{m-2} \rightarrow \mathbb{P}^{m-2} \times p t \times \mathbb{P}^{m-2}
$$

Claim 5.18. We have (i) $R v_{*} v^{*} \mathcal{O}_{\mathcal{V}} \cong R v_{*} \mathcal{O}_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{V}}$. (ii) In $\operatorname{Pic}(\mathcal{U})$ we have $v^{*} \omega_{\rho}=\omega_{\pi}$, while on $W$ we have $\psi_{y}=\omega_{\alpha_{y}}$. (iii) $g^{*} \delta_{T \cup\{x\}}=\delta_{T \cup\{x\}}+\delta_{T \cup\{y, x\}}$.
Proof. The map $v$ is birational and its image has rational singularities, which are in fact locally isomorphic to the product of the affine cone $x y=z t$ and a smooth variety (see e.g. [Kee92, page 548]). Part (i) follows. Part (iii) is immediate since $g$ is the map that forgets the marking $y$.

By definition $\psi_{y}=\sigma_{y}^{*} \omega_{\pi}$. Since $v^{*} \omega_{\rho}=\omega_{\pi}$, it follows that if $s_{y}=v \circ \sigma_{y}$, then $\psi_{y}=s_{y}^{*} \omega_{\rho}=s_{y}^{*} \phi^{*} \omega_{\alpha}=\omega_{\alpha}$, since $\phi \circ s_{y}=I d_{W}$. This proves (ii).

We have $F_{l, E}=R \pi_{*} L_{1}$, where $L_{1}:=N_{l, E}$, which is equal to

$$
\omega_{\pi^{\frac{e-l}{2}}}\left(E-\sum_{f_{T, E, l}<0}\left(-f_{T, E, l}\right) \delta_{T \cup\{x\}}+\left(-f_{T, E, l}-|E \cap\{y\}|\right) \delta_{T \cup\{y, x\}}\right) .
$$

where the sum is over all $T \subset N$ such that $\delta_{T, T^{c}} \subseteq \overline{\mathrm{M}}$ is a boundary component ( $T^{c}:=N \backslash T$ ). We now compute $\alpha^{*} F_{l^{\prime}, E^{\prime}}$, for $y \notin E^{\prime}$. Using (i), we have $\alpha^{*} F_{l^{\prime}, E^{\prime}}=\alpha_{y}^{*} R \alpha_{x *} N_{l^{\prime}, E^{\prime}}=R \rho_{*} \phi^{*} N_{l^{\prime}, E^{\prime}}=R \pi_{*} L_{2}$, where $L_{2}:=v^{*} \phi^{*} N_{l^{\prime}, E^{\prime}}=g^{*} N_{l^{\prime}, E^{\prime}}$, with $N_{l^{\prime}, E^{\prime}}$ is the usual line bundle on $W$ :

$$
\left.N_{l^{\prime}, E^{\prime}}=\omega_{\pi}^{\frac{e^{\prime}-l^{\prime}}{2}}\left(E^{\prime}-\sum_{f_{T, E^{\prime}, l^{\prime}}<0}\left(-f_{T, E^{\prime}, l^{\prime}}\right)\right) \delta_{T \cup\{x\}}\right) .
$$

Since $v$ has no exceptional divisors, it follows that

$$
L_{2}=\omega_{\pi}^{\frac{e^{\prime}-l^{\prime}}{2}}\left(E^{\prime}-\sum_{f_{T, E^{\prime}, l^{\prime}}<0}\left(-f_{T, E^{\prime}, l^{\prime}}\right)\left(\delta_{T \cup\{x\}}+\delta_{T \cup\{y, x\}}\right)\right) .
$$

Case $y \notin E, l^{\prime}=l, E^{\prime}=E$. We clearly have $L_{1}=L_{2}$, and this proves that when $y \notin E$, we have $F_{l, E}=\alpha^{*} F_{l, E}$.

Case $y \in E, l^{\prime}=l-1, E^{\prime}=E \backslash\{y\}$. As $\frac{e^{\prime}-l^{\prime}}{2}=\frac{e-l}{2},|E \cap\{y\}|=1$, we have $L_{1}=L_{2}\left(\delta_{y x}+\sum_{f_{T, E, l}<0} \delta_{T \cup\{y, x\}}\right)$, which implies exact sequences

$$
0 \rightarrow L_{2} \rightarrow L_{1}\left(-\delta_{y x}\right) \rightarrow \bigoplus_{f_{T, E, l}<0}\left(L_{1}\left(-\delta_{y x}\right)_{\mid \delta_{T \cup\{y, x\}}} \rightarrow 0\right.
$$

and $0 \rightarrow L_{1}\left(-\delta_{y x}\right) \rightarrow L_{1} \rightarrow\left(L_{1}\right)_{\mid \delta_{y x}} \rightarrow 0$. Assume $T$ is such that $f_{T, E, l}<0$. Then by Lemma5.14]we have $\left(L_{1}\left(-\delta_{y x}\right)\right)_{\mid \delta_{T \cup\{y, x\}}}=\mathcal{O}(-H) \boxtimes \mathcal{O}\left(-f_{T, E, l}-1\right)$, on $\delta_{T \cup\{y, x\}}=\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m} \times \mathbb{P}^{m-2}$. Since on $\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{m}$ we have $R \tilde{q}_{*} \mathcal{O}\left(-E_{2}\right)=$ 0 and $\tilde{q}^{*} \mathcal{O}(H)=\mathcal{O}\left(H-E_{2}\right)$ (Lemma 5.14), it follows that $R \tilde{q}_{*} \mathcal{O}(-H)=0$. Hence, $R \pi_{*}\left(L_{1}\left(-\delta_{y x}\right)\right)_{\mid \delta_{T \cup\{y, x\}}}=0, R \pi_{*} L_{2} \cong R \pi_{*} L_{1}\left(-\delta_{y x}\right)=F_{l-1, E \backslash\{y\}}$ (and here $\left.R \pi_{*}(-)=\pi_{*}(-)\right)$. Applying $R \pi_{*}(-)$ to the above two exact sequences, it follows that there is an exact sequence (5.6) with $Q_{l, E}^{y}=\sigma_{y}^{*} N_{l, E}$. Finally, we have $R \alpha_{*}\left(F_{l, E}^{\vee}\right)=R \alpha_{y_{*}} \circ R \pi_{*}\left(\omega_{\pi} \otimes L_{1}^{\vee}\right)=R \alpha_{x *} \circ R g_{*}\left(\omega_{\pi} \otimes L_{1}^{\vee}\right)$. Hence, it suffices to prove that $R g_{*}\left(\omega_{\pi} \otimes L_{1}^{\vee}\right)=0$. Since $\omega_{\pi}=g^{*} \omega_{\alpha}, L_{2}$ is a
pull-back by $g$, and $L_{1}=L_{2}\left(\delta_{y x}+D\right)$, where $\Sigma=\sum_{f_{T, E, l}<0} \delta_{T \cup\{y, x\}}$, it suffices to prove that $R g_{*} \mathcal{O}\left(-\delta_{y x}-D\right)=0$. Consider the exact sequence

$$
0 \rightarrow \mathcal{O}\left(-\delta_{y x}-D\right) \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O}(-D)_{\mid \delta_{y x}} \rightarrow 0
$$

It suffices to prove that $g_{*}(-)$ induces an isomorphism when applied to the restriction map $\mathcal{O}(-D) \rightarrow \mathcal{O}(-D)_{\mid \delta_{y x}}$ and all higher push forwards by $g$ of $\mathcal{O}(-D)$ and $\mathcal{O}(-D)_{\mid \delta_{y x}}$ are 0 . Since $\delta_{y x}=\sigma_{y}$ and $g \circ \sigma_{y}=$ Id, we have $R^{i} g_{*} \mathcal{O}(-D)_{\mid \delta_{y}}=0$ for all $i>0$ and $g_{*} \mathcal{O}(-D)_{\mid \delta_{y x}}=\mathcal{O}_{W}\left(-D^{\prime}\right)$, where $D^{\prime}=g(D)=\sum_{f_{T, E, l}<0} \delta_{T \cup\{x\}}$. Because of the condition $f_{T, E, l}<0$, the divisors in $D$ are disjoint. Similarly, the divisors in $D^{\prime}$ are disjoint. Using the exact sequence $0 \rightarrow \mathcal{O}\left(-\delta_{T \cup\{y, x\}}\right) \rightarrow \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{O}_{\delta_{T \cup\{y, x\}}} \rightarrow 0$, it suffices to prove that $R g_{*} \mathcal{O}_{\mathcal{U}}=\mathcal{O}_{W}$ and $R g_{*} \mathcal{O}_{\delta_{T \cup\{y, x\}}} \cong \mathcal{O}_{\delta_{T \cup\{x\}}}$. Since $g=\phi \circ v$, the first statement follows from $R v_{*} \mathcal{O}_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{V}}$ (part (i) of Claim 5.18) and $R \phi_{*} \mathcal{O}_{\mathcal{V}} \cong \mathcal{O}_{W}$, as $\phi$ is the pull-back of $\alpha_{y}: W \rightarrow \overline{\mathrm{M}}$, the universal family over $\overline{\mathrm{M}}$. The second statement follows as the map $g$ restricted to $\delta_{T \cup\{y, x\}}$ is the map $\pi_{\mid \delta}=(\tilde{q}$, Id $)$ from Lemma 5.14, where $\tilde{q}: \overline{\mathrm{M}}_{T \cup\{y, x, u\}}=\mathrm{Bl}_{1,2, \overline{12}} \rightarrow$ $\overline{\mathrm{M}}_{T \cup\{y, u\}}=\mathrm{Bl}_{p} \mathbb{P}^{m-1}$ is the universal family, in this case a flat family of rational, at worst nodal curves over a smooth base.

## 6. Perpendicularity of $F_{l, E}$ 's Versus the sheaves $\mathcal{O}_{\delta}(-a,-b)$

Let $p$ even, $q$ odd, $q+1 \geq 0$. Recall that in Section 5 we defined vector bundles $F_{l, E}$ on the Hassett spaces $\overline{\mathrm{M}}_{p, q+1}$ and on their universal families $W$, generalizing the definition of the vector bundles $F_{l, E}$ on the stacks $\mathcal{P}_{n}$ (defined in Section 3). In this section we verify that the bundles $F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+1}$ in Theorems 1.8, 1.9, Remark 1.11 are perpendicular to the torsion sheaves in the subcategory $\mathcal{A}$ (Corollary 6.2 ). We will prove a more general statement in Proposition 6.1, as this will be needed later. For the exceptionality part in Theorems 1.8, 1.9, Remark 1.11, we will compare RHom's between objects on $\overline{\mathrm{M}}_{p, q+1}$ to RHom's between similar objects on other Hassett spaces (e.g., $W$ ). We will need a general statement about perpendicularity of the bundles $F_{l, E}$ on $W$ to certain torsion sheaves (Proposition6.4).
Proposition 6.1. Assume $P$ (resp., $Q$ ) is the set of heavy (resp., light) indices and $|P|=p=2 r \geq 4,|Q|=q+1=2 s+2 \geq 0$ (so $s$ can be -1 ). Assume $l \geq 0, E \subseteq P \cup Q$ with $e=|E|$ such that $e+l$ is even. Let $\delta_{T}=\delta_{T, T^{c}}=$ $\mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-1} \subseteq \overline{\mathrm{M}}_{p, q+1}$ be a boundary divisor such that

$$
\begin{equation*}
\left|f_{T, E, l}\right|,\left|f_{T^{c}, E, l}\right| \leq \mu \leq(r+s) / 2 \tag{6.1}
\end{equation*}
$$

Then $\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l, E}^{\vee}, \mathcal{O}_{\delta_{T}}(-a,-b)\right)=0$, where by $\mathcal{O}_{\delta_{T}}(-a,-b)$ we denote $\mathcal{O}_{T}(-a) \boxtimes \mathcal{O}_{T^{c}}(-b)$, i.e., $-a$ is on the $T$-component) if $1 \leq a, b \leq r+s-1$, or $\mu<b<r+s, a=r+s$ (hence, any $a$ ) or $\mu<a<r+s, b=r+s$ (hence, any b). Similarly, $\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l, E}, \mathcal{O}_{\delta_{T}}(-a,-b)\right)=0$ if $1 \leq a, b \leq r+s-1$, or $a=0,0<b<r+s-\mu$, or $b=0,0<a<r+s-\mu$.

Corollary 6.2. Assume the pair $(l, E)$ is in group $1 A$ or $1 B$ on $\overline{\mathrm{M}}_{p, q+1}$. Let $\mathcal{O}_{\delta}(-a,-b)$ be one of the generators of the category $\mathcal{A}$ (Notation 1.6). Then $\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l, E}, \mathcal{O}_{\delta}(-a,-b)\right)=0$.

Proof. We apply Proposition 6.1 with $\mu=\frac{r+s}{2}$. By Lemma 4.5 we have that $\left|f_{T, E, l}\right|,\left|f_{T^{c}, E, l}\right| \leq(r+s) / 2$ and the statement follows from Proposition6.1,

Proof of Proposition 6.1. By Serre duality, if $E$ is a vector bundle on $\overline{\mathrm{M}}_{p, q+1}$, then

$$
\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q+1}}\left(E^{\vee}, \mathcal{O}_{\delta_{T}}(-a,-b)\right)=\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q+1}}\left(E, K_{\delta_{T}} \otimes \mathcal{O}_{\delta_{T}}(a, b)\right)^{\vee}
$$

So the second statement is equivalent to the first. We now prove the first statement. If $\alpha: W \rightarrow \overline{\mathrm{M}}_{p, q+1}$ is the universal family, let $\beta:=\alpha_{\mid \alpha^{-1}\left(\delta_{T}\right)}$ : $\alpha^{-1}\left(\delta_{T}\right)=\delta_{T} \cup \delta_{T^{c}} \rightarrow \delta_{T}$, where we denote $\delta_{T}:=\delta_{T, T^{c} \cup\{y\}}$ ( $y$ the new marking on $W$ ). Then $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l, E}^{\vee}, \mathcal{O}_{\delta_{T}}(-a,-b)\right)=R \Gamma_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l, E \mid \delta_{T}} \otimes\right.$ $\left.\mathcal{O}_{\delta_{T}}(-a,-b)\right)$. Since $F_{l, E}=R \alpha_{*}\left(N_{l, E}\right)$ (see Definition5.2), it follows by cohomology and base change that $F_{l, E \mid \delta_{T}}=\left(R \alpha_{*}\left(N_{l, E}\right)\right)_{\delta_{T}}=R \beta_{*}\left(N_{l, E \mid \delta_{T} \cup \delta_{T^{c}}}\right)$, and by projection formula
$R \Gamma_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l, E \mid \delta_{T}} \otimes \mathcal{O}_{\delta_{T}}(-a,-b)\right)=R \Gamma_{\overline{\mathrm{M}}_{p, q+1}}\left(R \beta_{*}\left(N_{l, E_{\mid \delta_{T} \cup \delta_{T c}}} \otimes \beta^{*} \mathcal{O}(-a,-b)\right)\right)$.
We have to show that the line bundle $\left.\tilde{N}:=\left(N_{l, E}{\mid \delta_{T} \cup \delta_{T^{c}}} \otimes \beta^{*} \mathcal{O}(-a,-b)\right)\right)$ on $\delta_{T} \cup \delta_{T^{c}}$ has no cohomology. Consider the following exact sequence: $0 \rightarrow \mathcal{O}_{\delta_{T^{c}}}\left(-\delta_{T}\right) \rightarrow \mathcal{O}_{\delta_{T} \cup \delta_{T^{c}}} \rightarrow \mathcal{O}_{\delta_{T}} \rightarrow 0$. Tensoring with $\tilde{N}$ gives an exact sequence: $0 \rightarrow \tilde{N}^{\prime \prime} \rightarrow \tilde{N} \rightarrow \tilde{N}^{\prime} \rightarrow 0$, where $\tilde{N}^{\prime}=\left(N_{l, E} \otimes \beta^{*} \mathcal{O}(-a,-b)\right)_{\left.\right|_{\delta^{\prime}}}$ $\tilde{N}^{\prime \prime}=\left(N_{l, E} \otimes \beta^{*} \mathcal{O}(-a,-b) \otimes \mathcal{O}\left(-\delta_{T}\right)\right)_{\mid \delta_{T c}}, N_{l, E}=\omega_{\pi^{\frac{e-l}{2}}}(E)\left(-\sum \alpha_{T, E, l} \delta_{T \cup\{y\}, T^{c}}\right)$ (see (4.2) for the definition if $\alpha_{T, E, l}$ ). We use Lemma 5.8 to compute $\tilde{N}^{\prime}$ and $\tilde{N}^{\prime \prime}$. As usual, for simplicity, denote $\alpha_{T}:=\alpha_{T, E, l}, f_{T}:=f_{T, E, l}$. Using the identification $\delta_{T}=\delta_{T, T^{c} \cup\{y\}}=\mathbb{P}^{r+s-1} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s}$, we have

$$
\tilde{N}^{\prime}:=\mathcal{O}\left(-a+\alpha_{T^{c}}\right) \boxtimes\left(\left(f_{T^{c}}+\alpha_{T^{c}}-b\right) H+\left(b-\alpha_{T}\right) \Delta\right) .
$$

Using the identification $\delta_{T^{c}}=\delta_{T^{c}, T \cup\{y\}}=\mathbb{P}^{r+s-1} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s}$, we have

$$
\tilde{N}^{\prime \prime}:=\mathcal{O}\left(-b+\alpha_{T}\right) \boxtimes\left(\left(f_{T}+\alpha_{T}-a\right) H+\left(a-\alpha_{T^{c}}-1\right) \Delta\right) .
$$

Here $H$, resp. $\Delta$, denotes $\mathcal{O}_{\mathbb{P}^{r+s}}(1)$, resp., the exceptional divisor on $\mathrm{Bl}_{p} \mathbb{P}^{r+s}$. We prove that both $\tilde{N}^{\prime}, \tilde{N}^{\prime \prime}$ are acyclic. Recall that, for all $T$, either $f_{T} \geq 0$, $\alpha_{T}=0$ or $f_{T}<0, f_{T}+\alpha_{T}=0$ and $f_{T}+f_{T^{c}}=l \geq 0$.

Case $f_{T}, f_{T^{c}} \geq 0$ : then $\alpha_{T}=\alpha_{T^{c}}=0$ and $\tilde{N}^{\prime}=\mathcal{O}(-a) \boxtimes\left(\left(f_{T^{c}}-b\right) H+b \Delta\right)$, $\tilde{N}^{\prime \prime}=\mathcal{O}(-b) \boxtimes\left(\left(f_{T}-a\right) H+(a-1) \Delta\right)$. Clearly, if $0<a \leq(r+s-1)$, $0<b \leq(r+s-1)$ then $\mathcal{O}(-a), \mathcal{O}(-b)$ are acyclic, hence so are $\tilde{N}^{\prime}, \tilde{N}^{\prime \prime}$. Assume that $a=r+s, \mu<b \leq r+s-1$. Then $\mathcal{O}(-b)$ and therefore $\tilde{N}^{\prime \prime}$ is acyclic. By (6.1), we have $-(r+s-1) \leq-b \leq f_{T^{c}}-b \leq \mu-b<0$. It follows that $\tilde{N}^{\prime}$ is acyclic by Lemma 5.10. Similarly, if $b=r+s, \mu<a \leq r+s-1$, then $\mathcal{O}(-a)$ and $\left(f_{T}-a\right) H+(a-1) \Delta$ are acyclic, as we have $-(r+s-1) \leq$ $-a \leq f_{T}-a \leq \mu-a<0$.

Case $f_{T} \geq 0, f_{T}^{c}<0$ : then $\alpha_{T}=f_{T^{c}}+\alpha_{T^{c}}=0$ and $\tilde{N}^{\prime}=\mathcal{O}\left(-a-f_{T^{c}}\right) \boxtimes(-$ $b H+b \Delta), \tilde{N}^{\prime \prime}=\mathcal{O}(-b) \boxtimes\left(\left(f_{T}-a\right) H+\left(a+f_{T^{c}}-1\right) \Delta\right)$. If $0<b \leq r+s-1$ then $\mathcal{O}(-b),-b H+b \Delta$ are acyclic, and the claim follows. If $b=r+s$, $\mu<a \leq r+s-1$ then by (6.1) $0 \leq \mu+f_{T^{c}}<a+f_{T^{c}}<a \leq r+s-1$,
$-(r+s-1) \leq-a \leq f_{T}-a \leq \mu-a<0$, so $\mathcal{O}\left(-a-f_{T^{c}}\right)$ and $\left(f_{T}-a\right) H+$ $\left(a+f_{T^{c}}-1\right) \Delta$ are acyclic.

Case $f_{T^{c}} \geq 0, f_{T}<0$ : then $\alpha_{T^{c}}=\alpha_{T}+f_{T}=0$ and $\tilde{N}^{\prime}=\mathcal{O}(-a) \boxtimes\left(\left(f_{T^{c}}-\right.\right.$ b) $\left.H+\left(b+f_{T}\right) \Delta\right), \tilde{N}^{\prime \prime}=\mathcal{O}\left(-b-f_{T}\right) \boxtimes(-a H+(a-1) \Delta)$. If $0<a \leq r+s-1$ then $\mathcal{O}(-a),-a H+(a-1) \Delta$ are acyclic, and the result follows. Assume now $a=r+s, \mu<b \leq r+s-1$. Note that $-a H+(a-1) \Delta$ is still acyclic. Furthermore, $\left(f_{T^{c}}-b\right) H+\left(b+f_{T}\right) \Delta$ is acyclic, as by (6.1) $0 \leq \mu+f_{T}<$ $b+f_{T}<b \leq r+s-1$ and $-(r+s-1) \leq-b \leq f_{T^{c}}-b \leq \mu-b<0$.
Corollary 6.3. The line bundle $F_{0, \Sigma}$ on $\overline{\mathrm{M}}_{p}$ for $p \geq 6$ even is the pull-back of the GIT polarization via the morphism $\phi: \overline{\mathrm{M}}_{p} \rightarrow \mathrm{X}_{p}$. When $p=4, F_{0, \Sigma} \simeq \mathcal{O}_{\mathbb{P}^{1}}(1)$.
Proof. The line bundle $\mathcal{O}(1, \ldots, 1)$ on $\left(\mathbb{P}^{1}\right)^{p}$ descends to $\mathrm{X}_{p}$ by the Kempf descent criterion giving a polarization $L$ of $\mathrm{X}_{p}$. Note that, away from the singularities, this agrees with our definition of $F_{0, \Sigma}$ as $R \alpha_{*} N_{0, \Sigma}$, where $N_{0, \Sigma}=\omega_{\alpha}^{2}(\Sigma)$ and $\alpha: W \rightarrow \overline{\mathrm{M}}_{p}$ is the universal family. It follows that $F_{0, \Sigma} \simeq \phi^{*} L\left(\sum a_{T, T^{c}} \delta_{T, T^{c}}\right)$, for some integers $a_{T, T^{c}}$. When $p=2 r \geq 6$, it remains to show that $a_{T, T^{c}}=0$ for every partition $P=T \sqcup T^{c},|T|=\left|T^{c}\right|=r$. For every $a=1, \ldots r-2$, we have $0=\operatorname{RHom}_{\bar{M}_{p}}\left(F_{l, E}, \mathcal{O}_{\delta_{T, T_{c}}}(-a,-a)\right)=$ $\oplus_{T, T^{c}} \mathrm{R} \Gamma\left(\mathbb{P}^{r-2} \times \mathbb{P}^{r-2}, \mathcal{O}_{\delta_{T}}\left(a_{T, T^{c}}-a, a_{T, T^{c}}-a\right)\right)$ by Prop. 6.1. It follows that all $a_{T, T^{c}}=0$ and hence, $F_{0, \Sigma} \simeq \phi^{*} L$. To see the last statement, recall from the proof of Lemma 5.12, that $N_{0, \Sigma}=\alpha^{*} F_{0, \Sigma}$. It follows that $F_{0, \Sigma}=\sigma_{1}^{*} \alpha^{*} F_{0, \Sigma}=\sigma_{1}^{*} N_{0, \Sigma}=\psi_{1}+\sum_{i=2} \sigma_{1}^{*} \sigma_{i}$. If $p=4$, we have that $\sigma_{1}^{*} \sigma_{i}=0$ if $i \neq 1$. Furthermore, $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{4}$ are isomorphic, with the same universal family, hence, $\psi_{1}=\mathcal{O}_{\mathbb{P}^{1}}(1)$. It follows that $F_{0, \Sigma}=\mathcal{O}_{\mathbb{P}^{1}}(1)$.
Proposition 6.4. Assume $|P|=p=2 r \geq 4,|Q|=q=2 s+1 \geq 1$. Let $y \in Q$ and consider the universal family $\alpha: W \rightarrow \overline{\mathrm{M}}_{p, q-1}=\overline{\mathrm{M}}_{P \cup Q \backslash\{y\}}$. Let $E \subseteq P \cup Q$ with $e=|E|, l \geq 0$ such that $e+l$ is even. Let $\delta_{T \cup\{y\}, T^{c}} \subseteq W$ be a boundary divisor $\left(T \sqcup T^{c}=P \cup(Q \backslash\{y\})\right)$ with the property that, for some $\mu$ and $\mu^{\prime}$,

$$
\begin{gather*}
m_{T^{c}}=m_{T^{c}, E, l} \leq \mu^{\prime} \leq \frac{r+s}{2}  \tag{6.2}\\
m_{T}=m_{T, E, l} \leq \mu-|E \cap\{y\}| \leq \frac{r+s}{2}-|E \cap\{y\}|, \tag{6.3}
\end{gather*}
$$

see (4.3) for $m_{T}$. Using an identification $\delta_{T \cup\{y\}}=\mathrm{Bl}_{p} \mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-2}$, we have $\operatorname{RHom}_{W}\left(F_{l, E},(-a H) \boxtimes \mathcal{O}(-b)\right)=0$ if $1 \leq a \leq r+s-1,1 \leq b \leq r+s-2$ or $a=0,0<b<r+s-1-\mu^{\prime}$ or $b=0,0<a<r+s-\mu$.
Proof. Note that when $f_{T^{c}, E, l} \leq 0$, then $\alpha_{T^{c}}=-f_{T^{c}, E, l}=|E \cap\{y\}|+f_{T, E, l}-l$ and so an immediate consequence of (6.3) is that

$$
\begin{equation*}
\alpha_{T^{c}} \leq \mu-l \tag{6.4}
\end{equation*}
$$

Similarly, because of (6.2), when $f_{T, E, l} \leq 0$ then we have

$$
\begin{equation*}
\alpha_{T}=-f_{T, E, l} \leq \mu^{\prime}+|E \cap\{y\}|-l . \tag{6.5}
\end{equation*}
$$

The proof is similar to that of Lemma6.1. Let $\pi: \mathcal{U} \rightarrow W$ be the universal family (with $x$ the new index on $\mathcal{U}$ ). Denote for simplicity $\delta:=\delta_{T \cup\{y\}, T^{c}} \subseteq$ $W, \delta_{1}=\delta_{T \cup\{y, x\}, T^{c}} \subseteq \mathcal{U}, \delta_{2}=\delta_{T \cup\{y\}, T^{c} \cup\{x\}} \subseteq \mathcal{U}, \beta:=\pi_{\mid \pi^{-1}(\delta)}: \pi^{-1}(\delta)=$ $\delta_{1} \cup \delta_{2} \rightarrow \delta, N:=N_{l, E}, F:=F_{l, E}=R \pi_{*} N_{l, E}$. Using Grothendieck-Verdier duality (Remark 3.10), it suffices to prove that letting $j: \pi^{-1}(\delta) \hookrightarrow \mathcal{U}$ be
the inclusion map, then $\tilde{N}:=j^{*}\left(N^{\vee} \otimes \omega_{\pi}\right) \otimes \beta^{*}(-a H \boxtimes \mathcal{O}(-b))$ (as a line bundle on $\pi^{-1}(\delta)$ ) is acyclic. Consider the exact sequence: $0 \rightarrow \mathcal{O}_{\delta_{2}}\left(-\delta_{1}\right) \rightarrow$ $\mathcal{O}_{\delta_{1} \cup \delta_{2}} \rightarrow \mathcal{O}_{\delta_{1}} \rightarrow 0$. Tensoring with $\tilde{N}$ gives an exact sequence $0 \rightarrow \tilde{N}^{\prime \prime} \rightarrow$ $\tilde{N} \rightarrow \tilde{N}^{\prime} \rightarrow 0$, where $\tilde{N}^{\prime}:=\left(N^{\vee} \otimes \omega_{\pi}\right)_{\mid \delta_{1}} \otimes \beta^{*}(-a H \boxtimes \mathcal{O}(-b)), \tilde{N}^{\prime \prime}:=$ $\left(N^{\vee} \otimes \omega_{\pi}\left(-\delta_{1}\right)\right)_{\mid \delta_{2}} \otimes \beta^{*}(-a H \boxtimes \mathcal{O}(-b))$. We prove that both $\tilde{N}^{\prime}$ and $\tilde{N}^{\prime \prime}$ are acyclic.

Using the identification $\delta_{2}=\delta_{T \cup\{y\}, T^{c} \cup\{x\}}=\mathrm{Bl}_{p} \mathbb{P}^{r+s-1} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s-1}$, (the first copy of $\mathrm{Bl}_{p} \mathbb{P}^{r+s-1}$ corresponds to $\left.T \cup\{y\}\right)$ by Lemma 5.14 we have $\tilde{N}^{\prime \prime}=M_{1}^{\prime \prime} \boxtimes M_{2}^{\prime \prime}, M_{1}^{\prime \prime}=\left(-\alpha_{T^{c}}-a\right) H+\alpha_{T^{c} \cup\{y\}} \Delta, M_{2}^{\prime \prime}=\left(-f_{T^{c}}-\right.$ $\left.\alpha_{T^{c}}-b-1\right) H+\left(\alpha_{T \cup\{y\}}+b-1\right) \Delta$. Similarly, using the identification $\delta_{1}=$ $\delta_{T \cup\{y, x\}, T^{c}}=\mathrm{Bl}_{1,2, \overline{12}} \mathbb{P}^{P+s} \times \mathbb{P}^{r+s-2}$, by Lemma 5.14 we have $\tilde{N}^{\prime}=M_{1}^{\prime} \boxtimes M_{2}^{\prime}$, where $M_{2}^{\prime}=\mathcal{O}\left(-\alpha_{T \cup\{y\}}-b\right), M_{1}^{\prime}=\left(-f_{T}-|E \cap\{y\}|-\alpha_{T \cup\{y\}}-a-1\right) H+$ $\left(f_{T}+\alpha_{T}+1\right) E_{1}+\left(\alpha_{T^{c}}+a\right) E_{2}+\alpha_{T^{c} \cup\{y\}} E_{12}$. We prove that one of $M_{1}^{\prime}, M_{2}^{\prime}$ and one of $M_{1}^{\prime \prime}, M_{2}^{\prime \prime}$ are acyclic using Lemma 5.10 (or an analogue).

Case 1) $f_{T}, f_{T^{c}} \geq 0$. Then $\alpha_{T}=\alpha_{T \cup\{y\}}=\alpha_{T^{c}}=\alpha_{T^{c} \cup\{y\}}=0, f_{T}=m_{T}$, $f_{T^{c}}=m_{T^{c}}, M_{1}^{\prime \prime}=-a H, M_{2}^{\prime \prime}=-d^{\prime \prime} H+(b-1) \Delta$, where $d^{\prime \prime}=m_{T^{c}}+b+1$, $M_{1}^{\prime}=-d^{\prime} H+\beta_{1} E_{1}+\beta_{2} E_{2}$, where $d^{\prime}=m_{T}+|E \cap\{y\}|+a+1, \beta_{1}=m_{T}+1$, $\beta_{2}=a, M_{2}^{\prime}=\mathcal{O}(-b)$. If $0<a \leq r+s-1$, then $M_{1}^{\prime \prime}$ is acyclic. Similarly, $M_{2}^{\prime}$ is acyclic when $0<b \leq r+s-2$. Furthermore, if $0<b \leq r+s-1-\mu^{\prime}$, then $M_{2}^{\prime \prime}$ is acyclic since (6.2) implies $d^{\prime \prime}=m_{T^{c}}+b+1 \leq \mu^{\prime}+b+1<r+s$. Similarly, if $0<a<r+s-\mu$ then $M_{1}^{\prime}$ is acyclic since (6.3) implies that $d^{\prime}=m_{T}+|E \cap\{y\}|+a+1 \leq \mu+a+1<r+s+1$, and $\beta_{2}=a \leq r+s-1$, $\beta_{1}=m_{T}+1 \leq \mu+1<r+s$.

Case 2) $f_{T} \geq 0, f_{T^{c}}<0$. Then $\alpha_{T}=\alpha_{T \cup\{y\}}=0, f_{T}=m_{T}, \alpha_{T^{c}}=-f_{T^{c}}>$ $0, \alpha_{T^{c} \cup\{y\}}=\alpha_{T^{c}}-|E \cap\{y\}| \geq 0, M_{1}^{\prime \prime}=\left(-\alpha_{T^{c}}-a\right) H+\left(\alpha_{T^{c}}-|E \cap\{y\}|\right) \Delta$, $M_{2}^{\prime \prime}=(-b-1) H+(b-1) \Delta, M_{1}^{\prime}=-d^{\prime} H+\beta_{1} E_{1}+\beta_{2} E_{2}+\beta_{12} E_{12}$, where $d^{\prime}=m_{T}+|E \cap\{y\}|+a+1, \beta_{1}=m_{T}+1, \beta_{2}=\alpha_{T^{c}}+a, \beta_{12}=\alpha_{T^{c}}-|E \cap\{y\}|$, $M_{2}^{\prime}=\mathcal{O}(-b)$. If $0<b \leq r+s-2, M_{2}^{\prime \prime}$ and $M_{2}^{\prime}$ are both acyclic. Assume $b=0,0<a<r+s-\mu$. Then $M_{1}^{\prime \prime}$ is acyclic since by (6.4) we have $0 \leq \alpha_{T^{c}}-|E \cap\{y\}| \leq \mu-l \leq \mu<r+s-1(r+s=2, \mu=1$ is not possible, as $0<a<r+s-\mu$ ) and $0<\alpha_{T^{c}}+a \leq \mu+a<r+s$. Furthermore, $M_{1}^{\prime}$ is acyclic since (6.3) implies that $d^{\prime}=m_{T}+|E \cap\{y\}|+a+1 \leq \mu+a+1 \leq r+s$. Furthermore, $\beta_{1}=m_{T}+1 \leq \mu+1 \leq \mu+a<r+s, \beta_{12} \leq \mu<r+s-1$, and by (6.4) we have $\beta_{2}=\alpha_{T^{c}}+a \leq \mu+a<r+s$.
Case 3) $f_{T}<0, f_{T^{c}} \geq 0$. Then $\alpha_{T^{c}}=\alpha_{T^{c} \cup\{y\}}=0, f_{T^{c}}=m_{T^{c}}, \alpha_{T}=$ $-\overline{f_{T}>0, \alpha_{T \cup\{y\}}=\alpha_{T}-\mid} E \cap\{y\} \mid \geq 0, M_{1}^{\prime \prime}=(-a) H, M_{2}^{\prime \prime}=-d^{\prime \prime} H+\beta \Delta$, where $d^{\prime \prime}=m_{T^{c}}+b+1, \beta=\alpha_{T}-|E \cap\{y\}|+b-1, M_{1}^{\prime}=(-a-1) H+E_{1}+a E_{2}$, $M_{2}^{\prime}=\mathcal{O}\left(-\alpha_{T}-b+|E \cap\{y\}|\right)$. If $0<a \leq r+s-1$, then $M_{1}^{\prime \prime}$ and $M_{1}^{\prime}$ are acyclic. Assume $a=0,0<b<r+s-1-\mu^{\prime}$. Note that $M_{1}^{\prime}$ is still acyclic. Then $M_{2}^{\prime \prime}$ is acyclic since by (6.5) $0<\beta=\alpha_{T}-|E \cap\{y\}|+b-1 \leq\left(\mu^{\prime}-l\right)+b-1 \leq$ $\mu^{\prime}+b-1<r+s-2$ and (6.2) implies $d^{\prime \prime}=m_{T^{c}}+b+1 \leq \mu^{\prime}+b+1<r+s$. This finishes the proof.

## 7. $\left\{F_{l, E}\right\}$ EXCEPTIONAL ON $\overline{\mathrm{M}}_{2 r, 2 s} \Rightarrow\left\{F_{l, E}\right\}$ EXCEPTIONAL ON $\overline{\mathrm{M}}_{2 r, 2 s+1}$

The goal of this section is to prove the following theorem:

Theorem 7.1. Assume $p=2 r \geq 4, q=2 s+1 \geq 1$. The bundles $\left\{F_{l, E}\right\}$ on $\overline{\mathrm{M}}_{p, q}$ from group $1 A$ of Theorem 1.5 (resp., $1 B$ of Remark1.11) form an exceptional collection conditionally on the same statement for $\overline{\mathrm{M}}_{p, q-1}$.

We fix: $P$ a set of cardinality $p=2 r \geq 4$ and $Q$ a set of cardinality $q=2 s+1 \geq 1$. We also choose an index $y \in Q$, which will be allowed to change later on. The analysis is similar to the proof of Theorem 3.4, except that instead of forgetful morphisms $\mathcal{P}_{n} \rightarrow \mathcal{P}_{n-1}$, we now have rational maps $\overline{\mathrm{M}}_{P, Q} \rightarrow \overline{\mathrm{M}}_{P, Q \backslash\{y\}}$. These maps have a convenient resolution.
Lemma 7.2. The rational map $\overline{\mathrm{M}}_{P, Q} \rightarrow \overline{\mathrm{M}}_{P, Q \backslash\{y\}}$ is resolved by the following diagram of morphisms, where $f$ is a birational reduction morphism between Hassett spaces and $\alpha$ is the universal family over $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}$.


Proof. We choose the weights of $p$ heavy points in $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}$ and $\overline{\mathrm{M}}_{P, Q}$ to be $\frac{1}{r}-\epsilon_{1}+\epsilon_{2}$ and the weights of light points to be $\frac{2 r}{2 s} \epsilon_{1}+\epsilon_{2}$ on $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}$ and $\frac{2 r}{2 s+1} \epsilon_{1}+\epsilon_{2}$ on $\overline{\mathrm{M}}_{P, Q}$. Here we first choose $\epsilon_{1}$ such that $\left(r+1+\frac{2 r}{2 s+1}\right) \epsilon_{1}<$ $\frac{1}{r}$ and then choose $0<\epsilon_{2} \ll 1$ that depends on $\epsilon_{1}$ (except if $q=1$, in which case the weights of points in $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}=\overline{\mathrm{M}}_{P}$ are given by $\frac{1}{r}+\epsilon_{2}$ ). By Notation 2.8 , these weights give the Hassett spaces $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}$ and $\overline{\mathrm{M}}_{P, Q}$. We choose the weights of points on $W$ as follows: the weights of points in $P$ and $Q \backslash\{y\}$ are the same as on $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}$ and the weight of the point $y$ is the same as on $\overline{\mathrm{M}}_{P, Q}$. It is clear that the weights of $W$ dominate the weights in $\overline{\mathrm{M}}_{P, Q}$, so we have the reduction morphism $f$ of Hassett spaces.

It remains to show that $\alpha$ is the universal family. We use Has03, Propostion 5.1, Proposition 5.4]. Since the weights of points in $P$ and $Q \backslash\{y\}$ are the same on $W$ and $\overline{\mathrm{M}}_{P, Q \backslash\{y\}}$, it suffices to check that the weight of $y$ on $W$ is sufficiently small. Indeed, $(r-1)\left(\frac{1}{r}-\epsilon_{1}\right)+2 s \frac{r}{s} \epsilon_{1}+\frac{2 r}{2 s+1} \epsilon_{1}<1$, so $r-1$ heavy points can coincide with all points in $Q \backslash\{y\}$ and $y$ on $W$. On the other hand, $r\left(\frac{1}{r}-\epsilon_{1}\right)+(s-1) \frac{r}{s} \epsilon_{1}+\frac{2 r}{2 s+1} \epsilon_{1}<1$, so $r$ heavy points can coincide with $s-1$ points in $Q \backslash\{y\}$ and $y$ on $W$.

We will apply to the morphism $f$ the following abstract lemma:
Lemma 7.3. Let $f: M \rightarrow N$ be a morphism of smooth projective varieties such that $R f_{*} \mathcal{O}_{M} \cong \mathcal{O}_{N}$. Let $D^{b}(M)=\langle\mathcal{A}, \mathcal{B}\rangle$ be a s.o.d. Let $\Phi: D^{b}(N) \rightarrow \mathcal{B}$ be the composition of the derived pullback $L f^{*}$ and the right adjoint functor $i^{!}$of the inclusion $i: \mathcal{B} \hookrightarrow D^{b}(M)$. Then $\Phi$ has a left adjoint functor $\Psi: \mathcal{B} \rightarrow D^{b}(N)$ given by $X \mapsto\left[R f_{*}\left(i(X)^{\vee}\right)\right]^{\vee}$ (derived duals). Furthermore, if $R f_{*} Z^{\vee}=0$ for every $Z \in \mathcal{A}$ then $\Phi$ is a fully faithful functor and $\Psi$ is its left quasi-inverse.

Proof. The adjointness part follows from the following calculation:
$\operatorname{Hom}_{\mathcal{B}}(X, \Phi(Y)) \cong \operatorname{Hom}_{D^{b}(M)}\left(i(X), L f^{*}(Y)\right) \cong \operatorname{Hom}_{D^{b}(M)}\left(L f^{*}(Y)^{\vee}, i(X)^{\vee}\right)$

$$
\cong \operatorname{Hom}_{D^{b}(N)}\left(Y^{\vee}, R f_{*}\left(i(X)^{\vee}\right)\right) \cong \operatorname{Hom}_{D^{b}(N)}(\Psi(X), Y) .
$$

Suppose that $R f_{*} Z^{\vee}=0$ for every $Z \in \mathcal{A}$. Take an object $Y \in D^{b}(N)$ and a triangle $Z \rightarrow i(\Phi(Y)) \rightarrow L f^{*}(Y) \rightarrow$ associated with the s.o.d. $\langle\mathcal{A}, \mathcal{B}\rangle$. The dual triangle is $L f^{*}\left(Y^{\vee}\right) \rightarrow i(\Phi(Y))^{\vee} \rightarrow Z^{\vee} \rightarrow$. Taking the push-forward, we get $Y^{\vee} \cong R f_{*}\left(i(\Phi(Y))^{\vee}\right)$ as $R f_{*} Z^{\vee}=0$. It follows that $Y \cong \Psi(\Phi(Y))$. Thus $\Psi$ is a left quasi-inverse of $\Phi$ and therefore $\Phi$ is fully faithful.

In the rest of this section, we use the set-up and the notations of Lemma 7.3. The morphism $f: W \rightarrow \overline{\mathrm{M}}_{P, Q}$ has the following properties. Let $T \subset$ $P \cup(Q \backslash\{y\})$ be a subset of $r$ heavy and $s$ light indices, and let $T^{c}$ denote its complement in $P \cup(Q \backslash\{y\})$. In $\overline{\mathrm{M}}_{P, Q}$ we denote by $\mathbb{P}_{T}^{r+s-1}$ the locus where the points from $T^{c}$ come together. Note that $\mathbb{P}_{T}^{r+s-1} \cong \mathbb{P}^{r+s-1}$ by [Has03, §6.2] (see (3) in Note 5.1), as the weight of the point corresponding to $T^{c}$ can be changed to 1 without changing the stability condition.

Different loci $\mathbb{P}_{T}^{r+s-1}$ are disjoint, except for pairs $\mathbb{P}_{T}^{r+s-1}$ and $\mathbb{P}_{T_{c}^{c}}^{r+s-1}$, which intersect transversely at the point in $\overline{\mathrm{M}}_{P, Q}$ where the markings from $T$ coincide and similarly the markings from $T^{c}$ coincide. The loci $\mathbb{P}_{T}^{r+s-1}$ are blown-up by the morphism $f$ (in any order), creating boundary divisors $\delta_{T \cup\{y\}}:=\delta_{T \cup\{y\}, T^{c}} \cong \mathrm{Bl}_{p} \mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-2} \subset W$. These divisors are contracted by the morphism $f$ via the composition of the first projection and the blow-down $\mathrm{Bl}_{p} \mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-2} \rightarrow \mathbb{P}^{r+s-1}$. The divisors $\delta_{T \cup\{y\}}$ are disjoint, except for pairs $\delta_{T \cup\{y\}}, \delta_{T^{c} \cup\{y\}}$, which intersect transversely along a codimension 2 locus isomorphic to $\mathbb{P}_{T}^{r+s-2} \times \mathbb{P}_{T c}^{r+s-2}$. By Lemma 2.4, we have an s.o.d. $D^{b}(W)=\left\langle\mathcal{C}, L f^{*} D^{b}\left(\overline{\mathrm{M}}_{P, Q}\right)\right\rangle$, where $\mathcal{C}$ has an exceptional collection of the following objects (in some order):

$$
\begin{aligned}
& \text { - } \mathcal{O}_{\mathbb{P}_{T}^{r+s-2} \times \mathbb{P}_{T c}^{r+s-2}}(-i,-j) \text { for } 1 \leq i \leq r+s-2,1 \leq j \leq r+s-2 \text {; } \\
& \text { - } \mathcal{O}_{\delta_{T \cup\{y\}}}(a H,-u) \text { for } 0 \leq a \leq r+s-1,1 \leq u \leq r+s-2 \text {. }
\end{aligned}
$$

On the other hand, we have an s.o.d. $D^{b}(W)=\langle\mathcal{A}, \mathcal{B}\rangle$, where $\mathcal{A} \subset D^{b}(W)$ is the admissible subcategory generated by the exceptional collection

$$
\begin{equation*}
\mathcal{O}_{\delta_{T \cup\{ \}\}}}(-a H, u), \quad 0<a \leq\left\lfloor\frac{r+s-1}{2}\right\rfloor, 1 \leq u<\left\lfloor\frac{r+s-1}{2}\right\rfloor . \tag{7.2}
\end{equation*}
$$

As $\mathcal{C}$ is an exceptional collection, it follows that $\mathcal{A}$ is an exceptional collection. Indeed, if $Z=j_{*} L$ for some line bundle $L$ on $\delta:=\delta_{T \cup\{y\}}(j: \delta \hookrightarrow W$ the inclusion morphism), then $Z^{\vee} \cong\left(j_{*} L^{\vee}\right) \otimes \mathcal{O}(\delta)[-1]$ by [Huy06, 3.41].
Lemma 7.4. The functor $\Phi: D^{b}\left(\overline{\mathrm{M}}_{P, Q}\right) \rightarrow \mathcal{B}$ of Lemma 7.3 is fully faithful.
Proof. We need to check that $R f_{*} Z^{\vee}=0$ if $Z=\mathcal{O}_{\delta_{T \cup\{y\}}}(-a H, u)$ is one of the generators of $\mathcal{A}$. Indeed, $Z^{\vee} \cong \mathcal{O}_{\delta_{T \cup\{y\}}}((a-1) H,-u-1)[-1]$ by Huy06, 3.41]. Since $-(r+s-2) \leq-u-1 \leq-1$, we have $R f_{*} Z^{\vee}=0$.
Definition 7.5. Let $\mathcal{Q}$ be a set of objects in a triangulated category $\mathcal{T}$. We say that objects $G$ and $G^{\prime}$ in $\mathcal{T}$ are related by quotients in $\mathcal{Q}$ if there are objects $Q_{i}$ in $\mathcal{Q}$ for $0 \leq i \leq k$ and a sequence of exact triangles $G_{i-1} \rightarrow$ $G_{i}\left[l_{i}\right] \rightarrow Q_{i}\left[l_{i}^{\prime}\right] \rightarrow G_{i-1}[1]$ for some $l_{i}, l_{i}^{\prime} \in \mathbb{Z}$ with $G_{0} \cong G, G_{k} \cong G^{\prime}$.

Lemma 7.6. Consider the set of vector bundles $F_{l, E}$ on $\overline{\mathrm{M}}_{P, Q}$ with

$$
\begin{equation*}
m_{T, E, l} \leq \min \left(\frac{r+s}{2}-|E \cap\{y\}|, \quad \frac{r+s-1}{2}\right) \tag{7.3}
\end{equation*}
$$

for every subset $T \subset P \cup(Q \backslash\{y\})$ of $r$ heavy and slight indices (see Notation 4.1 for the definition of $\left.m_{T, E, l}\right)$. For example, all bundles of type $1 A / 1 B$ and type 2 belong to this set by Lemma 4.8. Then $\Phi\left(F_{l, E}\right) \cong F_{l, E}$ (see Definition 5.3 for the definition of $F_{l, E}$ on $W$ ). In particular,

$$
\begin{equation*}
\operatorname{RHom}_{\overline{\mathrm{M}}_{P, Q}}\left(F_{l, E}, F_{l^{\prime}, E^{\prime}}\right) \cong \operatorname{RHom}_{W}\left(F_{l, E}, F_{l^{\prime}, E^{\prime}}\right) \tag{7.4}
\end{equation*}
$$

for vector bundles satisfying (7.3). Furthermore, $R f_{*}\left(F_{l, E}^{\vee}\right) \cong F_{l, E}^{\vee}$.
Proof. By taking $\mu=\mu^{\prime}=\frac{r+s}{2}$ in Proposition 6.4 (note that all conditions are satisfied because of (7.3)) we have that

$$
\begin{equation*}
\operatorname{RHom}_{W}\left(F_{l, E}, \mathcal{O}_{\delta_{T}}(-a H, u)\right)=0 \tag{7.5}
\end{equation*}
$$

for all $0<a \leq\left\lfloor\frac{r+s-1}{2}\right\rfloor, 0 \leq-u \leq r+s-2$. So (7.5) holds for all $u$, positive or negative. Therefore, the bundles $F_{l, E}$ on $W$ belong to $\mathcal{B}$. By Proposition 7.7, using that $m_{T^{c}}<\frac{r+s}{2}$ by Lemma4.8, the bundles $F_{l, E}$ and $L f^{*} F_{l, E}$ are related by sheaves in $\mathcal{A}$. Since $i^{!}(\mathcal{A})=0$ (as in Lemma 7.3, $i^{!}$is the right adjoint functor of the inclusion $i: \mathcal{B} \hookrightarrow D^{b}(W)$, we have

$$
\Phi\left(F_{l, E}\right)=i^{!}\left(L f^{*} F_{l, E}\right) \cong i^{!}\left(F_{l, E}\right) \cong F_{l, E}
$$

By Lemma 7.4, we have (7.4) and $R f_{*}\left(F_{l, E}^{\vee}\right) \cong R f_{*}\left(i\left(\Phi\left(F_{l, E}\right)\right)^{\vee}\right) \cong F_{l, E}^{\vee}$ (see the proof of Lemma 7.3).

Proposition 7.7. Let $l \geq 0$, and let $E \subseteq P \cup Q$ be such that $e+l$ even. The vector bundles $F_{l, E}$ and $f^{*} F_{l, E}$ on $W$ are related by sheaves $Q_{i} \cong \mathcal{O}(-j H) \boxtimes \mathcal{O}(u)$ supported on boundary divisors $\delta_{T \cup\{y\}, T^{c}}$ for $0<j \leq m_{T^{c}}, 0 \leq u<m_{T^{c}}$ (see Notation 4.1 for the definition of $m_{T^{c}}:=m_{T^{c}, E, l}$ ). We have

$$
Q_{i}^{\vee} \cong \mathcal{O}((j-1) H) \boxtimes \mathcal{O}(-u-1)
$$

Proof. We claim that two types of quotients $\mathcal{O}(-j H) \boxtimes \mathcal{O}(u)$ appear:
(I) $0<j \leq m_{T^{c}}, 0 \leq u<m_{T^{c}}$, if $f_{T}>0$;
(II) $0<j \leq m_{T^{c}}, 0 \leq u<\tilde{m}_{2}(T)$, if $f_{T}+|E \cap\{y\}|<0$, where $\tilde{m}_{2}(T):=-f_{T}-|E \cap\{y\}|$ (see Notation 4.1 for the definition of $\left.f_{T}, f_{T^{c}}, m_{T^{c}}\right)$. Note that in this case $f_{T}<0$, we have $f_{T^{c}}=l-f_{T}>0$ (as $l \geq 0$ ) and $\tilde{m}_{2}(T) \leq m_{T^{c}}=f_{T^{c}}$.
To prove the claim, recall that $F_{l, E}=R \pi_{*}\left(N_{1}\right)$, where (see Definition 5.3)

$$
N_{1}:=N_{l, E}=\omega_{\pi}^{\frac{e-l}{2}}(E)\left(-\sum_{T} \alpha_{T} \delta_{T \cup\{x\}}-\sum_{T} \alpha_{T \cup\{y\}} \delta_{T \cup\{y, x\}}\right)
$$

Here $x$ is the new marking on the universal family $\pi: \mathcal{U} \rightarrow W$ and either $\alpha_{T}=\alpha_{T \cup\{y\}}=0\left(\right.$ when $\left.f_{T} \geq 0\right)$ or $\alpha_{T}=-f_{T, E, l}, \alpha_{T \cup\{y\}}=\alpha_{T}-|E \cap\{y\}| \geq$ 0 (when $f_{T}<0$, see (4.2)) for the definition of $\alpha_{T}, \alpha_{T \cup\{y\}}$ ). If $\pi^{\prime}: \mathcal{U}^{\prime} \rightarrow \overline{\mathrm{M}}_{p, q}$ is the universal family, there is a commutative diagram with a cartesian second square:

and we have $f^{*} F_{l, E} \cong R \pi_{*}\left(N_{2}\right)$, where we denote

$$
\begin{equation*}
N_{2}=v^{*} \omega_{\rho}^{\frac{e-l}{2}}(E) \cong \omega_{\pi^{\frac{e-l}{2}}}(E)+\sum_{T} f_{T} \delta_{T \cup\{x\}} \tag{7.6}
\end{equation*}
$$

The latter equality holds because of the identities (5.3). Recall from Note 5.1(1) that $W$ and $\mathcal{U}$ are smooth. Furthermore, the spaces $\mathcal{U}^{\prime}$ and $\mathcal{V}$ are $\mathbb{P}^{1}$-bundles over Hassett spaces, in particular, they are smooth. Note that among the two types of boundary divisors $\delta_{T \cup\{x\}}, \delta_{T \cup\{y, x\}}$ of $\mathcal{U}$, only the divisors $\delta_{T \cup\{x\}}$ are $v$-exceptional. We have:

$$
N_{2}=N_{1}+\sum_{T}\left(f_{T}+\alpha_{T}\right) \delta_{T \cup\{x\}}+\sum_{T} \alpha_{T \cup\{y\}} \delta_{T \cup\{y, x\}}=N_{1}+\Sigma_{1}+\Sigma_{2},
$$

where $\Sigma_{1}=\sum_{f_{T}>0} f_{T} \delta_{T \cup\{x\}}, \Sigma_{2}=\sum_{f_{T}<0} \tilde{m}_{2}(T) \delta_{T \cup\{y, x\}}$. Any two distinct boundary divisors appearing in $\Sigma_{1}$ do not intersect since different boundary divisors of type $\delta_{T \cup\{x\}}$ do not intersect. Similarly, any two distinct boundary divisors appearing in $\Sigma_{2}$ do not intersect since $\delta_{T \cup\{y, x\}}$ intersects $\delta_{T^{\prime} \cup\{y, x\}}\left(T^{\prime} \neq T\right)$ only when $T^{\prime}=T^{c}$, but if $f_{T}<0$, we must have $f_{T^{c}}>0$. For the same reason, each $\delta_{T \cup\{y, x\}}$ that appears in $\Sigma_{2}$ intersects exactly one term from $\Sigma_{1}$, namely, $\delta_{T^{c} \cup\{x\}}$. We add successively to $N_{1}$ first terms from $\Sigma_{1}$, then $\Sigma_{2}$, and get two types of quotients.

Type I quotients have the form $Q_{1}=\left(N_{1}+j \delta_{T \cup\{x\}}\right)_{\left.\right|_{\delta_{T \cup\{x\}}}}$ for $0<j \leq$ $f_{T}=m_{T}$. By Lemma 5.14(1) we have
$Q_{1}=\left(\omega_{\pi}^{\frac{e-l}{2}}(E)+j \delta_{T \cup\{x\}}-\alpha_{T^{c} \cup\{y\}} \delta_{T^{c} \cup\{y, x\}}\right)_{\mid \delta_{T \cup\{x\}}} \cong \mathcal{O}(\alpha H-\beta \Delta) \boxtimes \mathcal{O}(-j H)$ for $\alpha:=f_{T}-j, \beta=\alpha_{T^{c} \cup\{y\}}=\alpha_{T^{c}}-|E \cap\{y\}|$ (Notation (4.2)). By Definition (4.2), we always have $\beta \geq 0$. Since $\alpha_{T^{c}}$ is either 0 or $-f_{T}^{c}$ and $f_{T}+f_{T^{c}}=$ $l \geq 0$, it follows that $\beta \leq f_{T}=m_{T}$. Clearly, $0 \leq \alpha<m_{T}, 0<j \leq m_{T}$. By Lemma 5.9, $R \pi_{*} Q_{1}$ is a direct sum of sheaves of the form $\mathcal{O}(u) \boxtimes(-j H)$ supported on $\delta_{T, T^{c} \cup\{y\}} \cong \mathbb{P}^{r+s-2} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s-1}$ for $u \geq \min (\beta, \alpha+1) \geq 0$, $u \leq \max (\alpha, \beta-1)<m_{T}, 0<j \leq m_{T}$.
 various $T$ (with $f_{T}<0$, hence, $f_{T^{c}}>0$ ). Namely,

$$
\begin{gathered}
Q_{2}=\left(N_{1}+f_{T^{c}} \delta_{T^{c} \cup\{x\}}+i \delta_{T \cup\{y, x\}}\right)_{\mid \delta_{T \cup\{y, x\}}}= \\
=\left(\omega_{\pi^{\frac{e-l}{2}}}(E)+f_{T} \delta_{T \cup\{x\}}+\left(-\tilde{m}_{2}(T)+i\right) \delta_{T \cup\{y, x\}}+f_{T^{c}} \delta_{T^{c} \cup\{x\}}\right)_{\mid \delta_{T \cup\{y, x\}}}
\end{gathered}
$$

for $0<i \leq \tilde{m}_{2}(T)$. By Lemma 5.14(2), $Q_{2} \cong \mathcal{O}\left(-i H+\beta E_{2}\right) \boxtimes \mathcal{O}(u)$ where $\beta:=f_{T^{c}}=l-f_{T} \geq \tilde{m}_{2}(T), u=\tilde{m}_{2}(T)-i<\tilde{m}_{2}(T)$. In particular, $\beta \geq i>0$ and $0 \leq u<\tilde{m}_{2}(T)$. By the projection formula, Remark 5.15 and Lemma 5.14(2) (and using its notations), we have $R \tilde{q}_{*} \mathcal{O}\left(-i H+\beta E_{2}\right) \cong$ $\left.R \tilde{q}_{*}\left(\tilde{q}^{*} \mathcal{O}(-i H)+(\beta-i) E_{2}\right)\right) \cong \mathcal{O}(-i H) \oplus \ldots \oplus \mathcal{O}(-\beta H)$. The result follows since $\beta=f_{T^{c}}=m_{T^{c}}$ (as $f_{T^{c}} \geq 0$ ).
Remark 7.8. Prop. 7.7 shows that when $l=0$ and $E=P \cup(Q \backslash\{y\})$, on $W$ we have $f^{*} F_{l, E} \cong F_{l, E}$ (because $m_{T}=m_{T^{c}}=0$ for all $T$ ). Note that the same is trivially true when $l=0, E=\emptyset$. In particular, $F_{l, E}^{\vee} \cong R f_{*} F_{l, E}^{\vee}$ is still true in these cases. For example, on $\overline{\mathrm{M}}_{p, q-1} \cong \overline{\mathrm{M}}_{4}$ (i.e., $p=4, q=1$ )
$\left\{F_{0, \varnothing}, F_{0, \Sigma}\right\}$ is the collection in Theorem 1.9 (group $1 A$ is the same as group $1 B$ in this case). This is a full exceptional collection on $\overline{\mathrm{M}}_{4} \cong \mathbb{P}^{1}$ because $F_{0, \emptyset} \cong \mathcal{O}, F_{0, \Sigma} \cong \mathcal{O}(1)$ (Corollary 6.3).

Proof of Theorem 7.1 Let $F_{l, E}, F_{l^{\prime}, E^{\prime}}$ be bundles from one of the two collections (group $1 A$ or $1 B$ ) on $\overline{\mathrm{M}}_{p, q}$. By Lemma 7.6, for any $y \in Q$, we have

$$
R \operatorname{Hom}_{\bar{M}_{p, q}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong R \operatorname{Hom}_{W}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) .
$$

We consider several cases and make different choices of $y \in Q$.
Case 1: $E_{q}^{\prime} \backslash E_{q} \neq \emptyset$. Fix $y \in E_{q}^{\prime} \backslash E_{q}$. By Lemma 5.17, $F_{l, E}^{\vee} \cong \alpha^{*} F_{l, E}^{\vee}$, $R \alpha_{*} F_{l^{\prime}, E^{\prime}}^{\vee}=0$. Therefore, $R \operatorname{Hom}_{W}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong R \operatorname{Hom}_{W}\left(F_{l, E}^{\vee}, F_{l^{\prime}, E^{\prime}}^{\vee}\right) \cong$ $R \operatorname{Hom}_{W}\left(\alpha^{*} F_{l, E}^{\vee}, F_{l^{\prime}, E^{\prime}}^{\vee}\right) \cong R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q-1}}\left(F_{l, E}^{\vee}, R \alpha_{*} F_{l^{\prime}, E^{\prime}}^{\vee}\right)=0$.

Case 2: $E_{q}=E_{q}^{\prime} \neq Q$. Choose $y \in Q \backslash E_{q}$. Since the range in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q}$ is precisely the range in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q-1}$, we are done by Lemma 5.17, since $F_{l, E} \cong \alpha^{*} F_{l, E}, F_{l^{\prime}, E^{\prime}} \cong \alpha^{*} F_{l^{\prime}, E^{\prime}}$ and $R \operatorname{Hom}_{W}\left(\alpha^{*} F_{l^{\prime}, E^{\prime}}, \alpha^{*} F_{l, E}\right) \cong R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q-1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$. The latter is equal to 0 or $\mathbb{C}\left(\right.$ if $\left.(l, E)=\left(l^{\prime}, E^{\prime}\right)\right)$ by Thm. 1.8 for $\overline{\mathrm{M}}_{p, q-1}$ when $q>1$ and Theorem 1.9 when $q=1$ (for group $1 A$, and Remark 1.11 for group $1 B$ ).

Case 3: $E_{q}=E_{q}^{\prime}=Q, q>1$. Let $z \in Q$. By Prop. [3.6, $F_{l, E} \cong F_{l, E_{p} \cup\{z\}} \otimes$ $F_{0, E_{q} \backslash\{z\}}, F_{l^{\prime}, E^{\prime}} \cong F_{l, E_{p}^{\prime} \cup\{z\}} \otimes F_{0, E_{q}^{\prime} \backslash\{z\}}$. It follows that $R \operatorname{Hom}_{\bar{M}_{p, q}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong$ $R \operatorname{Hom}_{\bar{M}_{p, q}}\left(F_{l^{\prime}, E_{p}^{\prime} \cup\{z\}}, F_{l, E_{p} \cup\{z\}}\right)$. Note that $F_{l, E_{p} \cup\{z\}}$ is still in group $1 A$ (resp., group $1 B$ ) on $\overline{\mathrm{M}}_{p, q}$, and since by assumption $Q \neq\{z\}$ we are in Case 2.

Case 4: $E_{q}=E_{q}^{\prime}=Q, q=1$. Let $Q=\{z\}$. Note that $\overline{\mathrm{M}}_{p, 1} \cong \overline{\mathrm{M}}_{p+1}$ and since $e_{p}^{\prime} \geq e_{p}$, we have $e^{\prime} \geq e$. Case 4(i): the bundles are in group 1B. Then they are in the exceptional collection of Theorem 1.2 (proved in Section (3), and we are done. Finally, consider the case 4(ii): the bundles are in group 1A. Take the line bundle $L=\mathcal{O}(-P-2 z)$ on $\overline{\mathrm{M}}_{p+1}$, i.e., we take -1 in the position of every heavy point and -2 in the position of $z$. By the projection formula and Corollary 3.3, $L \otimes F_{l, E} \cong \mathcal{O}(-P-2 z) \otimes$ $R \pi_{*} \mathcal{O}(E, l) \cong R \pi_{*}\left(-\left(P \backslash E_{p}\right)-z, l\right) \cong F_{l,\left(P \backslash E_{p}\right) \cup\{z\}}^{\vee}$ and similarly for $F_{l^{\prime}, E^{\prime}}$. On $\overline{\mathrm{M}}_{p, 1}$, we have $R \operatorname{Hom}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, 1}}\left(F_{l^{\prime}, E^{\prime}} \otimes L, F_{l, E} \otimes L\right) \cong$ $R \operatorname{Hom}\left(F_{l^{\prime},\left(P \backslash E_{p}^{\prime}\right) \cup\{z\}}^{\vee}, F_{l,\left(P \backslash E_{p}\right) \cup\{z\}}^{\vee}\right) \cong R \operatorname{Hom}\left(F_{l,\left(P \backslash E_{p}\right) \cup\{z\}}, F_{l^{\prime},\left(P \backslash E_{p}^{\prime}\right) \cup\{z\}}\right)$. These bundles are in the group 1B on $\overline{\mathrm{M}}_{p, 1}$ and are in the correct order: $p-e_{p} \geq p-e_{p}^{\prime}$, hence, we are done by the argument in the Case 4(i).

## 8. $\left\{F_{l, E}\right\}$ EXCEPTIONAL ON $\overline{\mathrm{M}}_{2 r, 2 s+1} \Rightarrow\left\{F_{l, E}\right\}$ EXCEPTIONAL ON $\overline{\mathrm{M}}_{2 r, 2 s+2}$

The goal of this section is to prove the following theorem.
Theorem 8.1. Vector bundles $F_{l, E}$ from group $1 A$ (resp., group 1B) from Theorems 1.5 1.8 1.9 and Remark 1.11 form an exceptional collection.
Throughout this section we assume that $P$ (resp., $\tilde{Q}$ ) is the set of heavy (resp., light) indices and we have $|P|=p=2 r \geq 4,|\tilde{Q}|=q+1=2 s+2 \geq 0$. Given Theorem 7.1, it suffices to prove that the bundles $\left\{F_{l, E}\right\}$ from group 1A (resp., 1B) form an exceptional collection on $\overline{\mathrm{M}}_{p, q+1}$ conditionally on the same statement for $\overline{\mathrm{M}}_{p, q}$, as well as to prove, unconditionally, that these bundles form an exceptional collection on $\overline{\mathrm{M}}_{p}$. We treat this last case by
allowing $q+1=0$ (i.e., $s=-1$ ), in which case we also assume that $r \geq 3$. One checks directly the case of $r=2, s=-1$, i.e., that the collection on $\overline{\mathrm{M}}_{p}=\overline{\mathrm{M}}_{4} \cong \mathbb{P}^{1}$ in Theorem 1.9 is exceptional: the collection is $\left\{F_{0, \emptyset}, F_{0, \Sigma}\right\}$, where $F_{0, \emptyset} \cong \mathcal{O}, F_{0, \Sigma} \cong \mathcal{O}(1)$ by Corollary 6.3 (see also Remark 7.8).

Notation 8.2. We choose an index $z \in \tilde{Q}$ unless $q+1=0$, in which case we choose an index $z \in P$. This index will be allowed to vary later. For every boundary divisor $\delta_{T, T^{c}}$ of $\overline{\mathrm{M}}_{p, q+1}$, we always assume that $z \in T$. We let $\overline{\mathrm{M}}_{p, q}=\overline{\mathrm{M}}_{P, \tilde{Q} \backslash\{z\}}$ if $q \geq 0$ (resp., $\overline{\mathrm{M}}_{p,-1}:=\overline{\mathrm{M}}_{p-1}=\overline{\mathrm{M}}_{P \backslash\{z\}}$ if $q=-1$ ). Let $f: \overline{\mathrm{M}}_{p, q+1} \longrightarrow \overline{\mathrm{M}}_{p, q}$ be the forgetful map that forgets the marking $z$.

Lemma 8.3. The forgetful map $f$ is regular and factors through the universal family $\pi: \mathcal{U}_{p, q} \rightarrow \overline{\mathrm{M}}_{p, q}$, which is a $\mathbb{P}^{1}$-bundle. We have a commutative diagram


Here $\alpha$ (resp., $\gamma$ ) is the universal family of the corresponding Hassett space. Furthermore, $\gamma$ is a $\mathbb{P}^{1}$-bundle and $\mathcal{U}_{p, q}$ is a GIT quotient of $\left(\mathbb{P}^{1}\right)^{p+q+1}$. The morphisms $g$ and $\beta$ are birational reduction morphisms of Hassett spaces.

Proof. Suppose first that $q+1>0$. We choose the same weight $a+\epsilon$, where $a$ is such that $r<\frac{1}{a}<r+\frac{2 s+1}{2 s+3}<r+1, \epsilon \ll 1$, for all $p$ heavy points for $\overline{\mathrm{M}}_{p, q+1}, \mathcal{U}_{p, q}$, and $\overline{\mathrm{M}}_{p, q}$; the same weight $b+\epsilon$, where $b=\frac{2-2 r a}{2 s+1}, \epsilon \ll 1$, for all light points on $\overline{\mathrm{M}}_{p, q+1}$ and $\overline{\mathrm{M}}_{p, q}$ and for all light points on $\mathcal{U}_{p, q}$ except for $z$. We assume that the weight of $z$ on $\mathcal{U}_{p, q}$ is sufficiently small. Then $p a+q b=2$. Furthermore, $(r+1) a>1$, so $r+1$ heavy points coinciding corresponds to an unstable pointed curve; $(r-1) a+(2 s+2) b<1$, so $r-1$ heavy and all light points coinciding corresponds to a stable point of $\overline{\mathrm{M}}_{p, q+1}$ and $\overline{\mathrm{M}}_{p, q} ; r a+s b<1$, so $r$ heavy and $s$ light points coinciding corresponds to a stable point of for $\overline{\mathrm{M}}_{p, q+1}$ and $\overline{\mathrm{M}}_{p, q}$ and, finally, $r a+(s+1) b>1$, so $r$ heavy and $s+1$ light points coinciding corresponds to an unstable point of $\overline{\mathrm{M}}_{p, q+1}$ and $\overline{\mathrm{M}}_{p, q}$. It follows that these weights define $\overline{\mathrm{M}}_{p, q+1}$ and $\overline{\mathrm{M}}_{p, q}$, that $\mathcal{U}_{p, q}$ is the universal family over $\overline{\mathrm{M}}_{p, q}$, and that we have reduction maps of Hassett spaces as claimed. Furthermore, by Lemma 2.7, $\overline{\mathrm{M}}_{p, q}$ and $\mathcal{U}_{p, q}$ are GIT quotients and $\pi$ and $\gamma$ are $\mathbb{P}^{1}$-bundles.

In the case $q+1=0$, we choose the same weight $a+\epsilon$, where $a=$ $\frac{2}{p-1}, \epsilon \ll 1$, for all points for $\overline{\mathrm{M}}_{p}$ and $\overline{\mathrm{M}}_{p,-1}=\overline{\mathrm{M}}_{p-1}$ and for all points for $\mathcal{U}_{p,-1}=\overline{\mathrm{M}}_{p-1,1}$ except for $z$, which we assume has a sufficiently small weight. Then $(p-1) a=2$. Furthermore, $(r-1) a<1$ and $r a>1$, so $r-1$ points coinciding corresponds to a stable point, but $r$ points coinciding corresponds to an unstable point on $\overline{\mathrm{M}}_{p}$ and $\overline{\mathrm{M}}_{p,-1}=\overline{\mathrm{M}}_{p-1}$. We finish the argument as above.

Notation 8.4. The morphism $\beta: \overline{\mathrm{M}}_{p, q+1} \rightarrow \mathcal{U}_{p, q}$ belongs to a general class of reduction morphisms of Hassett spaces $\beta: \overline{\mathrm{M}}_{p, q+1} \rightarrow \mathcal{U}$, which we describe now with an eye towards a different application in Section 10 . We will
return to $\mathcal{U}=\mathcal{U}_{p, q}$ in Lemma 8.9. The following diagram generalizes (8.1):


Reducible fibers of the universal family $\alpha$ are over the boundary divisors $\delta_{T, T^{c}} \cong \mathbb{P}_{T}^{r+s-1} \times \mathbb{P}_{T^{c}}^{r+s-1}$ of $\overline{\mathrm{M}}_{p, q+1}$. The subscripts indicate that the fibers of $\alpha$ are unions of two $\mathbb{P}^{1}$, with one component marked by $T$ and another by $T^{c}$. We suppose that the restriction of $\beta: \overline{\mathrm{M}}_{p, q+1} \rightarrow \mathcal{U}$ to $\delta_{T, T^{c}}$ is either an isomorphism or is isomorphic to the projection $\mathbb{P}_{T}^{r+s-1} \times \mathbb{P}_{T c}^{r+s-1} \rightarrow \mathbb{P}_{T^{c}}^{r+s-1}$, i.e. the sections marked by $T$ become identified. In particular, if $\delta_{T, T^{c}}$ is contracted then we choose one of $T, T^{c}$ and call it $T$ and the other $T^{c}$. The boundary divisors in $W$ have the form $\delta_{T \cup\{y\}}$ and $\delta_{T^{c} \cup\{y\}}$, where $y$ is the extra marking on $W$. They are isomorphic to $\mathbb{P}^{r+s-1} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s}$. The morphism $g$ is either an isomorphism near $\delta_{T \cup\{y\}} \cup \delta_{T^{c} \cup\{y\}}$ or contracts $\delta_{T^{c} \cup\{y\}}$ via the second projection $\mathbb{P}^{r+s-1} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s} \rightarrow \mathrm{Bl}_{p} \mathbb{P}^{r+s}$ and $\delta_{T \cup\{y\}}$ via the composition $\mathbb{P}^{r+s-1} \times \mathrm{Bl}_{p} \mathbb{P}^{r+s} \rightarrow \mathbb{P}^{r+s-1} \hookrightarrow \mathrm{Bl}_{p} \mathbb{P}^{r+s}$ of the first projection followed by the embedding into $\mathrm{Bl}_{p} \mathbb{P}^{r+s}$ as the exceptional divisor (the locus where the marking corresponding to $T$ coincides with $y$ ). Note that any $\mathcal{U}$ as above carries vector bundles $F_{l, E}$ defined in Section 5.
Proposition 8.5. Let $l \geq 0$, and let $E \subseteq \Sigma:=P \cup \tilde{Q}$ be such that $e+l$ even.
(i) The bundles $F_{l, E}$ and $\beta^{*} F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+1}$ are related by sheaves $Q_{i}$ supported on contracted divisors $\delta_{T, T^{c}}=\mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-1}$ having the form $\mathcal{O}(u,-j), 0<$ $j \leq m, 0 \leq u<m$, where $m:=m_{T, E, l}$ (see (4.3)) and where the component $\mathcal{O}(u)$ is the one corresponding to $T$.
(ii) The bundles $F_{l, E}^{\vee}$ and $\beta^{*} F_{l, E}^{\vee}$ are related by sheaves supported on contracted divisors $\delta_{T, T^{c}}$ and having the form $\mathcal{O}(-u-1, j-1), 0<j \leq m, 0 \leq u<m$, where $m:=m_{T, E, l}$.

Proof. Since $F_{l, E}$ and $\beta^{*} F_{l, E}$ are isomorphic near the boundary divisors $\delta_{T, T^{c}}$ that are not contracted by $\beta$, we can remove them and work on the remaining open set $\overline{\mathrm{M}}_{p, q+1}^{0}$ to simplify notation. In particular, all subsets $T$ that we refer in the proof correspond to $\delta_{T, T^{c}} \subset \operatorname{Exc}(\beta)$. We use that $\max _{T}\left(-f_{T^{c}, E, l}\right)=\max _{T}\left(-l+f_{T, E, l}\right) \leq m$. On $\overline{\mathrm{M}}_{p, q+1}^{0}, F_{l, E}=R \alpha_{*}\left(N_{1}\right)$, where

$$
N_{1}:=N_{l, E}=\omega_{\alpha}^{\frac{e-l}{2}}\left(E-\sum_{f_{T}<0}\left(-f_{T}\right) \delta_{T \cup\{y\}}-\sum_{f_{T^{c}}<0}\left(-f_{T^{c}}\right) \delta_{T^{c} \cup\{y\}}\right)
$$

and $\beta^{*} F_{l, E} \cong R \alpha_{*}\left(N_{2}\right)$, where

$$
\begin{equation*}
N_{2}=g^{*} \omega_{\pi}^{\frac{e-l}{2}}(E) \cong \omega_{\alpha}^{\frac{e-l}{2}}\left(E+\sum_{T} f_{T} \delta_{T \cup\{y\}}\right)=N_{1}+S_{1}+S_{2} . \tag{8.3}
\end{equation*}
$$

Here $S_{1}=\sum_{f_{T}>0} f_{T} \delta_{T \cup\{y\}}, S_{2}=\sum_{f_{T}<0}\left(-f_{T^{c}}\right) \delta_{T^{c} \cup\{y\}}$ and the last congruence follows by the identities (5.3). Recall that the markings in $T$ get identified when applying $\beta$. We further break $S_{1}$ into $S_{1}^{\prime}+S_{1}^{\prime \prime}$, where $S_{1}^{\prime}$ (respectively
$S_{1}^{\prime \prime}$ ), contains the terms with $f_{T^{c}}>0$ (respectively $f_{T^{c}} \leq 0$ ). We add successively to $N_{1}$ first terms from $S_{1}^{\prime}$, then $S_{1}^{\prime \prime}$, followed by $S_{2}$, to get three types of quotients on $W$.

Type I has direct summands of the form: $Q_{1}^{\prime}=N_{1}\left(i \delta_{T \cup\{y\}}\right)_{\mid \delta_{T \cup\{y\}}} \cong$ $\omega_{\alpha}^{\frac{e-1}{2}}\left(E+i \delta_{T \cup\{y\}}\right)_{\mid \delta_{T \cup\{y\}}} \cong(u H) \boxtimes \mathcal{O}(-i)$, where $u:=f_{T}-i$, where $\delta_{T \cup\{y\}} \cong$ $\mathrm{Bl}_{1} \mathbb{P}^{r+s} \times \mathbb{P}^{r+s-1}$ is of the type appearing in $S_{1}^{\prime}: 0<i \leq f_{T}, f_{T^{c}}>0$ (for various $T$ with this property). Clearly, $0 \leq u<m$ and $0<i \leq m$. By Lemma [5.9, $R \alpha_{*}(u H)=\mathcal{O} \oplus \mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(u)$, and the result follows.
$\underline{\text { Type II }}$ has direct summands of the form: $Q_{1}^{\prime \prime}=N_{1}\left(i \delta_{T \cup\{y\}}\right)_{\mid \delta_{T \cup\{y\}}} \cong$ $\omega_{\alpha}^{\frac{e-l}{2}}\left(E+f_{T^{c}} \delta_{T^{c} \cup\{y\}}+i \delta_{T \cup\{y\}}\right)_{\mid \delta_{T \cup\{y\}}} \cong(u H-v \Delta) \boxtimes \mathcal{O}(-i)$, where $u:=$ $f_{T}-i, v=-f_{T^{c}}$, supported on $\delta_{T \cup\{y\}} \cong \mathrm{Bl}_{1} \mathbb{P}^{r+s} \times \mathbb{P}^{r+s-1}$ of the type appearing in $S_{1}^{\prime \prime}: 0<i \leq f_{T}, f_{T^{c}}<0$ (for various $T$ with this property). Clearly, we have $0 \leq u<m, 0<i \leq m$ and $0<v=-f_{T^{c}} \leq f_{T}=m$, since $f_{T}+f_{T^{c}}=l \geq 0$. By Lemma 5.9, if $u \geq v, R \pi_{*}(u H-v \Delta) \cong \mathcal{O}(v) \oplus \mathcal{O}(v+$ 1) $\oplus \ldots \oplus \mathcal{O}(u)$, while if $u<v$, we have nevertheless that $R \pi_{*}(u H-v \Delta)$ is either 0 or generated by $\mathcal{O}(u+1), \ldots, \mathcal{O}(v-1)$ and the result follows.
Type III has direct summands $Q_{2}=\left(N_{1}+S_{1}+i \delta_{T^{c \cup\{y\}}}\right)_{\mid \delta_{T^{c} \cup\{y\}}} \cong$ $\omega_{\alpha}^{\frac{e-l}{2}}\left(E+\left(f_{T^{c}}+i\right) \delta_{T^{c \cup\{ }\{y\}}+f_{T} \delta_{T \cup\{y\}}\right)_{\mid \delta_{T c \cup\{y\}}} \cong \mathcal{O}(u) \boxtimes(-i H+\beta \Delta)$, where $\beta=f_{T}, u=-f_{T^{c}}-i$, supported on $\delta_{T^{c} \cup\{y\}}=\delta_{T, T^{c} \cup\{y\}} \cong \mathbb{P}^{r+s-1} \times \mathrm{Bl}_{1} \mathbb{P}^{r+s}$ (of the type appearing in $S_{2}$ ): $0<i \leq-f_{T^{c}} \leq f_{T}=\beta$ since $f_{T}+f_{T^{c}}=l \geq 0$ (for various $T$ with this property). Clearly, $0<i \leq m, 0<\beta \leq m$, $0 \leq u<m$. Since $\beta \geq i$, it follows by Lemma 5.9 that $R \alpha_{*}(-i H+\beta \Delta) \cong$ $\mathcal{O}(-i) \oplus \mathcal{O}(-i-1) \oplus \ldots \oplus \mathcal{O}(-\beta)$, and (i) follows. Part (ii) follows from (i) by dualizing the triangles.
Corollary 8.6. Let $F_{l, E}$ be a bundle from groups $1 A$ or $1 B$ on $\overline{\mathrm{M}}_{p, q+1}$. Then sheaves $Q_{i}$ of Proposition [8.5 (i) belong to the subcategory $\mathcal{A}$ of Notation 1.6 with the possible exceptions of $Q_{i}=\mathcal{O}_{\delta_{T, T c}}\left(u,-\frac{r+s}{2}\right)\left(0 \leq u<\frac{r+s}{2}\right)$ on divisors $\delta_{T, T^{c}}$ contracted by $\beta$, with $r+s$ even and for the following subsets $T$ :

$$
\begin{array}{llll}
\text { - } l+e_{p}=r-1, & e_{q}=s+1, & E_{p} \subseteq T_{p}, & T_{q}=E_{q} \\
\text { - } e_{p}=r+1+l, e_{q}=s+1, & T_{p} \subseteq E_{p}, & T_{q}=E_{q} & (\text { group } 1 A) ; ~ \\
\text { en }
\end{array}
$$

Proof. Note that if $0<b \leq \frac{r+s-1}{2}$, then $\mathcal{O}(u,-b) \in \mathcal{A}$ for all $u$, positive or negative. In the notations of Proposition [8.5, we have by Lemma 4.5 that $m=m_{T, E, l} \leq \frac{r+s}{2}$. The Corollary now follows by Proposition 8.5(i) and Lemma 4.5 ,
Corollary 8.7. In the notations of Prop. 8.5. if $m \leq r+s-1$ (for example, if $\left.m \leq \frac{r+s}{2}\right)$, then on $\mathcal{U}$ we have: $F_{l, E} \cong\left[R \beta_{*} F_{l, E}^{\vee}\right]^{\vee}$.
Proof. Since $R \beta_{*} \mathcal{O}_{\delta_{T, T^{c}}}(-u-1, j-1)=0$ for $0 \leq u \leq r+s-2$ if the divisor is contracted, we have Proposition8.5(ii) that $R \beta_{*} F_{l, E}^{\vee} \cong R \beta_{*} \beta^{*} F_{l, E}^{\vee} \cong F_{l, E}^{\vee}$.

Corollary 8.8. (Beta game) Let $F_{l, E}$ and $F_{l^{\prime}, E^{\prime}}$ be bundles from group $1 A$ (resp., group $1 B$ ) on $\overline{\mathrm{M}}_{p, q+1}$. Suppose $\left|E_{q}\right| \leq\left|E_{q}^{\prime}\right|$. Then

$$
R \operatorname{Hom}_{\bar{M}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong R \operatorname{Hom}_{\mathcal{U}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)
$$

unless $r+s$ is even and we have one of the following exceptions, which include existence of a critical subset $T=T_{p} \coprod T_{q}$ split into heavy and light markings
(1A) $l+e_{p}=l^{\prime}+e_{p}^{\prime}=r-1, \quad e_{q}=e_{q}^{\prime}=s+1, \quad E_{p} \subseteq T_{p}, \quad E_{q}=T_{q} \quad$ and either $E_{p}^{\prime} \subseteq T_{p}, \quad E_{q}^{\prime}=T_{q} \quad$ or $E_{p}^{\prime} \subseteq T_{p}^{c}, \quad E_{q}^{\prime}=T_{q}^{c}$;
(1B) $e_{p}=r+1+l, e_{p}^{\prime}=r+1+l^{\prime}, e_{q}=e_{q}^{\prime}=s+1, \quad T_{p} \subseteq E_{p}, \quad T_{q}=E_{q}$ and either $T_{p} \subseteq E_{p}^{\prime} \quad T_{q}=E_{q}^{\prime} \quad$ or $T_{p}^{c} \subseteq E_{p}^{\prime} \quad T_{q}^{c}=E_{q}^{\prime}$.

Proof. We have $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, \beta^{*}\left(F_{l, E}\right)\right)=R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(\beta^{*}\left(F_{l, E}^{\vee}\right), F_{l^{\prime}, E^{\prime}}^{\vee}\right)$, which by adjointness is the same as $R \operatorname{Hom}_{\mathcal{U}}\left(F_{l, E}^{\vee}, \beta_{*} F_{l^{\prime}, E^{\prime}}^{\vee}\right)$, and the latter is zero by Corollary 8.7, since $m \leq \frac{r+s}{2}$ by Lemma 4.5. It remains to show that $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, \beta^{*} F_{l, E}\right)$. By Proposition 8.5 and Corollary 8.6, the bundles $F_{l, E}$ and $\beta^{*} F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+1}$ are related by sheaves in $\mathcal{A}$, unless $F_{l, E}$ is listed in Corollary 8.6, Since the bundles $F_{l^{\prime}, E^{\prime}}$ are perpendicular to $\mathcal{A}$ by Proposition 6.1, we only need to show that $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, \mathcal{O}_{\delta_{T}}(-a,-b)\right)=0$, where $a$ is arbitrary, $b=\frac{r+s}{2}$. By Proposition 6.1 (with $\mu=\frac{r+s}{2}-1$ ), this holds unless $\left|f_{T, E^{\prime}, l^{\prime}}\right|$ or $\left|f_{T^{c}, E^{\prime}, l^{\prime}}\right|$ equals $\frac{r+s}{2}$. This means that $\left(l^{\prime}, E^{\prime}\right)$ should be listed in Lemma 4.5, with a critical subset either $T$ or $T^{c}$.

We now specialize to $\mathcal{U}=\mathcal{U}_{p, q}$ of Lemma 8.3. Recall that $\Sigma=P \cup Q \cup\{z\}$, and we denote as usual $E_{p}=E \cap P, E_{q}=E \cap(Q \cup\{z\})$.

Lemma 8.9. Assume $s \geq 0$. Consider the following collections of vector bundles $F_{l, E}$ on $\mathcal{U}_{p, q}$ (see Definition 3.1), where we assume that $l+e$ is even:

$$
\begin{array}{ll}
l+\min \left(e_{p}, p+1-e_{p}\right) \leq r-1 & (\text { group } 1 A) \\
l+\min \left(e_{p}+1, p-e_{p}\right) \leq r-1 & (\text { group } 1 B)
\end{array}
$$

The vector bundles from group $1 A$ (resp., 1B) form an exceptional collection on $\mathcal{U}_{p, q}$. The order is as follows: we put all subsets $E$ containing $z$ last and before them all subsets not containing $z$. Within each of these two blocks, we order first by increasing $e_{q}$, arbitrarily if $e_{q}$ is the same but $E_{q}$ 's are different, if $E_{q}$ 's are the same, in order of increasing $e_{p}$, and for a given $e_{p}$, arbitrarily.

Proof. It follows from Theorem 3.4 that we have
(i) If $z \notin E$, then $F_{l, E} \cong \pi^{*} F_{l, E}$;
(ii) If $z \in E, F_{l, E} \cong F_{l, E^{c}}^{\vee} \otimes F_{0, \Sigma} \cong \pi^{*} F_{l, E^{c}}^{\vee} \otimes F_{0, \Sigma}$.

In other words, the part of the collection with $z \notin E$ is identical to the one in Theorem 1.5 (and Remark 1.11), while the part with $z \in E$ corresponds to vector bundles $F_{l, E} \cong F_{l, E^{c}}^{\vee} \otimes F_{0, \Sigma}$ with the collection $\left\{F_{l, E^{c}}^{\vee}\right\}$ being the dual collection to the one in Theorem 1.5 (and Remark 1.11). By Orlov's theorem on the derived category of a projective bundle applied to $\pi: \mathcal{U}_{p, q} \rightarrow \overline{\mathrm{M}}_{p, q}$, we have that $D^{b}\left(\mathcal{U}_{p, q}\right)=\left\langle D^{b}\left(\overline{\mathrm{M}}_{p, q}\right), D^{b}\left(\overline{\mathrm{M}}_{p, q}\right) \otimes F_{0, \Sigma}\right\rangle$, since $F_{0, \Sigma}=\mathcal{O}_{\pi}(1) \otimes \pi^{*} L$, for some line bundle $L$ on $\overline{\mathrm{M}}_{p, q}$. This is because by Theorem 3.4 we have $R \pi_{*}\left(F_{0, \Sigma}^{\vee}\right)=0$. Now use the collection $F_{l, E}$ of Theorem 1.5 for the block $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$ and the dual collection $F_{l, E}^{\vee}$ (tensored with $\left.F_{0, \Sigma}\right)$ for the block $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right) \otimes F_{0, \Sigma}$.

Proof of Theorem 8.1 We will compare the vector bundles $F_{l, E}$ on $\mathcal{U}_{p, q}$ and $\overline{\mathrm{M}}_{p, q}$. The range for the pairs $(l, E)$ for collections $1 A$ and $1 B$ on $\overline{\mathrm{M}}_{p, q+1}$ in Thm. 1.8 is precisely the range for these collections on $\mathcal{U}_{p, q}$ in Lemma 8.9 .

Consider first the case $s \geq 0$. Let $F_{l, E}, F_{l^{\prime}, E^{\prime}}$ be bundles from one of the two collections (either $1 A$ or $1 B$ ) on $\overline{\mathrm{M}}_{p, q+1}$ (in the correct order). If $E_{q} \neq$ $E_{q^{\prime}}^{\prime}$ choose $z \in E_{q}^{\prime} \backslash E_{q}$. By Cor. 8.8, we have $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong$ $R \operatorname{Hom}_{\mathcal{U}_{p, q},}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$, since the only exceptions happen when $E_{q}=E_{q}^{\prime}$ or $E_{q}$ and $E_{q}^{\prime}$ are disjoint and have $s+1$ elements, but then $z$ must be in both $E_{q}$ and $E_{q}^{\prime}$ (use that $z \in T$ ). By Lemma 8.9 , we have $R \operatorname{Hom}_{\mathcal{U}_{p, q}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)=0$.

Suppose now that $E_{q}=E_{q}^{\prime}$. If $e_{q}<q+1$, let $z$ be any light index not in $E_{q}$, while if $e_{q}=q+1$, let $z$ be any light index. Then Cor. 8.8 applies: $R \operatorname{Hom}_{\bar{M}_{p, q+1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong R \operatorname{Hom}_{\mathcal{U}_{p, q}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$, since the only exception for $E_{q}=E_{q}^{\prime}$ is when $e_{q}=s+1$ and $E_{q}$ contains $z$. The latter group is equal to 0 or $\mathbb{C}$ when $(l, E)=\left(l^{\prime}, E^{\prime}\right)$, by Lemma 8.9 ,

Consider now the case $s=-1$. In this case $E:=E_{p}, E^{\prime}:=E_{p}^{\prime}$ are subsets of $P$. We first prove the statement for group $1 A$. Note that if $(l, E)$ is in group $1 A$ on $\overline{\mathrm{M}}_{p}$, it is also in group $1 A$ on $\overline{\mathrm{M}}_{p, 1}$, and in the collection on $\overline{\mathrm{M}}_{p+1}$. Consider the set-up in Section 7 let $\alpha: W \rightarrow \overline{\mathrm{M}}_{p}$ be the universal family and let $f: W \rightarrow \overline{\mathrm{M}}_{p, 1} \cong \overline{\mathrm{M}}_{p+1}$ be the birational morphism that contracts the boundary divisors $\delta_{T \cup\{y\}, T^{c}}$ by identifying the points in $T^{c}$. Using that $F_{l, E} \cong \alpha^{*} F_{l, E}$ (Lemma 5.17), $R \operatorname{Hom}_{\bar{M}_{p}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong$ $R \operatorname{Hom}_{W}\left(\alpha^{*} F_{l^{\prime}, E^{\prime}}, \alpha^{*} F_{l, E}\right) \cong \operatorname{RHom}_{W}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$, which is isomorphic to $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, 1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$ by Lemma 7.6. As $\overline{\mathrm{M}}_{p+1} \cong \overline{\mathrm{M}}_{p, 1}$ and the two spaces have the same universal family, we have $R \operatorname{Hom}_{\bar{M}_{p, 1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)=0$ if $e^{\prime} \geq$ $e,(l, E) \neq\left(l^{\prime}, E^{\prime}\right)$, and $\mathbb{C}$ when $(l, E)=\left(l^{\prime}, E^{\prime}\right)$, by Theorem 1.2.

We now prove the statement for group $1 B$ when $s=-1$. Recall that $z \in P, \overline{\mathrm{M}}_{p,-1}=\overline{\mathrm{M}}_{p-1}$ and $\pi: \mathcal{U}_{p,-1}=\mathcal{U}=\overline{\mathrm{M}}_{p-1,1} \rightarrow \overline{\mathrm{M}}_{p,-1}$ is the universal family. Note, the previous proof works only if none of $(l, E),\left(l^{\prime}, E^{\prime}\right)$ satisfy $l+(p-e)=r-1$. We therefore proceed with a different proof for group 1B. As in the case $q>0$, by Cor. 8.8, if $z \notin E$ we have $R \operatorname{Hom}_{\bar{M}_{p}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong$ $R \operatorname{Hom}_{\bar{M}_{p-1,1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$, since the only exceptions for group $1 B$ happen when $z \in E$. If $E^{\prime} \backslash E \neq \emptyset$, we pick $z \in E^{\prime} \backslash E$. Since $F_{l, E} \cong \pi^{*} F_{l, E}$ and $R \pi_{*}\left(F_{l^{\prime}, E^{\prime}}^{\vee}\right)=0$ by Theorem 3.4, we have $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p-1,1}}\left(F_{l, E}^{\vee}, F_{l^{\prime}, E^{\prime}}^{\vee}\right) \cong$ $R \operatorname{Hom}_{\bar{M}_{p-1,1}}\left(\pi^{*} F_{l, E}^{\vee}, F_{l^{\prime}, E^{\prime}}^{\vee}\right) \cong R \operatorname{Hom}_{\overline{\mathrm{M}}_{p-1}}\left(F_{l, E}^{\vee}, R \pi_{*}\left(F_{l^{\prime}, E^{\prime}}^{\vee}\right)\right)=0$. If $E^{\prime} \subseteq E$, since $e^{\prime} \geq e$, we must have $E^{\prime}=E$. If $E \neq \Sigma=P$, let $z \in \Sigma \backslash E$. Since $F_{l, E} \cong \pi^{*} F_{l, E}, F_{l^{\prime}, E^{\prime}} \cong \pi^{*} F_{l^{\prime}, E^{\prime}}$ we have $R \operatorname{Hom}_{\bar{M}_{p-1,1}}\left(F_{l, E}^{\vee}, F_{l^{\prime}, E^{\prime}}^{\vee}\right) \cong$ $R \operatorname{Hom}_{\bar{M}_{p-1}}\left(F_{l, E}^{\vee}, F_{l^{\prime}, E^{\prime}}^{\vee}\right)$. As $(l, E)$ is in group $1 B$, we have $l+\min \{e+1, p-$ $e\} \leq r-1$, i.e., $l+\min \{e,(p-1)-e\} \leq r-2$, which is the range of pairs $(l, E)$ in Theorem 1.2 with $E \subseteq P \backslash\{z\}$. The result follows in this case from Theorem 1.2

If $E=E^{\prime}=\Sigma$, by Cor. 8.8, we still have $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong$ $R \operatorname{Hom}_{\bar{M}_{p-1,1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$, unless $l=l^{\prime}=r-1$ (and $r$ is odd since $l+e$ is even). Assume this is not the case. We have $R \operatorname{Hom}_{\bar{M}_{p-1,1}}\left(F_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong$ $R \operatorname{Hom}_{\bar{M}_{p-1,1}}\left(F_{l^{\prime}, \emptyset}^{\vee}, F_{l, \emptyset}^{\vee}\right) \cong R \operatorname{Hom}_{\overline{\mathrm{M}}_{p-1}}\left(F_{l, \emptyset}, F_{l^{\prime}, \emptyset}\right)$ (use that $F_{l, E}=F_{l, E^{c}}^{\vee} \otimes F_{0, \Sigma}$ on $\overline{\mathrm{M}}_{p-1,1}$ by Corollarry 3.3 and $F_{l, \emptyset}=\pi^{*} F_{l, \emptyset}$ ) and we are done by Theorem
1.2 It remains to prove that the vector bundle $F_{r-1, \Sigma}(r$ odd) is exceptional on $\overline{\mathrm{M}}_{p}$. By Lemma5.12, we have that $F_{r-1, \Sigma} \cong F_{r-1, \emptyset} \otimes F_{0, \Sigma}$. Hence, it suffices to prove that $F_{r-1, \emptyset}$ is exceptional on $\overline{\mathrm{M}}_{p}$. But this bundle is in group 1 A , so we are done.

## 9. EQUIVARIANT EXCEPTIONAL COLLECTION ON $\overline{\mathrm{M}}_{2 r, 2 s+1}$

Notation 9.1. Throughout this section, let $|P|=p=2 r,|Q|=q=2 s+1$ $(r \geq 2, s \geq 0)$. Let $P=\{1, \ldots, p\}, R=\{1, \ldots, r\}, R^{\prime}=P \backslash R$. Let $Z_{R}$ be the locus in $\overline{\overline{\mathrm{M}}}_{p, q}$ where points in $R$ come together: $Z_{R}=\delta_{12} \cap \delta_{13} \cap \ldots \cap \delta_{1 r} \cong$ $\overline{\mathrm{M}}_{\{u\} \cup R^{\prime} \cup Q}$, where $u$ is the marking corresponding to the combined points in $R$. Note that $u$ can only coincide with at most $s$ light points and with none of the points in $R^{\prime}$. In $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$, we have a Koszul resolution

$$
\begin{equation*}
\mathcal{O}_{Z_{R}} \cong\left[\ldots \rightarrow \Lambda^{2} \bigoplus_{i=2}^{r} \mathcal{O}\left(-\delta_{1 i}\right) \rightarrow \bigoplus_{i=2}^{r} \mathcal{O}\left(-\delta_{1 i}\right) \rightarrow \mathcal{O}_{\overline{\mathrm{M}}_{p, q}}\right] \tag{9.1}
\end{equation*}
$$

Let $Y=Z_{R} \cap Z_{R^{\prime}} \cong \overline{\mathrm{M}}_{\{u, v\} \cup Q}$, where $u$ (resp., $v$ ) is the marking corresponding to $R$ (resp., $R^{\prime}$ ). We have a Cartesian diagram of embeddings


A torsion sheaf $\mathcal{T}_{l, E}$ (see Notation 1.4) has disjoint support from, and is therefore perpendicular to, any sheaf $\mathcal{T}_{l^{\prime}, E^{\prime}}$ unless $E_{p}=E_{p}^{\prime}$ or $E_{p} \cap E_{p}^{\prime}=\emptyset$. So semi-orthogonality of objects in Theorem 1.5 follows from Theorem 8.1 and the following theorem, which takes care of the torsion sheaves:

Theorem 9.2. Assume $p=2 r, q=2 s+1$. In the notations of Thm. 1.5.
(1) If $(l, E)$ is in group $1 A$ (resp., 1B) and $\left(l^{\prime}, E^{\prime}\right)$ is in group 2, we have $R \operatorname{Hom}_{\bar{M}_{p, q}}\left(F_{l, E}, \mathcal{T}_{l^{\prime}, E^{\prime}}\right)=0$ if $e_{q} \geq e_{q}^{\prime}$.
(2) If $(l, E)$ is in group 2 and $\left(l^{\prime}, E^{\prime}\right)$ is in group $1 A$ (resp., $1 B$ ), we have $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q}}\left(\mathcal{T}_{l, E}, F_{l^{\prime}, E^{\prime}}\right)=0$ if $e_{q}>e_{q}^{\prime}$ or if $e_{q}=e_{q^{\prime}}^{\prime} E_{q} \neq E_{q}^{\prime}$.
(3) $\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q}}\left(\mathcal{T}_{l, E}, \mathcal{T}_{l^{\prime}, E^{\prime}}\right)=0$ if $(l, E),\left(l^{\prime}, E^{\prime}\right)$ are both in group 2, $E_{p}=$ $E_{p}^{\prime}=R, e_{q} \geq e_{q}^{\prime}, E_{q} \neq E_{q}^{\prime}$ or $E_{q}=E_{q}^{\prime}, l<l^{\prime}$. Also, $R \operatorname{Hom}_{\bar{M}_{p, q}}\left(\mathcal{T}_{l, E}, \mathcal{T}_{l, E}\right)=\mathbb{C}$.
(4) If $(l, E),\left(l^{\prime}, E^{\prime}\right)$ are both in group 2 and $E_{p}=R, E_{p}^{\prime}=R^{\prime}$, we have $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q}}\left(\mathcal{T}_{l, E}, \mathcal{T}_{l^{\prime}, E^{\prime}}\right)=0$ if $e_{q} \geq e_{q}^{\prime}$.

We start by computing some tautological classes of $Z_{R}$.
Lemma 9.3. (i) $\psi_{j \mid Z_{R}} \sim \psi_{u}$ if $j \in R,-\psi_{u}$ if $j \in R^{\prime}$, and $-\psi_{u}-2 \delta_{j u}$ if $j \in Q$. (ii) $\delta_{i j \mid Z_{R}} \sim-\psi_{u}$ if $i, j \in R, \psi_{u}$ if $i, j \in R^{\prime}, 0$ if $i \in R, j \in R^{\prime}$, and $\delta_{i j}$ if $\{i, j\} \cap Q \neq \emptyset$ (where if $j \in R, i \in Q$, we identify $\delta_{i j}=\delta_{i u}$ ). Furthermore, (iii) On $Y$ we have $\psi_{u}=-\psi_{v}$. (iv) $K_{\mid Z_{R}} \sim K_{Z_{R}}+(r-1) \psi_{u}$, where $K=K_{\overline{\mathrm{M}}_{p, q}}$ is the canonical class of $\overline{\mathrm{M}}_{p, q}$. (v) If $E_{p}=R$ then $\mathcal{T}_{l, E}=i_{R *} \sigma_{u}^{*}\left(\omega_{\pi_{R}}^{\frac{e-l}{-2}}(E)\right) \cong$ $i_{R *}\left(\frac{e_{q}-r-l}{2} \psi_{u}+\sum_{j \in E_{q}} \delta_{j u}\right)=i_{R *}\left(-\frac{r+l}{2} \psi_{u}-\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}\right)$.

Proof. It follows from the definition that the restriction of $\psi_{j}$ to $Z_{R}$ is the class $\psi_{j}$ on the corresponding Hassett space for all $j$. Note that the universal family over $Z_{R}$ is a $\mathbb{P}^{1}$-bundle. It follows from [CT20b, Lemma 2.1] that $\psi_{i}+\psi_{j} \sim-2 \delta_{i j}$. Since $\delta_{u j}=\emptyset$ if $j \in R^{\prime}$, parts (i), (ii) follow. If $j \in R$, we have by (i) that $\psi_{j \mid Z_{R}}=\psi_{u}$, hence, $\psi_{j \mid Y}=\psi_{u}$, and similarly, $\psi_{j \mid Z_{R^{\prime}}}=-\psi_{v}$, $\psi_{j \mid Y}=-\psi_{v}$, which implies part (iii). Part (iv) follows from adjunction since $c_{1}\left(N_{Z_{R} / \overline{\mathrm{M}}_{p, q}}\right) \sim \sum_{j=2}^{r}\left(\delta_{1 j}\right)_{\mid Z_{R}} \sim-(r-1) \psi_{u}$. Part (v) follows from definition of $\mathcal{T}_{l, E}$ (Notation 1.4), previous parts and the fact that $\sigma_{u}^{*} \sigma_{u}=-\psi_{u}$.

We reduce Theorem 9.2 to a calculation of $R \Gamma$ on $Z_{R}$ or $Y$ :
Lemma 9.4. Theorem 9.2 follows from (1)-(4) below, where we abuse notation and denote by $\mathcal{T}_{l, E}$ both a line bundle on $Z_{E_{p}}$ and its pushforward to $\overline{\mathrm{M}}_{p, q}$.
(1) $R \Gamma\left(Z_{R},\left(F_{l, E \mid Z_{R}}\right)^{\vee} \otimes \mathcal{T}_{l^{\prime}, E^{\prime}}\right)=0$ if $(l, E)$ is in group $1 A$ (resp., $1 B$ ), ( $l^{\prime}, E^{\prime}$ ) is in group $2, e_{q} \geq e_{q}^{\prime}$, and $R=E_{p}^{\prime}$.
(2) $R \Gamma\left(Z_{R}, \mathcal{T}_{l, E}^{\vee} \otimes F_{l^{\prime}, E^{\prime} \mid Z_{R}} \otimes c_{1}\left(N_{Z_{R} \mid \overline{\mathrm{M}}_{p, q}}\right)\right)=0$ if $(l, E)$ is in group 2, $\left(l^{\prime}, E^{\prime}\right)$ is in group $1 A$ (resp., $1 B$ ) if $R=E_{p}$ and either $e_{q}>e_{q}^{\prime}$ or $e_{q}=e_{q^{\prime}}^{\prime}$ $E_{q} \neq E_{q}^{\prime}$.
(3) Suppose $(l, E),\left(l^{\prime}, E^{\prime}\right)$ are both in group $2, E_{p}=E_{p}^{\prime}=R$, and $J \subseteq R \backslash\{1\}$. Then $R \Gamma\left(Z_{R}, \mathcal{T}_{l, E}^{\vee} \otimes \mathcal{T}_{l^{\prime}, E^{\prime}} \otimes\left(\sum_{j \in J} \delta_{1 j}\right)\right)=0$ if $e_{q} \geq e_{q^{\prime}}^{\prime} E_{q} \neq E_{q^{\prime}}^{\prime}$ or if $E_{q}=E_{q^{\prime}}^{\prime}$ $l<l^{\prime}$. Furthermore, $R \Gamma\left(Z_{R}, \mathcal{T}_{l, E}^{\vee} \otimes \mathcal{T}_{l, E} \otimes\left(\sum_{j \in J} \delta_{1 j}\right)\right)=0$ if $J \neq \emptyset$.
(4) $R \Gamma\left(Y,\left(\frac{l+l^{\prime}}{2}+1\right) \psi_{u}+\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}-\frac{1}{2} \sum_{j \in E_{q}^{\prime}} \psi_{j}\right)=0$ if $(l, E),\left(l^{\prime}, E^{\prime}\right)$ are both in group 2, $E_{p}=R, E_{p}^{\prime}=R^{\prime}$, and $e_{q} \geq e_{q}^{\prime}$.
Proof. (1) $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q}}\left(F_{l, E}, \mathcal{T}_{l^{\prime}, E^{\prime}}\right) \cong R \Gamma\left(Z_{R},\left(F_{l, E \mid Z_{R}}\right)^{\vee} \otimes \mathcal{T}_{l^{\prime}, E^{\prime}}\right)$.
(2) Using that $i^{i} F_{l^{\prime}, E^{\prime}} \cong i^{*} F_{l^{\prime}, E^{\prime}} \otimes c_{1}\left(N_{Z_{R} \mid \overline{\mathrm{M}}_{p, q}}\right)$, we have (up to a shift) that $R \operatorname{Hom}_{\bar{M}_{p, q}}\left(\mathcal{T}_{l, E}, F_{l^{\prime}, E^{\prime}}\right) \cong R \operatorname{Hom}_{Z_{R}}\left(\mathcal{T}_{l, E}, F_{l^{\prime}, E^{\prime} \mid Z_{R}} \otimes c_{1}\left(N_{Z_{R} \mid \bar{M}_{p, q}}\right)\right) \cong$ $R \Gamma\left(Z_{R}, \mathcal{T}_{l, E}^{\vee} \otimes F_{l^{\prime}, E^{\prime} \mid Z_{R}} \otimes c_{1}\left(N_{Z_{R} \mid \overline{\mathrm{M}}_{p, q}}\right)\right)$ (Huy06, Cor. 3.38]).
(3) Follows from tensoring the Koszul resolution (9.1) of $Z_{R}$ with a line bundle $L$ on $\overline{\mathrm{M}}_{p, q}$ such that $i_{*}\left(L_{\mid Z_{R}}\right) \cong \mathcal{T}_{l, E}$ and applying $R \operatorname{Hom}\left(-, \mathcal{T}_{l^{\prime}, E^{\prime}}\right)$.
(4) For any line bundles $L$ on $Z_{R}$ and $L^{\prime}$ on $Z_{R^{\prime}}$, by cohomology and base change applied to the Tor-independent diagram (9.2), $R \operatorname{Hom}\left(i_{*} L, i_{*}^{\prime} L^{\prime}\right)=$ $R \operatorname{Hom}\left(L i^{*} i_{*} L, L^{\prime}\right) \cong R \operatorname{Hom}\left(R j_{*}^{\prime} j^{*} L, L^{\prime}\right) \cong R \operatorname{Hom}_{Y}\left(j^{*} L, j^{\prime!} L^{\prime}\right)$. It follows that $R \operatorname{Hom}_{\bar{M}_{p, q}}\left(\mathcal{T}_{l, E}, \mathcal{T}_{l^{\prime}, E^{\prime}}\right) \cong R \operatorname{Hom}_{Y}\left(\mathcal{T}_{l, E \mid Y}, \mathcal{T}_{l^{\prime}, E^{\prime} \mid Y} \otimes c_{1}\left(N_{Y \mid Z_{R^{\prime}}}\right)\right) \cong$ $R \Gamma\left(Y,\left(\mathcal{T}_{l, E \mid Y}\right)^{\vee} \otimes \mathcal{T}_{l^{\prime}, E^{\prime} \mid Y} \otimes\left(\sum_{j \in R \backslash\{1\}} \delta_{1 j}\right)_{\mid Y}\right)$. By Lemma 0.3, this complex is isomorphic to $R \Gamma\left(Y,\left(\frac{l+l^{\prime}}{2}+1\right) \psi_{u}+\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}-\frac{1}{2} \sum_{j \in E_{q}^{\prime}} \psi_{j}\right)$.

We prove the vanishing in Lemma 9.4 by windows calculations on $Z_{R}$ and $Y$.

Notation 9.5. $Z_{R}$ is isomorphic to a GIT quotient of $X=\mathbb{P}_{u}^{1} \times\left(\mathbb{P}^{1}\right)^{r} \times\left(\mathbb{P}^{1}\right)^{q}$ by $\mathrm{PGL}_{2}$, corresponding to the partition $\{u\} \sqcup R^{\prime} \sqcup Q$ of the markings on $Z_{R}=\overline{\mathrm{M}}_{\{u\} \cup R^{\prime} \cup Q}$. For $a \in \mathbb{Z}, A \subseteq R^{\prime}, B \subseteq Q$, consider on $X$ the line bundle $\mathcal{O}(a, A, B):=p r_{1}^{*} \mathcal{O}(a) \otimes p r_{R^{\prime}}^{*} \mathcal{O}(A) \otimes p r_{Q}^{*} \mathcal{O}(B)$ where we denote $\mathcal{O}(A)=\mathcal{O}\left(i_{1}, \ldots, i_{r}\right)$ with $i_{j}=1$ if $j \in A$ and 0 otherwise, and likewise for $\mathcal{O}(B)$. We denote by $\mathcal{O}(-A)$ the dual of $\mathcal{O}(A)$. We have the following
correspondence between vector bundles on $Z_{R}$ and $\mathrm{PGL}_{2}$-equivariant vector bundles on $X=\left(\mathbb{P}^{1}\right)^{r+q+1}: F_{l, E Z_{R}} \cong \mathcal{O}\left(\left|E_{p} \cap R\right|, E_{p} \cap R^{\prime}, E_{q}\right) \otimes V_{l}$ (by Lemma 3.2), $\mathcal{T}_{l, E} \cong \mathcal{O}\left(r+l, 0, E_{q}\right)$ if $E_{p}=R, \omega_{Z_{R}} \cong \mathcal{O}(-2,-2, \ldots,-2)$. Recall from Section 3 (see Remark (3.8) that $\psi_{i} \cong \mathcal{O}(0, \ldots, 0,-2,0, \ldots, 0)$ (with -2 in position $i$ ) and $\delta_{i j} \cong \mathcal{O}(0, \ldots 0,1,0, \ldots, 0,1,0, \ldots, 0)$ (with 1 in positions $i$ and $j$ ). Likewise, $Y$ is isomorphic to the GIT quotient of $X^{\prime}=\left(\mathbb{P}^{1}\right)^{q+2}=\mathbb{P}_{u}^{1} \times \mathbb{P}_{v}^{1} \times\left(\mathbb{P}^{1}\right)^{q}$, corresponding to the partition $\{u\} \sqcup\{v\} \sqcup Q$ of the markings on $\overline{\mathrm{M}}_{\{u, v\} \cup Q}$.

Instead of Thm. 3.13 we are going to use a related Theorem 9.6 proved for vector bundles by Teleman [Tel00]. The calculation of the Kempf-Ness stratification and the weights is like in Section 3. However, from now on, we will follow a convention of [Tel00] and take an opposite weight to (3.2):

$$
\begin{equation*}
\text { weight }_{\lambda} \mathcal{O}_{X}\left(a_{1}, \ldots, a_{k}\right)_{\mid z_{K}}=\sum_{i \in K} a_{i}-\sum_{i \in K^{c}} a_{i} \tag{9.3}
\end{equation*}
$$

We find this convention more natural since the ample polarization of the GIT quotient has positive weights on the unstable locus. Then we have

Theorem 9.6. [HL15, Th. 3.29] For any object $F \in D^{b}[X / G]$ such that

$$
\begin{equation*}
\mathcal{H}^{*}\left(\sigma_{i}^{*} F\right) \text { has weights }>-\eta_{i} \tag{9.4}
\end{equation*}
$$

for every Kempf-Ness stratum, we have $R \Gamma_{[X / G]}(F) \simeq R \Gamma_{\left[X^{s s} / G\right]}(F)$.
Remark 9.7. (The devil's trick) The line bundle $\mathbb{D}:=\mathcal{O}\left(r, R^{\prime}, 0\right)$ on $X$ is trivial on $Z_{R}$, since for any $j \in R^{\prime}$, on $Z_{R}=\overline{\mathrm{M}}_{\{u\} \cup R^{\prime} \cup Q}$ we have $\delta_{u j}=\emptyset$, so the line bundle $\mathcal{O}(1,0, \ldots, 0,1,0, \ldots, 0)$ (with 1 in the positions of $u$ and $j$ ) descends to the trivial line bundle on $Y$. Likewise, the line bundle $\mathbb{D}:=\mathcal{O}(1,1,0)$ on $X^{\prime}$ descends to the trivial line bundle on $Y$. Instead of proving vanishing in Lemma 9.4 directly, we will prove vanishing on $X$ (resp., $X^{\prime}$ ) after tensoring with a high multiple of $\mathbb{D}$ since it's this tensor product that will satisfy conditions of Theorem 9.6. This is a useful observation for any GIT quotient $X / / G$ such that the unstable locus contains a divisorial component.

Proof of Thm 9.2 We prove the vanishings in Lemma 9.4. First we prove the $\mathrm{PGL}_{2}$-invariant vanishing on $X=\left(\mathbb{P}^{1}\right)^{1+r+q}$ (cases (1), (2) and (3)) and on $X^{\prime}=\left(\mathbb{P}^{1}\right)^{2+q}$ (case (4)) after tensoring a vector bundle with the devil line bundle $\mathbb{D}^{N}, N \gg 0$. Later on we check the weight condition (9.4).

For (1), assuming condition (9.4), $R \Gamma\left(Z_{R},\left(F_{l, E}{\mid Z_{R}}\right)^{\vee} \otimes \mathcal{T}_{l^{\prime}, E^{\prime}} \otimes \mathbb{D}^{N}\right) \cong$ $R \Gamma\left(X, \mathcal{O}\left(r+l^{\prime}-\left|E_{p} \cap R\right|+N r,-E_{p} \cap R^{\prime}+N R^{\prime}, E_{q}^{\prime}-E_{q}\right) \otimes V_{l}\right)^{\mathrm{PGL}_{2}}$, which is clearly 0 if $E_{q} \nsubseteq E_{q}^{\prime}$. Since here we assume $e_{q} \geq e_{q}^{\prime}$, we have that $E_{q} \subseteq E_{q}^{\prime}$ if and only if $E_{q}=E_{q}^{\prime}$. Assume $E_{q}=E_{q}^{\prime}$. In this case, the $\mathrm{SL}_{2}$-module is equal to $V_{r+l^{\prime}-\left|E_{p} \cap R\right|+N r} \otimes V_{N-y_{1}} \otimes \ldots \otimes V_{N-y_{r}} \otimes V_{l}$, where $y_{i}=1$ if the corresponding index in $E_{p} \cap R^{\prime}$ and 0 otherwise. We claim that the $\mathrm{PGL}_{2}{ }^{-}$ invariant part is 0 . Since $(l, E)$ is in group $1 A$ or $1 B$, we have $l+\mid E_{p} \cap$ $R\left|-\left|E_{p} \cap R^{\prime}\right|<r\right.$ by Cor. 4.9. It follows that $r+l^{\prime}-\left|E_{p} \cap R\right|+N r>$ $N r-\left|E_{p} \cap R^{\prime}\right|+l$, which implies the vanishing by the Clebsch-Gordan formula (Lemma 3.7).

For (2), assuming (9.4), $R \Gamma\left(Z_{R}, \mathcal{T}_{l, E}^{\vee} \otimes F_{l^{\prime}, E^{\prime} \mid Z_{R}} \otimes c_{1}\left(N_{Z_{R} \mid \overline{\mathrm{M}}_{p, q}}\right) \otimes \mathbb{D}^{N}\right) \cong$ $R \Gamma\left(X, \mathcal{O}\left(r-l+\left|E_{p}^{\prime} \cap R\right|-2+N r, E_{p}^{\prime} \cap R^{\prime}+N R^{\prime}, E_{q}^{\prime}-E_{q}\right) \otimes V_{l^{\prime}}\right)^{\mathrm{PGL}_{2}}$ (use $\left.c_{1}\left(N_{Z_{R} \mid \overline{\mathrm{M}}_{p, q}}\right)=\mathcal{O}(2 r-2,0,0)\right)$. This is 0 since $E_{q} \nsubseteq E_{q}^{\prime}$.

For (3), assuming condition (9.4), $R \Gamma\left(Z_{R}, \mathcal{T}_{l, E}^{\vee} \otimes \mathcal{T}_{l^{\prime}, E^{\prime}} \otimes \sum_{j \in J} \delta_{1 j} \otimes \mathbb{D}^{N}\right) \cong$ $R \Gamma\left(X, \mathcal{O}\left(2|J|-l+l^{\prime}+N r,+N R^{\prime}, E_{q}^{\prime}-E_{q}\right)\right)^{\mathrm{PGL}_{2}}$ for all $J \subseteq R \backslash\{1\}$. This is 0 if $e_{q} \geq e_{q}^{\prime}, E_{q} \nsubseteq E_{q}^{\prime}$, while if $E_{q}=E_{q}^{\prime}$ the $\mathrm{PGL}_{2}$-invariant part is 0 when $l<l^{\prime}$ or when $l=l^{\prime},|J|>0$ by the Clebsch-Gordan formula (Lemma 3.7), since in these cases we have $2|J|-l+l^{\prime}+N r>N r$.

For (4), assuming (9.4), the question is equivalent to the vanishing for $N \gg 0$ of the $\mathrm{PGL}_{2}$-invariant part of $R \Gamma\left(\mathcal{O}\left(-l-l^{\prime}-2+N, N, E_{q}^{\prime}-E_{q}\right)\right)$. This is clear if $e_{q} \geq e_{q}^{\prime}, E_{q} \nsubseteq E_{q}^{\prime}$. If $E_{q}=E_{q}^{\prime}$ then this follows from the Clebsch-Gordan formula (Lemma 3.7), since $N>N-l-l^{\prime}-2$.

We now check that for each stratum, each of the cases (1)-(4) fall under the assumption (9.4) on weights of Thm. 9.6. Up to symmetry, the unstable loci in $X$ have the following form:
(The locus $K_{I}$ ), for $I \subseteq Q,|I| \geq s+1$, where $u$ and the indices in $I$ come together. In this case, $\eta=2|I|$.
(The locus $K_{J, I}$ ), for $J \subseteq R^{\prime}, I \subseteq Q, J \neq \emptyset,|I| \geq 0$, where $u$ and the indices in $J$ and $I$ come together. In this case, $\eta=2|I|+2|J|$.
(The locus $L_{I}$ ), for $I \subseteq Q,|I| \geq s+1$, where the indices in $R^{\prime}$ and $I$ come together. In this case, $\eta=2|I|+2 r-2$.

The devil line bundle $\mathcal{O}\left(r, R^{\prime}, 0\right)$ has weight $r+|J|-\left|R^{\prime} \backslash J\right|=2|J|>0$ on $K_{J, I}$, while its weight for the other strata is 0 . Therefore, the condition (9.4) for the stratum $K_{J, I}$ can be achieved by tensoring with a high enough multiple of this line bundle. We only need to consider the remaining strata.

Strata $K_{I}$. Assume $|I| \geq s+1$. In Case (1) we need to verify that the weights of $\mathcal{O}\left(r+l^{\prime}-\left|E_{p} \cap R\right|,-E_{p} \cap R^{\prime}, E_{q}^{\prime}-E_{q}\right) \otimes V_{l}$ are $>-2|I|$. Since the weights of $V_{l}$ are between $-l$ and $l$, it suffices to prove that $r+l^{\prime}-\left|E_{p} \cap R\right|+$ $\left|E_{p} \cap R^{\prime}\right|+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|-l>-2|I|$. By (4.10) for the pair $\left(l, E_{p}\right)$, it suffices to show $l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right| \geq$ $-2|I|$. Since the left hand side equals $l^{\prime}+\left(e_{q}-e_{q}^{\prime}\right)-2\left|E_{q} \cap I\right|+2\left|E_{q}^{\prime} \cap I\right|$ and we assume $e_{q} \geq e_{q}^{\prime}$, the result follows from $-2\left|E_{q} \cap I\right| \geq-2|I|$.

In Case (2), we need to check that $\mathcal{O}\left(r-l+\left|E_{p}^{\prime} \cap R\right|-2, E_{p}^{\prime} \cap R^{\prime}, E_{q}^{\prime}-E_{q}\right) \otimes V_{l^{\prime}}$ has weights $>-2|I|$. It suffices to prove that $r-l+\left|E_{p}^{\prime} \cap R\right|-2-\left|E_{p}^{\prime} \cap R^{\prime}\right|+$ $\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|-l^{\prime}>-2|I|$. Using (4.10) for the pair $\left(l^{\prime}, E_{p}^{\prime}\right)$, it suffices to prove that $-l+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right| \geq$ $-2|I|+2$. The left hand side is greater than $-l-\left|E_{q} \cap I\right|+\left|E_{q} \cap I^{c}\right|-\left|I^{c}\right|$, and this is $\geq-2|I|+2$ by Lemma 4.10 applied to the pair $\left(l, E_{q}\right)$.

In Case (3) we need to verify that the weight of $\mathcal{O}\left(2|J|-l+l^{\prime}, 0, E_{q}^{\prime}-E_{q}\right)$ is $>-2|I|$ for all $0 \leq|J| \leq r-1$. Equivalently, we need to prove that $-l+l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|>-2|I|$. The left hand side is greater or equal than $-l-\left|E_{q} \cap I\right|+\left|E_{q} \cap I^{c}\right|-\left|I^{c}\right|$ and this is $>-2|I|$ by Lemma4.10 applied to the pair $\left(l, E_{q}\right)$.

Strata $L_{I}$. In Case (1), we need to show that $-r-l^{\prime}+\left|E_{p} \cap R\right|-\left|E_{p} \cap R^{\prime}\right|+$ $\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|-l>-2|I|-2 r+2$. Using (4.10) for the pair $\left(l, E_{p}\right)$, it suffices to prove that $-l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\mid E_{q} \cap$
$I^{c}|\geq-2| I \mid+2$. The left hand side is greater than $-l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|-|I|$, hence, it suffices to show that $-l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right| \geq-|I|+2$, but this follows from Lemma 4.10 applied to the pair $\left(l^{\prime}, E_{q}^{\prime}\right)$.
In Case (2), we need $-\left(r-l+\left|E_{p}^{\prime} \cap R\right|-2\right)+\left|E_{p}^{\prime} \cap R^{\prime}\right|+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-$ $\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|-l^{\prime}>-2|I|-2 r+2$. Using (4.10) for the pair ( $l^{\prime}, E_{p}^{\prime}$ ), it suffices to prove that $l+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right| \geq-2|I|$. The left hand side is clearly greater than $-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right| \geq-|I|-\left|I^{c}\right|=-2 s-1$, and the inequality follows since $|I| \geq s+1$.

In Case (3), we need to verify that for all $0 \leq|J| \leq r-1$ we have $-2|J|+$ $l-l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|>-2|I|-2 r+2$, or equivalently, $l-l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|>-2|I|$. The left hand side is clearly greater than $-l^{\prime}+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|-|I|$, which is $>-2|I|$ by Lemma 4.10 applied to the pair $\left(l^{\prime}, E_{q}^{\prime}\right)$ and $|I| \geq s+1$.

We now consider Case (4). Up to symmetry, the unstable loci in $X^{\prime}=$ $\left(\mathbb{P}^{1}\right)^{q+2}=\mathbb{P}_{u}^{1} \times \mathbb{P}_{v}^{1} \times\left(\mathbb{P}^{1}\right)^{q}$ have the following form:
(The locus $K_{I}^{\prime}$ ), for $I \subseteq Q,|I| \geq s+1$, where $u$ and the indices in $I$ come together. In this case, $\eta=2|I|$.
(The locus $K_{I}^{\prime \prime}$ ), for $I \subseteq Q,|I| \geq s+1$, where $v$ and the indices in $I$ come together. In this case, $\eta=2|I|$.
(The locus $K_{I}^{\prime \prime \prime}$ ), for $I \subseteq Q,|I| \geq 0$, where $u, v$ and the indices in $I$ come together. In this case, $\eta=2|I|+2$.

The devil line bundle $\mathcal{O}(1,1,0)$ has weight $2>0$ on $K_{I}^{\prime \prime \prime}$ but its weight on other strata is 0 . Therefore we only have to consider the strata $K_{I}^{\prime}, K_{I}^{\prime \prime}$.
For $K_{I}^{\prime}$, we need to show that the weight of $\mathcal{O}\left(-\left(l+l^{\prime}+2\right), 0, E_{q}^{\prime}-E_{q}\right)$ is $>-2|I|$. Equivalently, $-l-l^{\prime}-2+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|>$ $-2|I|$. This follows from Lemma 4.10 applied to the pairs $\left(l, E_{q}\right)$ and $\left(l^{\prime}, E_{q}^{\prime}\right)$. For $K_{I}^{\prime \prime}$, we need $l+l^{\prime}+2+\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right|+\left|E_{q} \cap I^{c}\right|>-2|I|$, which is clearly satisfied as the left hand side is greater than $-\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I^{c}\right| \geq$ $-|I|-\left|I^{c}\right|>-2|I|$. This completes the proof.

## 10. EQUivariant exceptional collection on $\overline{\mathrm{M}}_{2 r, 2 s+2}$

Here we will prove exceptionality of the collection of Theorem 1.8 on $\overline{\mathrm{M}}_{p, q+1}$ (see also Remark 1.11), for $p=2 r \geq 4$ and $q=2 s+1 \geq 1$, which contains objects of three types:

- Torsion sheaves $\mathcal{O}_{\delta}(-a,-b)$ that generate the subcategory $\mathcal{A}$ (see Notation 1.6). These sheaves form an exceptional collection (for example when arranged in order of decreasing $a+b$ ) by [CT20b, Lemma 4.9]. In particular, the subcategories $\mathcal{A}$ and $\mathcal{B}=\perp \mathcal{A}$ are admissible. We denote by $T_{\mathcal{B}}$ the projection of an object $T$ in $D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ to $\mathcal{B}$.
- Vector bundles $F_{l, E}$. They form an exceptional collection by Theorem8.1 and belong to the subcategory $\mathcal{B}$ by Corollary 6.2
- Complexes $\tilde{\mathcal{T}}_{l, E}$ (Definition 1.7). These complexes are projections of torsion sheaves $\mathcal{T}_{l, E}$ (Notation 1.4) to the subcategory $\mathcal{B}$.

In Corollary 10.10, we will prove the remaining semi-orthogonality of complexes $\tilde{\mathcal{T}}_{l, E}$ among themselves and with $F_{l^{\prime}, E^{\prime}}$. In Proposition 10.4, we will
reduce this calculation to a computation on a different Hassett space $\overline{\mathrm{M}}_{R^{\prime}}$. The semi-orthogonality in $D^{b}\left(\overline{\mathrm{M}}_{R^{\prime}}\right)$ will be checked in Theorem 10.9 ,
Lemma-Definition 10.1. Let $R \subseteq P$ be a subset of heavy indices, $|R|=r$, $R^{\prime}=P \backslash R$. There exists a reduction morphism $\beta_{R^{\prime}}: \overline{\mathrm{M}}_{p, q+1} \rightarrow \overline{\mathrm{M}}_{R^{\prime}}$ of Hassett spaces that contracts boundary divisors $\delta_{T, T^{c}} \cong \mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-1} \subseteq \overline{\mathrm{M}}_{p, q+1}$ with $T_{p}=R^{\prime}$, via the projection onto the second factor and is an isomorphism elsewhere.
Proof. For $\overline{\mathrm{M}}_{p, q+1}$, we use weights $\frac{1}{r}-\epsilon_{1}+\epsilon_{2}$ for heavy points and $\frac{2 r}{2 s+2} \epsilon_{1}+\epsilon_{2}$ for light points (Notation 2.8). For $\overline{\mathrm{M}}_{R^{\prime}}$, points in $R^{\prime}$ have weight $\frac{1}{r}-\epsilon_{1}$, points in $R$ have weight $\frac{1}{r}-\epsilon_{1}+\epsilon_{2}$, and light points have weight $\frac{2 r}{2 s+2} \epsilon_{1}$. It easily follows that $\beta_{R^{\prime}}$ exists and has required properties.
Notation 10.2. Let $\tilde{Q}$ be the set of light indices, $|\tilde{Q}|=2 s+2$. For $E \subseteq P \cup \tilde{Q}$ and $l \geq 0$, we let $F_{l, E}$ be the vector bundle on $\overline{\mathrm{M}}_{R^{\prime}}$ defined in Section 5 (Definition 5.2). For $e_{p}=r$, let $\overline{\mathcal{T}}_{l, E}$ be the torsion sheaf on $\overline{\mathrm{M}}_{R^{\prime}}$ defined as in Notation 1.4 (see also Lemma 9.3 (v)).
Lemma 10.3. Let $(l, E)$ be in groups $1 A$ or $1 B$. Then (1) $\left(L \beta_{R^{\prime}}^{*} F_{l, E}\right)_{\mathcal{B}} \cong F_{l, E}$ except when either $E_{p} \subseteq R^{\prime}, l+e_{p}=r-1, e_{q}=s+1\left(\right.$ case 1A) or $R^{\prime} \subseteq E_{p}$, $e_{p}=r+1+l$, $e_{q}=s+1$ (case $1 B$ ), in which cases there is an exact triangle

$$
\begin{equation*}
F_{l, E} \rightarrow\left(L \beta_{R^{\prime}}^{*} F_{l, E}\right)_{\mathcal{B}} \rightarrow Q_{\mathcal{B}} \rightarrow \tag{10.1}
\end{equation*}
$$

where $Q Q_{\mathcal{B}}$ is the projection into $\mathcal{B}$ of an object $Q \in D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ which is generated by the sheaves $\mathcal{O}_{\delta_{T, T^{c}}}\left(u,-\frac{r+s}{2}\right)$ such that $T_{p}=R^{\prime}, T_{q}=E_{q}, 0 \leq u<$ $\frac{r+s}{2}$. (2) $\left[R \beta_{R^{\prime} *} F_{l, E}^{\vee}\right]^{\vee} \cong F_{l, E}$.
Proof. Part (1) follows from Proposition 8.5 and its proof, as well as Corollary 8.6, Part (2) is a particular case of Corollary 8.7, since $m_{T, E, l} \leq(r+s) / 2$ for all $T$ by (4.6).
Proposition 10.4. $R \operatorname{Hom}_{\overline{\mathrm{M}}_{p, q+1}}\left(G^{\prime}, G\right)=R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\bar{G}^{\prime}, \bar{G}\right)$ in any of the cases:
(1) $E_{p}=R, G=\tilde{\mathcal{T}}_{l, E}, G^{\prime}=F_{l^{\prime}, E^{\prime}}, \bar{G}=\overline{\mathcal{T}}_{l, E}, \bar{G}^{\prime}=F_{l^{\prime}, E^{\prime}}$;
(2) $E_{p}^{\prime}=R^{\prime}, G=F_{l, E}, G^{\prime}=\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \bar{G}=F_{l, E}, \bar{G}^{\prime}=\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}$.

In the remaining cases, $G=\tilde{\mathcal{T}}_{l, E}, G^{\prime}=\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \bar{G}=\overline{\mathcal{T}}_{l, E}, \bar{G}^{\prime}=\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}$ and
(3) $E_{p}=R, E_{p}^{\prime}=R$;
(4) $E_{p}=R, E_{p}^{\prime}=R^{\prime}$;

Here all pairs $(l, E)$ are in group $1 A$ or $1 B$ for vector bundles $F_{l, E}$ and in group $2 A$ or $2 B$ for torsion objects $\tilde{\mathcal{T}}_{l, E}$ and $\overline{\mathcal{T}}_{l, E}$, and similar for $\left(l^{\prime}, E^{\prime}\right)$.
Proof. In cases (1), (3) and (4), $G=\tilde{\mathcal{T}}_{l, E}$, which is isomorphic to $\left(L \beta_{R^{\prime}}^{*} \overline{\mathcal{T}}_{l, E}\right)_{\mathcal{B}}$ by Proposition 10.8 (iii). For each $G^{\prime}$ in cases (1), (3) and (4), we also have $G^{\prime} \in \mathcal{B}$. By Lemma 7.3, it follows that

$$
\begin{aligned}
& \operatorname{RHom}_{\bar{M}_{p, q+1}}\left(G^{\prime}, G\right) \cong \operatorname{RHom}_{\bar{M}_{p, q+1}}\left(G^{\prime},\left(L \beta_{R^{\prime}}^{*} \overline{\mathcal{T}}_{l, E}\right)_{\mathcal{B}}\right) \cong \\
& \cong \operatorname{RHom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\left(R \beta_{R^{\prime} *} G^{\prime \vee}\right)^{\vee}, \overline{\mathcal{T}}_{l, E}\right) .
\end{aligned}
$$

For each $G^{\prime}$ in cases (1), (3) and (4), $\left(R \beta_{R^{\prime} *} G^{\prime \vee}\right)^{\vee} \cong \bar{G}^{\prime}$ by Lemma 10.3(2) for $G^{\prime}=F_{l^{\prime}, E^{\prime}}$ and Proposition 10.8 (ii) and (iv)) for $G^{\prime}=\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}$.

In case (2), we distinguish two cases. Assume first that $(l, E)$ is not one of the exceptions in Lemma 10.3(1). Then, $\left(L \beta_{R^{\prime}}^{*} F_{l, E}\right)_{\mathcal{B}} \cong F_{l, E}$ by this lemma. Lemma 7.3]gives RHom $\bar{M}_{p, q+1}\left(G^{\prime}, F_{l, E}\right) \cong \operatorname{RHom}_{\bar{M}_{R^{\prime}}}\left(\left(R \beta_{R^{\prime} *} G^{\prime \vee}\right)^{\vee}, F_{l, E}\right)$. As by Proposition $10.8(\mathrm{ii}),\left(R \beta_{R^{\prime} *} G^{\prime \vee}\right)^{\vee} \cong \bar{G}^{\prime}$ for $G^{\prime}=\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}$, the result follows.

Assume now that $(l, E)$ is one of the exceptions in Lemma 10.3(1). Arguing as above, we have $\operatorname{RHom}_{\overline{\mathrm{M}}_{p, q+1}}\left(\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, F_{l, E}\right) \cong \operatorname{RHom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}, F_{l, E}\right)$ by (10.1) and provided that we can prove the following claim:

Claim 10.5. RHom $_{\overline{\mathrm{M}}_{p, q+1}}\left(\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \mathcal{O}_{\delta_{T, T^{c}}}\left(u,-\frac{r+s}{2}\right)\right)=0$ if $\left(l^{\prime}, E^{\prime}\right)$ is in group $2 A$ or $2 B, E_{p}^{\prime}=R^{\prime}, 0 \leq u<\frac{r+s}{2}$ and boundary $\delta_{T, T^{c}}$ satisfies $T_{p}=R^{\prime}$.

We will prove Claim 10.5 and Proposition 10.8 which are used in this proof after writing an explicit resolution of $\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}$. These statements aside, this finishes the proof of Proposition 10.4.

Lemma 10.6. Let $(l, E)$ be in group $2 A$ or $2 B$ with $E_{p}=R=\{1, \ldots, r\}$. The complex $\tilde{\mathcal{T}}_{l, E}=\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}}$ is isomorphic to the following complex, call it $\mathcal{S}_{l, E}$ :

$$
\begin{equation*}
\tilde{L}_{R \backslash\{1\}} \rightarrow \ldots \rightarrow \bigoplus_{J \subseteq R \backslash\{1\},|J|=r-2} \tilde{L}_{J} \rightarrow \ldots \rightarrow \bigoplus_{j \in R \backslash\{1\}} \tilde{L}_{j} \rightarrow \tilde{L}_{\emptyset} \tag{10.2}
\end{equation*}
$$

(with differentials described in the proof), where for any subset $J \subseteq R \backslash\{1\}$, we let
$\tilde{L}_{J}=L^{\prime}\left(-\sum_{T_{p}=R} \alpha_{J, T, E, l} \delta_{T, T^{c}}-\sum_{j \in J} \delta_{1 j}\right) \quad$ and $\quad L^{\prime}=\frac{e_{q}-r-l}{2} \psi_{1}+\sum_{k \in E_{q}} \delta_{1 k}$.
Here $\alpha_{J, T, l, E}=\max \left\{\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T_{q}\right|+|J|, 0\right\}$. Furthermore, we have an exact triangle $\tilde{\mathcal{T}}_{l, E} \rightarrow \mathcal{T}_{l, E} \rightarrow Q \rightarrow$, with $Q$ generated by generators of the subcategory $\mathcal{A}$ supported on the boundary divisors $\delta_{T, T^{c}}, T_{p}=R$.

The $\alpha_{J, T, l, E}$ in Lemma 10.6 is a variant of $\alpha_{T, E, l}=\max \left\{\frac{e-l}{2}-|E \cap T|, 0\right\}$ (see (4.2)), as when $E_{p}=R$ we have $\frac{e-l}{2}-|E \cap T|=\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T_{q}\right|$. The main property of the line bundle $L^{\prime}$ that we use is that $\tau_{l, E}=i_{R *}\left(L_{\mid Z_{R}}^{\prime}\right)$ (to see this, use the analogue of Lemma 9.3).
Proof of Lemma 10.6 A sheaf $\mathcal{T}_{l, E}$ with $E_{p}=R$ is isomorphic in $D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ to the Koszul resolution (9.1) of the stratum $Z_{R}$ tensored with the line bundle $L^{\prime}$ :

$$
\begin{equation*}
L_{R \backslash\{1\}} \rightarrow \cdots \rightarrow \bigoplus_{J \subseteq R \backslash \backslash 1\},|J|=r-2} L_{J} \rightarrow \cdots \rightarrow \bigoplus_{j \in R \backslash\{1\}} L_{j} \rightarrow L_{\emptyset} . \tag{10.3}
\end{equation*}
$$

Here $L_{J}=L^{\prime}\left(-\sum_{j \in J} \delta_{1 j}\right)$. Let $\mathcal{A}_{R} \subset D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ be the admissible subcategory generated by sheaves in $\mathcal{A}$ supported on $\delta_{T, T^{c}}$ with $T_{p}=R$. Let $\mathcal{B}_{R}={ }^{\perp} \mathcal{A}_{R}$. Since $L_{J}=\tilde{L}_{J}+\sum_{T_{p}=R} \alpha_{J, T, l, E} \delta_{T, T^{c}}$, the canonical morphisms $\tilde{L}_{J} \rightarrow L_{J}$ have the cokernels generated by sheaves supported on boundary $\delta_{T, T^{c}}$, with $T_{p}=R$. We claim that these cokernels are in $\mathcal{A}_{R}$. Indeed, by Lemma 5.7, quotients relating $\tilde{L}_{J}$ and $L_{J}$ have the form $Q=$ $\left(\tilde{L}_{J}+i \delta\right)_{\mid \delta} \cong \mathcal{O}\left(-i, \alpha_{J, T, E, l}-i\right)$ for $0<i \leq \alpha_{J, T, E, l}$, where $\alpha_{J, T, E, l}=$
$\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T\right|+|J| \leq \frac{e_{q}-l}{2}-\left|E_{q} \cap T\right|+\frac{r}{2}-1$, and this is $<\frac{r+s}{2}$ by Corollary 4.6(ii). It follows that $\left(\tilde{L}_{J}\right)_{\mathcal{B}_{R}} \cong\left(L_{J}\right)_{\mathcal{B}_{R}}$.

Next we claim that $\tilde{L}_{J} \in \mathcal{B}_{R}$, i.e., that $R \operatorname{Hom}\left(\tilde{L}_{J}, \mathcal{O}_{\delta}(-a,-b)\right)=0$ for any $\mathcal{O}_{\delta}(-a,-b)$ as in Theorem 1.8 with $T_{p}=R$. We have $\left(\tilde{L}_{J}\right)^{\vee} \otimes \mathcal{O}_{\delta}(-a,-b) \cong$ $\mathcal{O}_{\delta}\left(\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T\right|+|J|-\alpha_{J, T, l, E}-a,-\alpha_{J, T, l, E}-b\right)$. Consider the case when $\alpha_{J, T, l, E}=0$. The sheaf is clearly acyclic if $b \neq 0$. Assume $b=0$, in which case $0<a<\frac{r+s}{2}$. Since $\alpha_{J, T, l, E}=0$, we have that $\left|E_{q} \cap T\right|-$ $\frac{e_{q}-r-l}{2}-|J| \geq 0$. By Corollary 4.6(i), $\left|E_{q} \cap T\right|-\frac{e_{q}-r-l}{2} \leq \frac{r+s}{2}$. It follows that $0<\left|E_{q} \cap T\right|-\frac{e_{q}-r-l}{2}-|J|+a \leq r+s-1$ and the sheaf is acyclic.

Consider now the case when $\alpha_{J, T, l, E}>0$. In this case $\left(\tilde{L}_{J}\right)^{\vee} \otimes \mathcal{O}_{\delta}(-a,-b) \cong$ $\mathcal{O}_{\delta}\left(-a,-\alpha_{J, T, l, E}-b\right)$. Clearly, this is acyclic when $a>0$. Assume $a=0$, in which case $0<b<\frac{r+s}{2}$. Then it suffices to prove that $\alpha_{J, T, l, E}=$ $\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T\right|+|J| \leq \frac{r+s}{2}$ for all $J \subseteq R \backslash\{1\}$, or equivalently that $\frac{e_{q}-l}{2}-\left|E_{q} \cap T\right| \leq \frac{s}{2}+1$. This holds by Corollary 4.6.

Now we use the following simple lemma:
Lemma 10.7. Let $A$ and $B$ be abelian categories. Let $F: D^{b}(A) \rightarrow D^{b}(B)$ be an exact functor. Let $L^{\bullet} \in D^{b}(A)$ be a complex $\left[L_{0} \xrightarrow{d_{1}} \ldots \xrightarrow{d_{r}} L_{r}\right]$. Suppose $F\left(L_{1}\right), \ldots, F\left(L_{r}\right) \in B$. Then $F\left(L^{\bullet}\right) \cong\left[F\left(L_{0}\right) \xrightarrow{F\left(d_{1}\right)} \ldots \xrightarrow{F\left(d_{r}\right)} F\left(L_{r}\right)\right]$.

Proof. Since $B$ is a fully faithful subcategory of $D^{b}(B)$, the morphism $F\left(d_{i}\right)$ is a morphism in $B$ for every $i=1, \ldots, r$ and so $\left[F\left(L_{0}\right) \xrightarrow{F\left(d_{1}\right)} \ldots \xrightarrow{F\left(d_{r}\right)} F\left(L_{r}\right)\right]$ is a complex in $D^{b}(B)$. It is isomorphic to $F\left(L^{\bullet}\right)$ by induction on $r$ and by applying the functor $F$ to the naive truncation of $L^{\bullet}$.

We apply Lemma 10.7 to the projection functor $D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right) \rightarrow \mathcal{B}_{R} \hookrightarrow$ $D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$, where we recall that $\mathcal{B}_{R}={ }^{\perp} \mathcal{A}_{R}$ and $\mathcal{A}_{R} \subset D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ is the admissible subcategory generated by sheaves in $\mathcal{A}$ supported on $\delta_{T, T^{c}}$ with $T_{p}=R$. It follows that $\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}_{R}}$ is isomorphic to the complex (10.2) with differentials obtained by applying the functor $T \rightarrow T_{\mathcal{B}_{R}}$ to differentials of the complex $\mathcal{S}_{l, E}$ from (10.3). Concretely, for $j \in J$, we have $L_{J \backslash\{j\}}=$ $L_{J}\left(\delta_{1 j}\right)$ and the differentials in $\mathcal{T}_{l, E}$ are built from the maps $\sigma: L_{J} \rightarrow L_{J \backslash\{j\}}$ given by multiplication with a canonical section of $\mathcal{O}\left(\delta_{1 j}\right)$. On the other hand, $\tilde{L}_{J \backslash\{j\}}=\tilde{L}_{J}\left(\delta_{1 j}+\sum_{\substack{T_{p}=R, \alpha_{J, T, l, E}>0}} \delta_{T, T^{c}}\right)$, since $\alpha_{J \backslash\left\{j_{0}\right\}, T, l, E}=\alpha_{J, T, l, E}-1$ if $\alpha_{J, T, l, E}>0$ and $\alpha_{J, T, l, E}=0$ otherwise. We claim that differentials in $\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}_{R}}$ are built from the maps $\tilde{\sigma}: \tilde{L}_{J} \rightarrow \tilde{L}_{J \backslash\{j\}}$ given by multiplication with a canonical section of $\mathcal{O}\left(\delta_{1 j}+\sum_{\substack{T_{p}=R, \alpha_{J, T, l, E}>0}} \delta_{T, T^{c}}\right)$ (note that these maps are in $\mathcal{B}_{R}$ since $\mathcal{B}_{R}$ is a full subcategory). Indeed, consider the commutative diagram

where the left, resp., right, vertical maps are injections given by a multiplication with $\sum_{\alpha_{J, T, l, E}>0} \alpha_{J, T, l, E} \delta_{T, T^{c}}$, resp., $\sum_{\alpha_{J, T, l, E}>0}\left(\alpha_{J, T, l, E}-1\right) \delta_{T, T^{c}}$. Application of the functor $T \rightarrow T_{\mathcal{B}_{R}}$ gives a commutative diagram

where the first line is a differential in the complex $\mathcal{S}_{l, E}$ of $((\underline{10.3}))$ and the second in $\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}_{R}}$. This proves our above claim about differentials. In particular, we have an exact triangle of a s.o.d.

$$
\begin{equation*}
\mathcal{S}_{l, E} \rightarrow \mathcal{T}_{l, E} \rightarrow\left(\mathcal{T}_{l, E}\right)_{\mathcal{A}_{R}} . \tag{10.4}
\end{equation*}
$$

We claim that $\mathcal{S}_{l, E} \in \mathcal{B}$. For this, we need to prove that $\mathcal{S}_{l, E}$ is perpendicular to all the generators of $\mathcal{A}$. Since $\mathcal{S}_{l, E}=\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}_{R}}, \mathcal{S}_{l, E}$ is perpendicular to all the generators of $\mathcal{A}_{R}$. It remains to check that $\mathcal{S}_{l, E}$ is perpendicular on the generators of $\mathcal{A}$ supported on $\delta=\delta_{T, T^{c}}$ in $\overline{\mathrm{M}}_{p, q+1}$ with $T_{p} \neq R, R^{\prime}$. For this, using (10.4), it suffices to check that $\mathcal{T}_{l, E}$ is perpendicular on such generators. Since $\mathcal{T}_{l, E}$ is supported on $Z_{R}$ and $Z_{R}$ does not intersect any of the boundary $\delta=\delta_{T, T^{c}}$ in $\overline{\mathrm{M}}_{p, q+1}$ with $T_{p} \neq R, R^{\prime}$, we have $\operatorname{RHom}\left(\mathcal{T}_{l, E}, \mathcal{O}_{\delta}(-a,-b)\right)=0$, for all $a, b$. Hence, $\operatorname{RHom}\left(\tilde{\mathcal{T}}_{l, E}, \mathcal{O}_{\delta}(-a,-b)\right)=$ 0 by (10.4) since supports of objects in $\mathcal{A}_{R}$ are also disjoint from $\delta$. This proves that $\mathcal{S}_{l, E} \in \mathcal{B}$. Consider now the sequence of projection functors $D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right) \rightarrow \mathcal{B}_{R} \rightarrow \mathcal{B}$. By definition, $\tilde{\mathcal{T}}_{l, E}=\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}}$. But $\left(\mathcal{T}_{l, E}\right)_{\mathcal{B}_{R}} \cong \mathcal{S}_{l, E}$ and since $\mathcal{S}_{l, E} \in \mathcal{B}$, it follows that $\left(\mathcal{S}_{l, E}\right)_{\mathcal{B}} \cong \mathcal{S}_{l, E}$ and $\tilde{\mathcal{T}}_{l, E} \cong \mathcal{S}_{l, E}$. This finishes the proof of Lemma 10.6 .

Proof of Claim 10.5 Let $R^{\prime}=\{1, \ldots, r\}$. By Lemma 10.6 and Lemma 10.7 , it suffices to prove that for all subsets $J \subseteq R^{\prime} \backslash\{1\}$ we have $R \Gamma\left(\tilde{L}_{J}^{\vee} \otimes\right.$ $\left.\mathcal{O}\left(u,-\frac{r+s}{2}\right)\right)=0$. For $\delta=\delta_{T, T^{c}}$, with $T_{p}=R^{\prime}$, letting $\alpha:=\alpha_{J, T, E^{\prime}, l^{\prime}}$, we have $\left(\tilde{L}_{J}\right)_{\mid \delta}=\mathcal{O}\left(-\frac{e_{q}^{\prime}-r-l^{\prime}}{2}+\left|E_{q}^{\prime} \cap T_{q}\right|-|J|+\alpha, \alpha\right)=\mathcal{O}(0, \alpha)$ if $\alpha>0$ and $\mathcal{O}\left(-\frac{e_{q}^{\prime}-r-l^{\prime}}{2}+\left|E_{q}^{\prime} \cap T_{q}\right|-|J|, 0\right)$ if $\alpha=0$. If $\alpha=0$, the second component of $\tilde{L}_{J}^{\vee} \otimes \mathcal{O}\left(u,-\frac{r+s}{2}\right)$ is $\mathcal{O}\left(-\frac{r+s}{2}\right)$, which is acyclic on $\mathbb{P}^{r+s-1}$, and the result follows. Assume now $\alpha>0$. We have to prove that $\tilde{L}_{J}^{\vee} \otimes \mathcal{O}\left(u,-\frac{r+s}{2}\right)=$ $\mathcal{O}\left(u,-\alpha-\frac{r+s}{2}\right)$ is acyclic, or equivalently, that $\alpha+\frac{r+s}{2} \leq r+s-1$, i.e., that $\alpha=\frac{e_{q}^{\prime}-r-l^{\prime}}{2}-\left|E_{q}^{\prime} \cap T_{q}\right|+|J| \leq \frac{r+s}{2}-1$. As $|J| \leq r-1$, it suffices to prove that $\frac{e_{q}^{\prime}-l^{\prime}}{2}-\left|E_{q}^{\prime} \cap T_{q}\right| \leq \frac{s}{2}$. This follows from Cor. 4.6(ii) applied to the pair $\left(E^{\prime}, l^{\prime}\right)$.

Proposition 10.8. Let $(l, E)$ be in group $2 A$ or $2 B$ on $\overline{\mathrm{M}}_{p, q+1}$, with $E_{p}=R=$ $\{1, \ldots, r\}$. Let $R^{\prime}=P \backslash R$ and $\beta_{R}: \overline{\mathrm{M}}_{p, q+1} \rightarrow \overline{\mathrm{M}}_{R}, \beta_{R^{\prime}}: \overline{\mathrm{M}}_{p, q+1} \rightarrow \overline{\mathrm{M}}_{R^{\prime}}$ be the morphisms in Lemma-Definition 10.1 .

Then (i) $\left(L \beta_{R}^{*} \overline{\mathcal{T}}_{l, E}\right)_{\mathcal{B}}=\tilde{\mathcal{T}}_{l, E}$, except when $r+s$ is even and either $e_{q}=l+s+2$, $T_{q} \subseteq E_{q}$ (case $2 A$ ) or $e_{q}+l=s, E_{q} \subseteq T_{q}$ (case $2 B$ ). In the latter cases there
exists an exact triangle in $\mathcal{B}$ :

$$
\begin{equation*}
\tilde{\mathcal{T}}_{l, E} \rightarrow\left(L \beta_{R}^{*} \overline{\mathcal{T}}_{l, E}\right)_{\mathcal{B}} \rightarrow Q_{\mathcal{B}}, \tag{10.5}
\end{equation*}
$$

where $Q_{\mathcal{B}}$ is the projection into $\mathcal{B}$ of an object $Q \in D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ which is generated by the sheaves $\mathcal{O}_{\delta_{T, T c}}\left(0,-\frac{r+s}{2}\right)$ with $T_{q} \subseteq E_{q}($ case $2 A)$ and $E_{q} \subseteq T_{q}$ (case $2 B$ ).

$$
\text { (ii) } R \beta_{R *}\left(\tilde{\mathcal{T}}_{l, E}^{\vee}\right)=\overline{\mathcal{T}}_{l, E}^{\vee} ; \text { (iii) }\left(L \beta_{R^{\prime}}^{*} \overline{\mathcal{T}}_{l, E}\right)_{\mathcal{B}}=\tilde{\mathcal{T}}_{l, E} ; \text { (iv) } R \beta_{R^{\prime} *}\left(\tilde{\mathcal{T}}_{l, E}^{\vee}\right)=\overline{\mathcal{T}}_{l, E}^{\vee} \text {. }
$$

Proof. We note that $\overline{\mathcal{T}}_{l, E}$ isomorphic in $D^{b}\left(\overline{\mathrm{M}}_{R}\right)$ to the Koszul resolution of the stratum $Z_{R}=\bigcap_{j \in R \backslash\{1\}} \delta_{1 j}$ tensored with the line bundle $\bar{L}^{\prime}$ :

$$
0 \rightarrow \bar{L}_{R \backslash\{1\}} \rightarrow \ldots \rightarrow \bigoplus_{J \subseteq R \backslash\{1\},|J|=r-2} \bar{L}_{J} \rightarrow \ldots \rightarrow \bigoplus_{j \in R \backslash\{1\}} \bar{L}_{j} \rightarrow \bar{L}_{\emptyset} \rightarrow 0,
$$

where for every subset $J \subseteq R \backslash\{1\}$, we let $\bar{L}_{J}=\bar{L}^{\prime}\left(-\sum_{j \in J} \delta_{1 j}\right)$ and $\bar{L}^{\prime}=\frac{e_{q}-r-l}{2} \psi_{1}+\sum_{k \in E_{q}} \delta_{1 k}$, where all tautological classes are on $\overline{\mathrm{M}}_{R}$. As in Lemma 10.6, the main property of the line bundle $\bar{L}^{\prime}$ is that $\overline{\mathcal{T}}_{l, E}=$ $i_{R *}\left(\bar{L}_{\mid Z_{R}}^{\prime}\right)$.

We identify $\tilde{\mathcal{T}}_{l, E}$ with $S_{l, E}=\left(\tilde{L}_{J}\right)$ as in (10.2). To show (i), we prove that there is an exact triangle $\tilde{\mathcal{T}}_{l, E} \rightarrow L \beta_{R}^{*} \overline{\mathcal{T}}_{l, E} \rightarrow Q$, with $Q \in \mathcal{A}$ (and so in fact $Q_{\mathcal{B}}=0$ ), with exceptions listed in part (i), in which case $Q$ is generated by the sheaves $\mathcal{O}_{\delta_{T, T^{c}}}\left(0,-\frac{r+s}{2}\right)$ with the required properties.

For $J \subseteq R \backslash\{1\}$, using (5.1) and (5.2), we have when $E_{p}=R$ that

$$
\beta_{R}{ }^{*} \bar{L}_{J}=L_{J}+\sum_{T_{p}=R} \alpha_{J, T, E, l^{\prime}}^{\prime} \delta_{T, T^{c}}=\tilde{L}_{J}+\sum_{T_{p}=R, \alpha_{J, T, l, E}^{\prime}>0} \alpha_{J, T, l, E^{\prime}}^{\prime} \delta_{T, T^{c}},
$$

where we denote $\alpha_{J, T, E, l}^{\prime}=\left|E_{q} \cap T_{q}\right|-\frac{e_{q}-r-l}{2}-|J|$ and $\tilde{L}_{J}$ is defined as in Lemma 10.6, By definition of $\alpha_{J, T, E, l}$ (see Lemma 10.6) if $\alpha_{J, T, E, l}>0$ we have $\alpha_{J, T, E, l}=-\alpha_{J, T, E, l}^{\prime}$. $)$ Let $Q_{J}=\operatorname{Coker}\left(\tilde{L}_{J} \rightarrow \beta_{R}^{*} \bar{L}_{J}\right)$. Then $Q_{J}$ is generated by the successive quotients

$$
\begin{equation*}
Q_{J}^{i}:=\left(\tilde{L}_{J}+i \delta_{T, T^{c}}\right)_{\mid \delta_{T, T^{c}}}=\mathcal{O}_{T}\left(\alpha_{J, T, l, E}^{\prime}-i\right) \boxtimes \mathcal{O}_{T^{c}}(-i), \tag{10.6}
\end{equation*}
$$

for $0<i \leq \alpha_{J, T, l, E}^{\prime}$. Then $Q_{J}^{i} \in \mathcal{A}$ iff $i<\frac{r+s}{2}$. By Corollary 4.6(i),

$$
\begin{equation*}
i \leq \alpha_{J, T, l, E}^{\prime} \leq\left|E_{q} \cap T_{q}\right|-\frac{e_{q}-l}{2}+\frac{r}{2} \leq \frac{r+s}{2} \tag{10.7}
\end{equation*}
$$

with equality $i=\frac{r+s}{2}$ only when $J=\emptyset$, and either $e_{q}=l+s+2, T_{q} \subseteq$ $E_{q}$ (case $2 A$ ), or $e_{q}+l=s, E_{q} \subseteq T_{q}$ (case 2B), in which case $Q_{J}^{i}=$ $\mathcal{O}\left(0,-\frac{r+s}{2}\right)$, with 0 the component corresponding to markings from $T$. This gives an exact triangle (10.5) with the required properties as in the proof of Lemma 10.6

We now prove (ii). It suffices to prove that $R \beta_{R_{*}}\left(Q_{J}^{i}\right)^{\vee}=0$ for all quotients (10.6). Up to a shift, $\left(Q_{J}^{i}\right)^{\vee}=\left(\tilde{L}_{J}+i \delta_{T, T^{c}}\right)^{\vee} \otimes \mathcal{O}\left(\delta_{T, T^{c}}\right)_{\mid \delta_{T, T^{c}}}=$ $\mathcal{O}_{T}\left(-\alpha_{J, T, l, E}^{\prime}+i-1\right) \boxtimes \mathcal{O}_{T^{c}}(i-1)$. As $\beta_{R}$ contracts the $T$-component, it suffices to prove that $0<\alpha_{J, T, l, E}^{\prime}-i+1 \leq r+s-1$ for all $0<i \leq\left|\alpha_{J, T, l, E}^{\prime}\right|$. Indeed, by (10.7), $\alpha_{J, T, l, E}^{\prime} \leq \frac{r+s}{2} \leq r+s-1$ as $r+s \geq 2$.

We prove (iii) and (iv). Using (5.1) and (5.2), we have $\beta_{R^{\prime}}^{*} \bar{L}_{J}=\frac{e_{q}-r-l}{2} \psi_{1}+$ $\sum_{k \in E_{q}} \delta_{1 k}-\sum_{j \in J} \delta_{1 j}=\tilde{L}_{J}+\sum_{T_{p}=R,} \alpha_{J, T, l, E}>0$ 作 ${ }_{J, l, E} \delta_{T, T^{c}}$. As before, let $Q_{J}=$ $\operatorname{Coker}\left(\tilde{L}_{J} \rightarrow \beta_{R^{\prime}}^{*} \bar{L}_{J}\right)$. Then $Q_{J}$ is generated by the successive quotients $Q_{J}^{i}:=\left(\tilde{L}_{J}+i \delta_{T, T^{c}}\right)_{\mid \delta_{T, T^{c}}} \cong \mathcal{O}_{T}(-i) \boxtimes \mathcal{O}_{T^{c}}\left(\alpha_{J, T, l, E}-i\right)$ for all $0<i \leq$ $\alpha_{J, T, l, E}=\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T_{q}\right|+|J|$. Then $Q_{J}^{i} \in \mathcal{A}$ if and only if $i<\frac{r+s}{2}$. As $|J| \leq r-1$, we have
$\alpha_{J, T, l, E}=\frac{e_{q}-r-l}{2}-\left|E_{q} \cap T_{q}\right|+|J| \leq \frac{e_{q}+r-l}{2}-\left|E_{q} \cap T_{q}\right|-1 \leq \frac{r+s}{2}-1$,
by Corollary 4.6 (ii). Therefore, $Q \in \mathcal{A}$ and this proves (iii).
To prove (iv), it suffices to show that $\left(Q_{J}^{i}\right)^{\vee}$, which up to a shift is equal to $\left(\tilde{L}_{J}+i \delta_{T, T^{c}}\right)^{\vee}\left(\delta_{T, T^{c}}\right)_{\mid \delta_{T, T^{c}}} \cong \mathcal{O}_{T}(i-1) \boxtimes \mathcal{O}_{T^{c}}\left(-\left(\alpha_{J, T, l, E}-i\right)-1\right)$, pushes forward to 0 by $\beta_{R^{\prime}}$. As $\beta_{R^{\prime}}$ contracts the $T^{c}$-component, it suffices to prove that $0<\alpha_{J, T, l, E}-i+1 \leq r+s-1$ for all $0<i \leq \alpha_{J, T, l, E}$, or equivalently that $\alpha_{J, T, l, E} \leq r+s-1$, which follows from (10.8) as $\frac{r+s}{2}-1 \leq r+s-1$.

Theorem 10.9. On $\overline{\mathrm{M}}_{R^{\prime}}$, consider vector bundles $F_{l, E}$ for pairs $(l, E)$ in group $1 A$ or $1 B$, and torsion sheaves $\overline{\mathcal{T}}_{l, E}$ for $(l, E)$ in group $2 A$ or $2 B$. Then
(1) $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(F_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)=0$ if $E_{p}=R, e_{q}^{\prime} \geq e_{q}$.
(2) $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}, F_{l, E}\right)=0$ if $E_{p}^{\prime}=R^{\prime}$ and $e_{q}^{\prime}>e_{q}$ or $e_{q}^{\prime}=e_{q}, E_{q}^{\prime} \neq E_{q}$.
(3) $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)=0$ if $E_{p}=E_{p}^{\prime}=R$ and either $e_{q}^{\prime} \geq e_{q}, E_{q} \neq E_{q}^{\prime}$ or $E_{q}=E_{q}^{\prime}$ and $l>l^{\prime}$. Furthermore, $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l, E}, \overline{\mathcal{T}}_{l, E}\right)=\mathbb{C}$.
(4) $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)=0$ if $E_{p}=R, E_{p}^{\prime}=R^{\prime}, e_{q}^{\prime} \geq e_{q}$.

Corollary 10.10. The collection of Theorem 1.8 is exceptional.
Proof. As discusses in the beginning of this section, we only need to check exceptionality of complexes $\tilde{\mathcal{T}}_{l, E}$ and their semi-orthogonality with each other and vector bundles $F_{l, E}$, where, recall that the order is as in Theorem 1.5 i.e., as follows: the objects are arranged in blocks indexed by a subset $E_{q}$ and ordered by increasing $e_{q}$ and arbitrarily if $e_{q}$ is the same (but the set $E_{q}$ is different). Within each block with the same $E_{q}$ we put the complexes $\left\{\tilde{\mathcal{T}}_{l, E}\right\}$ first, in arbitrary order if $E_{p} \neq E_{p}^{\prime}$ and in order of decreasing $l$ when $E_{p}=E_{p}^{\prime}$. After the complexes we put the bundles $\left\{F_{l, E}\right\}$ in a certain order.

By Proposition 10.4 and Theorem 10.9 , by choosing a subset $R$ of $r$ heavy indices appropriately, we see that, indeed, $\operatorname{RHom}\left(F_{l^{\prime}, E^{\prime}}, \tilde{\mathcal{T}}_{l, E}\right)=0$ if $e_{q}^{\prime} \geq e_{q}$; $\operatorname{RHom}\left(\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, F_{l, E}\right)=0$ if $e_{q}^{\prime}>e_{q}$ or $e_{q}^{\prime}=e_{q}, E_{q}^{\prime} \neq E_{q} ; \operatorname{RHom}\left(\tilde{\mathcal{T}}_{l, E}, \tilde{\mathcal{T}}_{l, E}\right)=\mathbb{C}$; $\operatorname{RHom}\left(\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)=0$ if $E_{p}=E_{p}^{\prime}$ and either $e_{q}^{\prime} \geq e_{q}, E_{q} \neq E_{q}^{\prime}$ or $E_{q}=E_{q}^{\prime}$ and $l>l^{\prime} ; \operatorname{RHom}\left(\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \tilde{\mathcal{T}}_{l, E}\right)=0$ if $e_{q}^{\prime} \geq e_{q}$ and $E_{p}$ is disjoint from $E_{p}^{\prime}$. It remains to note that $\operatorname{RHom}\left(\tilde{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)=0$ if $E_{p}$ and $E_{p}$ are neither equal nor disjoint because these complexes have disjoint support.

In the remainder of this section we prove Theorem 10.9 .
Notation 10.11. Set $R=\{1, \ldots, r\}, R^{\prime}=\{r+1, \ldots, p\}$ and consider the loci $Z_{R}, Z_{R^{\prime}} \hookrightarrow \overline{\mathrm{M}}_{R^{\prime}}, Y=Z_{R} \cap Z_{R^{\prime}}$.

We reduce Theorem 10.9 to a calculation of $R \Gamma$ on loci $Z_{R}, Z_{R^{\prime}}$ and $Y$ :
Lemma 10.12. Let all pairs $(l, E)$ be in group $1 A$ or $1 B$ for $F_{l, E}$ and in group $2 A$ or $2 B$ for $\bar{T}_{l, E}$. In order to prove Theorem [10.9] it suffices to prove the following:
(1) $R \Gamma\left(Z_{R},\left(F_{l^{\prime}, E^{\prime} \mid Z_{R}}\right)^{\vee} \otimes \overline{\mathcal{T}}_{l, E}\right)=0$ if $e_{q}^{\prime} \geq e_{q}, R=E_{p}$.
(2) $R \Gamma\left(Z_{R^{\prime}}, \overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}^{\vee} \otimes F_{l, E \mid Z_{R^{\prime}}} \otimes c_{1}\left(N_{Z_{R^{\prime}} \mid \bar{M}_{R^{\prime}}}\right)\right)=0$ if $R^{\prime}=E_{p}^{\prime}$ and either $e_{q}^{\prime}>e_{q}$ or $e_{q}^{\prime}=e_{q}$ but $E_{q}^{\prime} \neq E_{q}$.
(3) $R \Gamma\left(Z_{R}, \overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}^{\vee} \otimes \overline{\mathcal{T}}_{l, E} \otimes\left(\sum_{j \in J} \delta_{1 j}\right)\right)=0$ if $R=E_{p}=E_{p^{\prime}}^{\prime}$ for all $J \subseteq$ $R \backslash\{1\}, e_{q}^{\prime} \geq e_{q}$, and either $E_{q} \neq E_{q}^{\prime}$ or $E_{q}=E_{q^{\prime}}^{\prime} l>l^{\prime}$. Furthermore, $R \Gamma\left(Z_{R}, \overline{\mathcal{T}}_{l, E}^{\vee} \otimes \overline{\mathcal{T}}_{l, E} \otimes\left(\sum_{j \in J} \delta_{1 j}\right)\right)=0$ for all $\emptyset \neq J \subseteq R \backslash\{1\}$.
(4) $R \Gamma\left(Y,-\left(\frac{l+l^{\prime}}{2}+1\right) \psi_{u}+\frac{1}{2} \sum_{j \in E_{q}^{\prime}} \psi_{j}-\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}\right)=0$ if $R=E_{p}$, $R^{\prime}=E_{p}^{\prime}, e_{q}^{\prime} \geq e_{q}$.
Proof. This is very similar to the proof of Lemma 9.4.
(1) $R \operatorname{Hom}_{\overline{\mathrm{m}}_{R^{\prime}}}\left(F_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)=R \Gamma\left(Z_{R},\left(F_{l^{\prime}, E^{\prime} \mid Z_{R}}\right)^{\vee} \otimes \overline{\mathcal{T}}_{l, E}\right)$.
(2) As $i^{i} F_{l, E}=i^{*} F_{l, E} \otimes c_{1}\left(N_{Z_{R^{\prime}} \overline{\mathrm{M}}_{R^{\prime}}}\right)$, where $i: Z_{R^{\prime}} \hookrightarrow \overline{\mathrm{M}}_{R^{\prime}}$ is the embedding, we have by Huy06, Cor. 3.38] that $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}, F_{l, E}\right)=$ $R \Gamma\left(Z_{R^{\prime}}, \overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}^{\vee} \otimes F_{l, E}{ }_{\mid Z_{R^{\prime}}} \otimes c_{1}\left(N_{Z_{R^{\prime}} \mid} \overline{\mathrm{M}}_{R^{\prime}}\right)\right)$ (up to a shift).
(3) Follows from tensoring the Koszul resolution of $Z_{R}$ with a line bundle $L$ such that $i_{R *}\left(L_{\mid Z_{R}}\right)=\mathcal{T}_{l^{\prime}, E^{\prime}}$ and applying $R \operatorname{Hom}\left(-, \mathcal{T}_{l, E}\right)$.
(4) Recall that $R^{\prime}=\{r+1, \ldots, p\}$. By an analogue of Lemma 9.3(v) we have that $\overline{\mathcal{T}}_{l, E}=i_{R *}\left(-\frac{r+l}{2} \psi_{u}-\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}\right)$, where $i_{R}: Z_{R} \hookrightarrow \overline{\mathrm{M}}_{R^{\prime}}$. As in the proof of Lemma 9.4, we have that $R \operatorname{Hom}_{\overline{\mathrm{M}}_{R^{\prime}}}\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}, \overline{\mathcal{T}}_{l, E}\right)$ equals $R \Gamma\left(Y,\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime} \mid Y}\right)^{\vee} \otimes \overline{\mathcal{T}}_{l, E_{\mid Y}} \otimes c_{1}\left(N_{Y \mid Z_{R}}\right)\right)=R \Gamma\left(Y,\left(\mathcal{T}_{l^{\prime}, E^{\prime} \mid Y}\right)^{\vee} \otimes \mathcal{T}_{l, E \mid Y} \otimes\right.$ $\left.\left(\sum_{j \in R^{\prime} \backslash\{p\}} \delta_{j p}\right)_{\mid Y}\right)=R \Gamma\left(Y,-\left(\frac{l+l^{\prime}}{2}+1\right) \psi_{u}+\frac{1}{2} \sum_{j \in E_{q}^{\prime}} \psi_{j}-\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}\right)$ using an analogue of Lemma 9.3 (i), (ii), (iii).

Notation 10.13. We prove vanishing in Lemma 10.12 by a windows calculation on $Z_{R}, Z_{R^{\prime}}$ and $Y$. Since $Z_{R}$ and $Z_{R^{\prime}}$ intersect only boundary $\delta_{T, T^{c}}$ with $T_{p}=R$ or $R^{\prime}$, which get contracted in $\overline{\mathrm{M}}_{R^{\prime}}$, they are smooth GIT quotients of $\left(\mathbb{P}^{1}\right)^{r+q+1}$ by $\mathrm{PGL}_{2}$ by Lemma 2.7. More precisely, let $X=\mathbb{P}_{u}^{1} \times\left(\mathbb{P}^{1}\right)^{r} \times\left(\mathbb{P}^{1}\right)^{q}$, corresponding to the partition $\{u\} \sqcup R^{\prime} \sqcup Q$ of the markings on $Z_{R}=\overline{\mathrm{M}}_{\{u\} \cup R^{\prime} \cup Q}$. Then $Z_{R}$ is a GIT quotient of $X$ by $\mathrm{PGL}_{2}$. Likewise, let $X^{\prime}=\left(\mathbb{P}^{1}\right)^{r} \times \mathbb{P}_{v}^{1} \times\left(\mathbb{P}^{1}\right)^{q}$, corresponding to the partition $R \sqcup\{v\} \sqcup Q$ of the markings on $Z_{R^{\prime}}=\overline{\mathrm{M}}_{R \cup\{v\} \cup Q}$. Then $Z_{R^{\prime}}$ is a GIT quotient of $X^{\prime}$ by $\mathrm{PGL}_{2}$. In addition, let $X^{\prime \prime}=\left(\mathbb{P}^{1}\right)^{q+2}=\mathbb{P}_{u}^{1} \times \mathbb{P}_{v}^{1} \times\left(\mathbb{P}^{1}\right)^{q}$, corresponding to the partition $\{u\} \sqcup\{v\} \sqcup Q$ of the markings on $\overline{\mathrm{M}}_{\{u, v\} \cup Q}$. Then $Y$ is the GIT quotient of $X^{\prime \prime}$ by $\mathrm{PGL}_{2}$.
Vector bundles on $Z_{R}$ (resp., $Z_{R^{\prime}}$ ) correspond to $\mathrm{PGL}_{2}$-linearized vector bundles on $X$ (resp., $X^{\prime}$ ) as follows: $F_{l, E_{\mid Z_{R}}}=\mathcal{O}_{X}\left(\left|E_{p} \cap R\right|, E_{p} \cap\right.$ $\left.R^{\prime}, E_{q}\right) \otimes V_{l} \overline{\mathcal{T}}_{l, E}=\mathcal{O}_{X}\left(r+l, 0, E_{q}\right)$ if $E_{p}=R, K_{Z_{R}}=\mathcal{O}(-2,-2, \ldots,-2)$, $F_{l, E \mid Z_{R^{\prime}}}=\mathcal{O}_{X^{\prime}}\left(E_{p} \cap R,\left|E_{p} \cap R^{\prime}\right|, E_{q}\right) \otimes V_{l}, \delta_{1 j \mid Z_{R}}=\mathcal{O}(2,0,0, \ldots)(j \in R \backslash\{1\})$, $\delta_{p j \mid Z_{R^{\prime}}}=\mathcal{O}(0, \ldots, 0,2,0, \ldots, 0)\left(j \in R^{\prime} \backslash\{p\}\right)$ (use Lemma[3.2, Remark 3.8 and an analogue of Lemma 9.3 (v)).

Remark 10.14. (The devil's trick reloaded.) Since for any $j \in R^{\prime}$ (resp., $j \in R)$, we have $\psi_{u}+\psi_{j}=0$ in $Z_{R}\left(\psi_{v}+\psi_{j}=0\right.$ in $\left.Z_{R^{\prime}}\right)$ as $\delta_{u j}=\emptyset$ (resp., $\delta_{v j}=\emptyset$ ). Therefore, as in Remark 9.7, the line bundles $\mathbb{D}:=\mathcal{O}\left(r, R^{\prime}, 0\right)$ on $X=\left(\mathbb{P}^{1}\right)^{r+q+1}, \mathbb{D}:=\mathcal{O}(R, r, 0)$ on $X^{\prime}=\left(\mathbb{P}^{1}\right)^{r+q+1}$, descend to trivial line bundles on $Z_{R}$ and $Z_{R^{\prime}}$. Likewise, the line bundle $\mathbb{D}:=\mathcal{O}(1,1,0)$ on $X^{\prime \prime}$ descends to the trivial line bundle on $Y$.

Proof of Theorem 10.9. We prove the vanishings in Lemma 10.12 As in the proof of Theorem 9.2, we will first prove the $\mathrm{PGL}_{2}$-invariant vanishing holds on $X$ (cases (1) and (3)), on $X^{\prime}$ (case (2)) and on $X^{\prime \prime}$ (case (4)) after tensoring with the devil line bundle $\mathbb{D}^{N}, N \gg 0$. Later on we check the weight condition (9.4) in Theorem 9.6 .

For (1), assuming condition (9.4), $R \Gamma\left(Z_{R},\left(F_{l^{\prime}, E^{\prime} \mid Z_{R}}\right)^{\vee} \otimes \overline{\mathcal{T}}_{l, E} \otimes \mathbb{D}^{N}\right)=$ $R \Gamma\left(X, \mathcal{O}\left(r+l-\left|E_{p}^{\prime} \cap R\right|+N r,-E_{p}^{\prime} \cap R^{\prime}+N R^{\prime},-E_{q}^{\prime}+E_{q}\right) \otimes V_{l^{\prime}}\right)^{\mathrm{PGL}_{2}}$, which is clearly 0 if $E_{q}^{\prime} \nsubseteq E_{q}$. Since here we assume $e_{q}^{\prime} \geq e_{q}$, we have that $E_{q}^{\prime} \subseteq E_{q}$ if and only if $E_{q}=E_{q}^{\prime}$. Assume $E_{q}=E_{q}^{\prime}$. In this case, we need to consider $V_{r+l-\left|E_{p}^{\prime} \cap R\right|+N r} \otimes V_{N-y_{1}} \otimes \ldots \otimes V_{N-y_{r}} \otimes V_{l^{\prime}}$, where $y_{i}=1$ if the corresponding index is in $E_{p}^{\prime} \cap R^{\prime}$ and 0 otherwise. We claim that the $\mathrm{PGL}_{2}$-invariant part is 0 by the Clebsch-Gordan formula (Lemma 3.7). It suffices to check that $r+l-\left|E_{p}^{\prime} \cap R\right|+N r>N r-\left|E_{p}^{\prime} \cap R^{\prime}\right|+l^{\prime}$. This follows if $l^{\prime}+\left|E_{p}^{\prime} \cap R\right|-\left|E_{p}^{\prime} \cap R^{\prime}\right|<r$, which holds by Corollary 4.9 (since $\left(l^{\prime}, E^{\prime}\right)$ is in group $1 A$ or $1 B$ ).

For (2), assuming (9.4), $R \Gamma\left(Z_{R^{\prime}}, \overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}^{\vee} \otimes F_{l, E_{\mid Z_{R^{\prime}}}} \otimes c_{1}\left(N_{Z_{R^{\prime}}} \overline{\mathrm{M}}_{R^{\prime}}\right) \otimes \mathbb{D}^{N}\right)=$ $R \Gamma\left(X^{\prime}, \mathcal{O}\left(E_{p} \cap R+N R,\left|E_{p} \cap R^{\prime}\right|-r-l^{\prime}+2(r-1)+N r, E_{q}-E_{q}^{\prime}\right) \otimes V_{l}\right)^{\mathrm{PGL}_{2}}$. This is 0 since in our case $E_{q}^{\prime} \nsubseteq E_{q}$.

For (3), assuming (9.4), $R \Gamma\left(Z_{R}, \overline{\mathcal{T}}_{l^{\prime}, E^{\prime}}^{\vee} \otimes \overline{\mathcal{T}}_{l, E} \otimes\left(\sum_{j \in J} \delta_{1 j}\right) \otimes \mathbb{D}^{N}\right)$ equals $R \Gamma\left(X, \mathcal{O}\left(2|J|+l-l^{\prime}+N r, N R^{\prime}, E_{q}-E_{q}^{\prime}\right)\right)^{\mathrm{PGL}_{2}}$ for all $J \subseteq R \backslash\{1\}$. This is 0 if $E_{q}^{\prime} \nsubseteq E_{q}$. Assume now $E_{q}=E_{q}^{\prime}$. We need to consider $V_{2|J|+l-l^{\prime}+N r} \otimes$ $V_{N}^{\otimes r}$, whose $\mathrm{PGL}_{2}$-invariant part is 0 when $l>l^{\prime}$ or when $l=l^{\prime},|J|>0$ by the Clebsch-Gordan formula (Lemma 3.7), since in these cases we have $2|J|+l-l^{\prime}+N r>N r$.

For (4), assuming (9.4), $R \Gamma\left(Y,\left(\overline{\mathcal{T}}_{l^{\prime}, E^{\prime} \mid Y}\right)^{\vee} \otimes \overline{\mathcal{T}}_{l, E \mid Y} \otimes c_{1}\left(N_{Y \mid Z_{R}}\right) \otimes \mathbb{D}^{N}\right)=$ $R \Gamma\left(X^{\prime \prime}, \mathcal{O}\left(l+l^{\prime}+2+N, N, E_{q}-E_{q}^{\prime}\right)\right)^{\mathrm{PGL}}$. This is 0 if $E_{q}^{\prime} \nsubseteq E_{q}$. If $E_{q}=E_{q}^{\prime}$ we have to consider $V_{l+l^{\prime}+2+N} \otimes V_{N}$, whose $\mathrm{PGL}_{2}$-invariant part is 0 by the Clebsch-Gordan formula (Lemma 3.7), as $l+l^{\prime}+2+N>N$.

We now check that for each stratum, each of the cases (1)-(4) fall under the assumption on weights of Thm. 9.6. The unstable loci in $X$ (resp., $X^{\prime}$ ) corresponding to the loci $Z_{R}$ (resp., $Z_{R^{\prime}}$ ) in $\overline{\mathrm{M}}_{R^{\prime}}$ have the following form:
(The locus $K_{I}$ ), for $I \subseteq Q,|I| \geq s+1$ (resp., $|I| \geq s+2$ ), where $u$ (resp., $v$ ) and the indices in $I$ come together. In this case, $\eta=2|I|$.
(The locus $K_{J, I}$ ), for $J \subseteq R^{\prime}$ (resp., $J \subseteq R$ ), $I \subseteq Q, J \neq \emptyset,|I| \geq 0$, where $u$ (resp., $v$ ) and the indices in $J$ and $I$ come together. In this case, $\eta=2|I|+2|J|$.
(The locus $L_{I}$ ), for $I \subseteq Q,|I| \geq s+2$ (resp., $|I| \geq s+1$ ), where the indices in $R^{\prime}$ and $I$ (resp., $R$ and $I$ ) come together. In this case, $\eta=2|I|+2 r-2$.

The devil line bundle $\mathcal{O}\left(r, R^{\prime}, 0\right)$ on $X$ has the property that its weight for $K_{J, I}$ is $r+|J|-\left|R^{\prime} \backslash J\right|=2|J|>0$ while its weight for the other strata is 0 (similarly for $\mathcal{O}(R, r, 0)$ on $X^{\prime}$ ). Therefore, the condition (9.4) for the stratum $K_{J, I}$ can be achieved by tensoring with a high enough multiple of this line bundle. We only need to consider the remaining strata.

Consider first cases (1) and (3) involving $Z_{R} \subseteq \overline{\mathrm{M}}_{R^{\prime}}$. For case (1) we need to verify that the vector bundle $\mathcal{O}\left(r+l-\left|E_{p}^{\prime} \cap R\right|,-E_{p}^{\prime} \cap R^{\prime}, E_{q}-E_{q}^{\prime}\right) \otimes V_{l^{\prime}}$ on $X$ has the weights $>-\eta$ : For the stratum $K_{I}(|I| \geq s+1)$, we need to prove that $r+l-l^{\prime}-\left|E_{p}^{\prime} \cap R\right|+\left|E_{p}^{\prime} \cap R^{\prime}\right|+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\mid E_{q} \cap$ $I^{c}\left|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2\right| I \mid$. By (4.10) for the pair ( $l^{\prime}, E_{p}^{\prime}$ ), it suffices to prove that $l+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq-2|I|$. Since the left hand side is greater than $-\left|I^{c}\right|-|I|=-q-1=-2 s-2$ and $|I| \geq s+1$, the result follows. For the stratum $L_{I}(|I| \geq s+2)$, we need to prove that $-r-l-l^{\prime}+\left|E_{p}^{\prime} \cap R\right|-\left|E_{p}^{\prime} \cap R^{\prime}\right|+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\mid E_{q}^{\prime} \cap$ $I^{c}|>-2| I \mid-2 r+2$. By (4.10) for the pair $\left(l^{\prime}, E_{p}^{\prime}\right)$, it suffices to prove that $-l+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq-2|I|+2$. As $-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq$ $-|I|$, it suffices to prove that $-l+\left|E_{q} \cap I\right|-\left|E_{q} \cap I^{c}\right| \geq-|I|+2$. This follows from Lemma 4.11 since $|I| \geq s+2$ and the pair $(l, E)$ is in group $2 A$ or $2 B$.

For case (3) we need to verify that the line bundle $\mathcal{O}\left(2|J|+l-l^{\prime}, 0, E_{q}-E_{q}^{\prime}\right)$ on $X$ has weights $>-\eta$ for all $0 \leq|J| \leq r-1$. For the stratum $K_{I}$, we need to prove that $l-l^{\prime}+2|J|+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2|I|$. Note that it suffices to prove that $-l^{\prime}-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right|-\left|I^{c}\right|>-2|I|$. This follows from Lemma 4.11 since $|I| \geq s+1$ for the locus $K_{I}$ in $X$. For the stratum $L_{I}(|I| \geq s+2)$, we need to prove that $-l+l^{\prime}-2|J|+$ $\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2|I|-2 r+2$, or equivalently, $-l+l^{\prime}+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2|I|$. As the left hand side is greater or equal than $-l+\left|E_{q} \cap I\right|-\left|E_{q} \cap I^{c}\right|-|I|$, it suffices to prove $-l+\left|E_{q} \cap I\right|-\left|E_{q} \cap I^{c}\right|>-|I|$. This follows from Lemma4.11 since $|I| \geq s+2$ and the pair $(l, E)$ is in group $2 A$ or $2 B$.

For case (2), involving $Z_{R^{\prime}} \subseteq \overline{\mathrm{M}}_{R^{\prime}}$, the relevant vector bundle on $X^{\prime}$ is $\mathcal{O}\left(E_{p} \cap R,\left|E_{p} \cap R^{\prime}\right|+r-l^{\prime}-2, E_{q}-E_{q}^{\prime}\right) \otimes V_{l}$. For the stratum $K_{I}(|I| \geq s+2)$, we need to prove that $r-l-l^{\prime}-2+\left|E_{p} \cap R^{\prime}\right|-\left|E_{p} \cap R\right|+\left|E_{q} \cap I\right|-\mid E_{q}^{\prime} \cap$ $I\left|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2\right| I \mid$. By (4.10) for the pair $\left(l, E_{p}\right)$, it suffices to prove that $-l^{\prime}-1+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2|I|$. As the left hand side is greater or equal than $-l^{\prime}-1-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right|-\left|I^{c}\right|$, it suffices to prove that $-l^{\prime}-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq\left|I^{c}\right|-2|I|+2$. This follows from Lemma 4.11]since $|I| \geq s+2$. For the stratum $L_{I}(|I| \geq s+1)$, we need to prove that $-r+l^{\prime}-l+2+\left|E_{p} \cap R\right|-\left|E_{p} \cap R^{\prime}\right|+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-$ $\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2|I|-2 r+2$. By (4.10) for the pair ( $l, E_{p}$ ), it suffices to prove that $l^{\prime}+\left|E_{q} \cap I\right|-\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq-2|I|$. As the left hand side is greater or equal than $-|I|-\left|I^{c}\right|=-q-1$, the inequality follows since $|I| \geq s+1$.

We now consider Case (4). Up to symmetry, the unstable loci in $X^{\prime}=$ $\mathbb{P}_{u}^{1} \times \mathbb{P}_{v}^{1} \times\left(\mathbb{P}^{1}\right)^{q+1}$ corresponding to $Y=Z_{R} \cap Z_{R^{\prime}} \subseteq \overline{\mathrm{M}}_{R^{\prime}}$ are
(The locus $K_{I}^{\prime}$ ), for $I \subseteq Q,|I| \geq s+1$, where $u$ and the indices in $I$ come together. In this case, $\eta=2|I|$.
(The locus $K_{I}^{\prime \prime}$ ), for $I \subseteq Q,|I| \geq s+2$, where $v$ and the indices in $I$ come together. In this case, $\eta=2|I|$.
(The locus $K_{I}^{\prime \prime \prime}$ ), for $I \subseteq Q,|I| \geq 0$, where $u, v$ and the indices in $I$ come together. In this case, $\eta=2|I|+2$.

As before, the devil line bundle $\mathcal{O}(1,1,0)$ on $X^{\prime \prime}$ has the property that its weight at $K_{I}^{\prime \prime \prime}$ is $2>0$, while its weight for the other strata is 0 . Hence, as before, we only have to consider the strata $K_{I}^{\prime}, K_{I}^{\prime \prime}$.

We verify that the weights of $\mathcal{O}\left(l+l^{\prime}+2,0, E_{q}-E_{q}^{\prime}\right)$ are $>-2|I|$. For the stratum $K_{I}^{\prime}(|I| \geq s+1)$, we need to prove that $l+l^{\prime}+2+\left|E_{q} \cap I\right|-\mid E_{q}^{\prime} \cap$ $I\left|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2\right| I \mid$. This is clear, as the left hand side is greater or equal than $2-|I|-\left|I^{c}\right|=2-(q+1)=-2 s$ and $|I| \geq s+1$.

For the stratum $K_{I}^{\prime \prime}(|I| \geq s+2)$, we need to prove that $-l-l^{\prime}+\left|E_{q} \cap I\right|-$ $\left|E_{q}^{\prime} \cap I\right|-\left|E_{q} \cap I^{c}\right|+\left|E_{q}^{\prime} \cap I^{c}\right|>-2|I|+2$. We apply Lemma4.11] to the pairs $\left(l, E_{q}\right)$ and $\left(l^{\prime}, E_{q}^{\prime}\right)$. Recall that $e_{q}^{\prime} \geq e_{q}$. If in case $2 A$ and $e_{q}^{\prime} \leq s$ (resp., case $2 B$ and $\left.e_{q}^{\prime} \leq s+1\right)$, then Lemma 4.11implies that $-l+\left|E_{q} \cap I\right|-\left|E_{q} \cap I^{c}\right| \geq$ $-s+1$, (resp., $\geq-s$ ), $-l^{\prime}-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq-s+1$, (resp., $\geq-s$ ), and the inequality follows by summing up, as $-2 s>-2|I|+2$.

If in case $2 A$ and $e_{q}^{\prime}>s \geq e_{q}$ (resp., case $2 B$ and $e_{q}^{\prime}>s+1 \geq e_{q}$ ), then Lemma 4.11implies that $-l+\left|E_{q} \cap I\right|-\left|E_{q} \cap I^{c}\right| \geq-s+1$, (resp., $\geq-s$ ), $-l^{\prime}-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq-2|I|+s+2$, (resp., $\geq-2|I|+s+3$ ), and the inequality follows again by summing up.

If in case $2 A$ and $e_{q}>s$ (resp., case $2 B$ and $e_{q}>s+1$ ), then Lemma 4.11 implies that $-l+\left|E_{q} \cap I\right|-\left|E_{q} \cap I^{c}\right| \geq-2\left|I^{c}\right|+s+2$ (resp., $\geq 2\left|I^{c}\right|+s+3$ ), $-l^{\prime}-\left|E_{q}^{\prime} \cap I\right|+\left|E_{q}^{\prime} \cap I^{c}\right| \geq-2|I|+s+2$ (resp., $\geq 2\left|I^{c}\right|+s+3$ ), and the inequality follows again by summing up, since $-2|I|-2\left|I^{c}\right|+2 s+4>-2|I|+2$, (as $\left|I^{c}\right|<s+1$, since $|I| \geq s+2$ ).

## 11. Fullness of the exceptional collection on $\overline{\mathrm{M}}_{2 r, 2 s+1}$

In this section we finish the proof of Theorem 1.5 (see also Remark 1.11), which gives a full equivariant exceptional collection in $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$ for $p=$ $2 r \geq 4, q=2 s+1 \geq 1$. Exceptionality of the collection was proved in Section 9 , here we prove fullness. Since $\overline{\mathrm{M}}_{p, q}$ is a GIT quotient, its derived category is generated by vector bundles $F_{l, E}$ for all pairs $(l, E)$ such that $l+e$ is even (Proposition 3.11). So it remains to prove the following theorem.
Theorem 11.1. Assume $p=2 r \geq 4, q=2 s+1 \geq 1$. The vector bundles $\left\{F_{l, E}\right\}$ for all $l \geq 0, E \subseteq P \cup Q, e=|E|$, with $l+e$ even, are generated by any of the two exceptional collections in $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$ given by Theorem 1.5 and Remark 1.11 In particular, these exceptional collections are full.
Proof. We prove equivalently that the dual of the collection in Thm. 1.5 (groups $1 A$ and 2, resp. $1 B$ and 2 , see Remark 1.11), call it $\mathcal{C}$, generates all the dual vector bundles $F_{l, E}^{\vee}$. We will do an induction based on the score $S(l, E)$ (see Notation 4.2). For equal scores, we do an induction on $l$. If $S(l, E) \leq r-2$, then $l+\min \left\{e_{p}+1, p+1-e_{p}\right\} \leq r-1$, i.e., the pair $(l, E)$ belongs to both groups $1 A$ and $1 B$. This ensures the base case of the induction.

We now assume that we have a pair $(l, E)$ (not in group $1 A$, resp. $1 B$ ) such that the bundle $F_{l^{\prime}, E^{\prime}}^{\vee}$ is generated by $\mathcal{C}^{\vee}$ for any $\left(l^{\prime}, E^{\prime}\right)$ with $S\left(l^{\prime}, E^{\prime}\right)<$
$S(l, E)$, or if $S(l, E)=S\left(l^{\prime}, E^{\prime}\right)$ and $l^{\prime}<l$. Like in the proof of Claim 3.16, we will use constructions based on the Koszul complex. We will call a set $I \subseteq P \cup Q$ stable if $i_{p} \leq r-1$ or if $i_{p}=r, i_{q} \leq s$, where as usual we denote $I_{p}=I \cap P, I_{q}=I \cap Q, i_{p}=\left|I_{p}\right|, i_{q}=\left|I_{q}\right|$. Otherwise, $I$ is called unstable (the usual notions of GIT stability giving $\overline{\mathrm{M}}_{p, q}$ ). Let $i:=|I|$. Given a subset $I \subseteq P \cup Q$, we consider the Koszul complex for $\Delta=\cap_{k \in I} \delta_{k x} \subseteq\left(\mathbb{P}^{1}\right)^{p+q} \times \mathbb{P}_{x}^{1}$,

$$
\begin{equation*}
0 \cong\left[\mathcal{O}_{\Delta} \leftarrow \mathcal{O} \leftarrow \bigoplus_{k \in I} \mathcal{O}\left(-\mathbf{e}_{\mathbf{k}}-\mathbf{e}_{\mathbf{x}}\right) \leftarrow \ldots \leftarrow \mathcal{O}\left(-\sum_{k \in I} \mathbf{e}_{\mathbf{k}}-i \mathbf{e}_{\mathbf{x}}\right)\right] . \tag{11.1}
\end{equation*}
$$

Here $\left\{\mathbf{e}_{\mathbf{i}}\right\}$ for $i \in P \cup Q \cup\{x\}$ is the standard basis of $\operatorname{Pic}\left(\mathbb{P}^{1}\right)^{p+q+1} \cong \mathbb{Z}^{p+q+1}$. Koszul Game 1: Assume $I$ is an unstable set disjoint from $E \subseteq P \cup Q$. Assume $e+l$ even. Tensoring (11.1) with $\mathcal{O}\left(-\sum_{k \in E} \mathbf{e}_{\mathbf{k}}+l \mathbf{e}_{\mathbf{x}}\right)$, pushing forward to $\left(\mathbb{P}^{1}\right)^{p+q}$, and restricting to the semistable locus gives by Corollary 3.3 and Grothendieck-Verdier duality (3.1) the following objects:

where the second line appears only if $i \geq l+2$. If $i \leq l$, the first line stops at $F_{l-i, E \cup I}^{\vee}$. By Lemma 3.17, any of these objects is generated by the rest.

Koszul Game 2: Assume $I$ is an unstable set disjoint from $E^{\prime} \subseteq P \cup Q$. Let $E=E^{\prime} \cup I, e+l$ even. Tensoring (11.1) with $\mathcal{O}\left(-\sum_{k \in E^{\prime}} \mathbf{e}_{\mathbf{k}}+(i-2-l) \mathbf{e}_{\mathbf{x}}\right)$, pushing forward, and restricting to the semistable locus gives by Corollary 3.3 and Grothendieck-Verdier duality (3.1) the following objects:

$$
\begin{array}{cccccccc}
0 & F_{i-2-l, E^{\prime}}^{\vee} & \cdots & \bigoplus_{J \subseteq I,|J|=j} F_{i-2-l-j, E^{\prime} \cup J}^{\vee} & \cdots & \bigoplus_{J \subseteq I,|J|=i-2-l} & F_{0, E^{\prime} \cup J}^{\vee} \\
0 & \bigoplus_{J \subseteq I,|J|=i-l} F_{0, E^{\prime} \cup J}^{\vee}[-1] & \ldots & \bigoplus_{J \subseteq I,|J|=j} F_{j-i+l, E^{\prime} \cup J}^{\vee}[-1] & \ldots & F_{l, E^{\prime} \cup I}^{\vee}[-1],
\end{array}
$$

where the first line appears only if $i \geq l+2$, while if $i \leq l$, the second line starts at $F_{l-i, E^{\prime}}^{\vee}$. By Lemma 3.17, any of these objects is generated by the rest.

Koszul Game 3: Assume $I=E_{p}=R \subseteq P, E \subseteq P \cup Q,|R|=r, e+l$ even. We tensor(11.1) with $\mathcal{O}\left(-\sum_{j \in E_{q}} \mathbf{e}_{\mathbf{j}}+(r-2-l) \mathbf{e}_{\mathbf{x}}\right)$, push forward, and restrict to the semistable locus. Unlike in the previous games, the torsion object has support in the semistable locus. Note that if $\mathcal{U}_{R} \rightarrow Z_{R}$ is the universal family over $Z_{R}$, then $\Delta$ corresponds to the section $\sigma_{u}\left(Z_{R}\right)$. Identifying $\Delta \cong(\mathbb{P})_{u}^{1} \times\left(\mathbb{P}^{1}\right)^{r+q}$, where $u$ is the marking corresponding to indices in $R$ we have $\mathcal{O}\left(-\sum_{j \in E_{q}} \mathbf{e}_{\mathbf{j}}+(r-2-l) \mathbf{e}_{\mathbf{x}}\right)_{\mid \Delta} \cong \mathcal{O}\left(-\sum_{j \in E_{q}} \mathbf{e}_{\mathbf{j}}+(r-2-l) \mathbf{e}_{\mathbf{u}}\right)$, which descends to $Z_{R} \subseteq \overline{\mathrm{M}}_{p, q}$ as the line bundle $-\frac{r-2-l}{2} \psi_{u}+\frac{1}{2} \sum_{j \in E_{q}} \psi_{j}=$ $-\frac{r-2-l+e_{q}}{2} \psi_{u}-\sum_{j \in E_{q}} \delta_{j u}$ (use Remark 3.8). Using Lemma 11.2 and GrothendieckVerdier duality (3.1), the terms have the following derived push-forwards:

$$
\mathcal{T}_{l, E}^{\vee}[r-1] \quad F_{r-l-2, E_{q}}^{\vee} \quad \cdots \quad \bigoplus_{\substack{J \subseteq R \\|J|=j}} F_{r-2-l-j, E_{q} \cup J}^{\vee} \quad \cdots \quad \bigoplus_{\substack{J \subseteq R \\|J|=r-2-l}} F_{0, E_{q} \cup J}^{\vee}
$$

0

$$
\bigoplus_{\substack{J \subseteq R \\|J|=r-l}} F_{0, E_{q} \cup J}^{\vee}[-1] \ldots \bigoplus_{\substack{J \subseteq R \\|J|=j}} F_{l-r+j, E_{q} \cup J}^{\vee}[-1] \ldots F_{l, E_{q} \cup R}^{\vee}[-1]=F_{l, E}^{\vee}[-1]
$$

where the first part of the sequence of bundles $F_{r-2-l-j, E_{q} \cup J}^{\vee}$ appears only when $r \geq l+2$, while if $r \leq l$, then the second part of the sequence starts at $F_{l-r, E_{q}}^{\vee}$. Note that the score of every $F_{\tilde{l}, \tilde{E}}^{\vee}$ with $(\tilde{l}, \tilde{E}) \neq(l, E)$ in the Koszul game 3 is strictly lower than the score of $F_{l, E}^{\vee}$ :
(a) For $(\tilde{l}, \tilde{E})=\left(r-2-l-j, E_{q} \cup J\right)(0 \leq j \leq r-2-l)$, the score $\tilde{S}$ is $\tilde{S}=(r-2-l-j)+j+\min \left\{e_{q}, q-e_{q}\right\}=r-2-l+\min \left\{e_{q}, q-e_{q}\right\}$, and clearly, we have $\tilde{S}<S:=S(l, E)=l+r+\min \left\{e_{q}, q-e_{q}\right\}$.
(b) For $(\tilde{l}, \tilde{E})=\left(l-r+j, E_{q} \cup J\right)(0 \leq j \leq r)$, the score $\tilde{S}$ is $\tilde{S}=$ $(l-r+j)+j+\min \left\{e_{q}, q-e_{q}\right\}=l-r+2 j+\min \left\{e_{q}, q-e_{q}\right\}$, and clearly, we have again $\tilde{S}<S$ if $J \neq R$, i.e., $\tilde{E} \neq E$.
In particular, if our starting pair $(l, E)$ is in group 2 , since $\mathcal{T}_{l, E}$ is in $\mathcal{C}$, the bundle $F_{l, E}^{\vee}$ is generated by $\mathcal{C}^{\vee}$ by induction, since all other terms in the above Koszul resolution have lower score. Hence we may assume that $(l, E)$ is not in group 2 (in addition to not being in group $1 A$ or $1 B$ ).

There are four cases in the induction argument.
Case 1: $e_{p}<r$. Since we assume that $(l, E)$ is not in group $1 A$ (resp., $1 B$ ), we have $l+e_{p} \geq r$ (resp., $l+e_{p} \geq r-1$ ). We play the Koszul game 1 with a set $I=I_{p}, i_{p}=r+1, i_{q}=0$ ("minimal" unstable $I$ disjoint from $E$ ). We verify that, for every $F_{\tilde{l}, \tilde{E}}^{\vee}$ in the list, its score $S(\tilde{l}, \tilde{E})$ is less or equal than $S:=S(l, E)=l+e_{p}+\min \left\{e_{q}, q-e_{q}\right\}$.
(a) Let $(\tilde{l}, \tilde{E})=(l-j, E \cup J), 0<j \leq l$. Note that $l=0$ cannot occur for this type of $F_{\tilde{l}, \tilde{E}}^{\vee}$. The score $S(\tilde{l}, \tilde{E})$ is $(l-j)+\min \left\{e_{p}+j, p-e_{p}-j\right\}+$ $\min \left\{e_{q}, q-e_{q}\right\}=l+\min \left\{e_{p}, p-e_{p}-2 j\right\}+\min \left\{e_{q}, q-e_{q}\right\} \leq l+e_{p}+$ $\min \left\{e_{q}, q-e_{q}\right\}=S$. If $S(\tilde{l}, \tilde{E})=S$ then, as $j>0$, we are done by induction on $l$.
(b) For $(\tilde{l}, \tilde{E})=(j-l-2, E \cup J), l+2 \leq j \leq i=r+1$, the score $S(\tilde{l}, \tilde{E})$ is $(j-l-2)+\min \left\{e_{p}+j, p-e_{p}-j\right\}+\min \left\{e_{q}, q-e_{q}\right\}$. Since $e_{p}+j \geq e_{p}+l+2 \geq$ $r+1$, we have $\min \left\{e_{p}+j, p-e_{p}-j\right\}=p-\left(e_{p}+j\right)$. When working with group $1 A$, since $l+e_{p} \geq r$, we have $S(\tilde{l}, \tilde{E})=p-e_{p}-l-2+\min \left\{e_{q}, q-e_{q}\right\}<$ $S$. When working with group $1 B$, we still have to consider the situation $l+e_{p}=r-1$. But then $\tilde{l}+\left(p-\tilde{e}_{p}\right)=(j-l-2)+\left(p-e_{p}-j\right)=r-1$, and therefore $F_{\tilde{l}, \tilde{E}}$ is in $1 B$.

Case 2: $e_{p}=r, e_{q} \leq s$. As $(E, l)$ is not in group 2, we must have $l+e_{q} \geq s$.
We now play the Koszul game 1 with a set $I$ with $i_{p}=r, i_{q}=s+1$ ("minimal" unstable $I$ ). We verify that the the score of every $F_{\tilde{l}, \tilde{E}}^{\vee}$ in the complex is less or equal than $S:=S(l, E)=l+r+e_{q}$.
a) Let $(\tilde{l}, \tilde{E})=(l-j, E \cup J), 0<j \leq l$. Note again that $l=0$ cannot occur for this type of $F_{\tilde{l}, \tilde{E}}^{\vee}$. The score $S(\tilde{l}, \tilde{E})=(l-j)+\left(r-j_{p}\right)+\min \left\{e_{q}+j_{q}, q-\right.$ $\left.e_{q}-j_{q}\right\} \leq(l-j)+\left(r-j_{p}\right)+\left(e_{q}+j_{q}\right)=l+r+e_{q}-2 j_{p} \leq S$. If $S(\tilde{l}, \tilde{E})=S$ then, as $j>0$, we are done by induction on $l$.
(b) For $(\tilde{l}, \tilde{E})=(j-l-2, E \cup J)(l+2 \leq j \leq i)$, the score $S(\tilde{l}, \tilde{E})=$ $(j-l-2)+\left(r-j_{p}\right)+\min \left\{e_{q}+j_{q}, q-e_{q}-j_{q}\right\} \leq(j-l-2)+\left(r-j_{p}\right)+\left(q-e_{q}-j_{q}\right)<$ $l+r+e_{q}=S$ since we assume $l+e_{q} \geq s$.

Case 3: $e_{p} \geq r+1$. Recall that we may assume that $(E, l)$ is not in group $1 A($ resp., $1 B)$. Then $l+\left(p+1-e_{p}\right) \geq r$ (resp., $\left.l+\left(p-e_{p}\right) \geq r\right)$, or equivalently, $e_{p} \leq l+r+1$ (resp., $e_{p} \leq l+r$ ). We now play the Koszul game 2 with the set $I=I_{p} \subseteq E_{p}, i=i_{p}=r+1\left(i_{q}=0\right)$ and with $E_{p}^{\prime}=E_{p} \backslash I_{p}, E_{q}^{\prime}=E_{q}$ ( $e_{p}^{\prime}=e_{p}-(r+1), e_{q}^{\prime}=e_{q}$ - so "minimal" unstable $\left.I\right)$. We verify that the score of every $F_{\tilde{l}, \tilde{E}}^{\vee}$ in the complex is less than or equal to $S:=S(l, E)=$ $l+\left(p-e_{p}\right)+\min \left\{e_{q}, q-e_{q}\right\}$.
a) For $(\tilde{l}, \tilde{E})=\left(i-l-2-j, E^{\prime} \cup J\right), 0 \leq j \leq i-l-2$, the score $S(\tilde{l}, \tilde{E})=$ $(i-l-2-j)+\min \left\{e_{p}^{\prime}+j, p-e_{p}^{\prime}-j\right\}+\min \left\{e_{q}, q-e_{q}\right\} \leq(i-l-2-j)+\left(e_{p}^{\prime}+j\right)+$ $\min \left\{e_{q}, q-e_{q}\right\}=e_{p}-l-2+\min \left\{e_{q}, q-e_{q}\right\}$. In case $1 B$, since $e_{p} \leq l+r$, we have $S(\tilde{l}, \tilde{E})<S$. In case $1 A$, since $e_{p} \leq l+r+1$, we have $S(\tilde{l}, \tilde{E})<S$, unless $e_{p}=l+r+1$, when $S(\tilde{l}, \tilde{E})=S$. However, if $e_{p}=l+r+1$, we can move on as $F_{\tilde{l}, \tilde{E}}^{\vee}$ is in group $1 A: \tilde{l}+\min \left\{e_{p}^{\prime}+j, p+1-e_{p}^{\prime}-j\right\} \leq(i-l-2-j)+\left(e_{p}^{\prime}+j\right)=r-1$.
b) For $(\tilde{l}, \tilde{E})=\left(j-i+l, E^{\prime} \cup J\right), \max \{i-l, 0\} \leq j<i$, the score $S(\tilde{l}, \tilde{E})=$ $(j-i+l)+\min \left\{e_{p}^{\prime}+j, p-e_{p}^{\prime}-j\right\}+\min \left\{e_{q}, q-e_{q}\right\} \leq(j-i+l)+\left(p-e_{p}^{\prime}-\right.$ $j)+\min \left\{e_{q}, q-e_{q}\right\}=l+p-e_{p}+\min \left\{e_{q}, q-e_{q}\right\}=S\left(\right.$ since $\left.e_{p}^{\prime}=e_{p}-i\right)$. If $S(\tilde{l}, \tilde{E})=S$, we are done by induction on $l$ since $j-i+l<l$. Note again that in this case $l>0$.

Case 4: $e_{p}=r, e_{q} \geq s+1$. Recall that we may assume that $(E, l)$ is in group 2, i.e., $l+\left(q-e_{q}\right) \geq s$, or equivalently, $e_{q} \leq l+s+1$. We now play the Koszul game 2 with the set $I$ with $I_{p}=E_{p}, I_{q}=E_{q}$, i.e., $E^{\prime}=\emptyset$ ("maximal" unstable $I$ with $I \subseteq E$ ). In particular, we have $i=e$. We verify that the score of every $F_{\tilde{l}, \tilde{E}}^{\vee}$ in the complex is less or equal than $S:=S(l, E)=$ $l+r+q-e_{q}$.
a) For $(\tilde{l}, \tilde{E})=(e-l-2-j, J), 0 \leq j \leq e-l-2$, the score $S(\tilde{l}, \tilde{E})=$ $(e-l-2-j)+j_{p}+\min \left\{j_{q}, q-j_{q}\right\} \leq\left(e-l-2-j_{q}\right)+j_{q}=e-l-2=$ $r+e_{q}-l-2<l+r+q-e_{q}=S$, since by assumption $e_{q} \leq l+s+1$.
b) $(\tilde{l}, \tilde{E})=(j-e+l, J)(\max \{e-l, 0\} \leq j<e)$, the score $S(\tilde{l}, \tilde{E})=$ $(j-e+l)+j_{p}+\min \left\{j_{q}, q-j_{q}\right\} \leq j_{q}-e+l+2 j_{p}+\left(q-j_{q}\right)=q-e+l+2 j_{p}=$ $q-e_{q}-r+l+2 j_{p} \leq l+r+q-e_{q}$, with equality if and only if $j_{q} \geq s+1$, $j_{p}=r$. If $S(\tilde{l}, \tilde{E})=S$ then, since $j<e$, we are done by induction on $l$ since $j-e+l<l$. Note again that in this case $l>0$.

Lemma 11.2. Let $\overline{\mathrm{M}}$ be one of $\overline{\mathrm{M}}_{p, q}$ or $\overline{\mathrm{M}}_{p, q+1}$. Let $R=\{1, \ldots, r\} \subseteq P,|R|=r$ and let $i: Z_{R} \rightarrow \overline{\mathrm{M}}$ denotes the inclusion map. If $(l, E)$ is a pair with $E_{p}=R$, the derived dual of $\mathcal{T}_{l, E}$ is given by $\mathcal{T}_{l, E}^{\vee}=i_{*}\left(\frac{2-e_{q}-r+l}{2} \psi_{u}-\sum_{j \in E_{q}} \delta_{j u}\right)[1-r]$.

Proof. If $L$ is a line bundle such that $L_{\mid Z_{R}}=\mathcal{T}_{l, E}=i_{*}\left(\frac{e_{q}-r-l}{2} \psi_{u}+\sum_{j \in E_{q}} \delta_{j u}\right)$ then $\mathcal{T}_{l, E}^{\vee}=L^{\vee} \otimes \mathcal{O}_{Z_{R}}^{\vee}=L^{\vee} \otimes \operatorname{det} N_{Z_{R}}[-c]$ [Huy06, 3.40], where $c=r-1$ is the codimension of $Z_{R}$ and $\operatorname{det} N_{Z_{R}}=\sum_{j \in R \backslash\{1\}}\left(\delta_{1 j}\right)_{\mid Z_{R}}=-(r-1) \psi_{u}$ (use Lemma 9.3 for $\overline{\mathrm{M}}_{p, q}$ and its analogue for $\overline{\mathrm{M}}_{p, q+1}$ ) is the determinant of the normal bundle.

Corollary 11.3. $D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$ is generated by the bundles $F_{l, E}$ with $(l, E)$ in group $1 A$ (resp., 1B) and group 2.
Proof. This follows immediately from the proof of Thm. 11.1 ,
We finish this section by analyzing fullness on $\mathcal{U}_{p, q}$, the universal family over $\overline{\mathrm{M}}_{p, q}$. This will be used in the next section. Recall from Section 8 that $\mathcal{U}_{p, q}$ is a Hassett space for the set of markings $P \cup \tilde{Q}$, where $\tilde{Q}=Q \cup\{z\}$ and $z$ is an extra light index, and since $\mathcal{U}_{p, q}$ is a GIT quotient (Lemma 8.3) the space $\mathcal{U}_{p, q}$ carries vector bundles $F_{l, E}$ by Definition 3.1.
Theorem 11.4. $D^{b}\left(\mathcal{U}_{p, q}\right)$ is generated by the bundles $F_{l, E}$ with $(l, E)$ in one of the following groups: (i) $1 A$ (resp., $1 B$ ), (ii) $2 A$ with $z \notin E$, (iii) $2 B$ with $z \in E$. Here we use the same groups $1 A, 1 B, 2 A$ and $2 B$ as in Thm. 1.8
Proof. This is a consequence of Orlov's Theorem for the universal family $\mathcal{U}_{p, q} \xrightarrow{\pi} \overline{\mathrm{M}}_{p, q}$, which is a $\mathbb{P}^{1}$ bundle, and is similar to the proof of Lemma 8.9. Recall that if $z \notin E$, then $F_{l, E}=\pi^{*} F_{l, E}$ by Theorem 3.4. If $z \notin E$, the range $2 A$ on $\mathcal{U}_{p, q}$ is exactly the range of group 2 on $\overline{\mathrm{M}}_{p, q}$ and the range $1 A$ (resp., $1 B$ ) on $\mathcal{U}_{p, q}$ corresponds to $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q}$. Hence, by Corollary 11.3 the $F_{l, E}$ 's with $z \notin E$ generate $L \pi^{*} D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$. If $z \in E$, then $(l, E)$ is in range $2 B$ on $\mathcal{U}_{p, q}$ if and only if $\left(l, E^{c}\right)$ is in the group 2 on $\overline{\mathrm{M}}_{p, q}$, and similarly, $(l, E)$ is in range $1 A$ (resp., $1 B$ ) on $\mathcal{U}_{p, q}$ if and only if $\left(l, E^{c}\right)$ is in the group $1 B$ (resp., $1 A$ ) on $\overline{\mathrm{M}}_{p, q}$. By Corollary 11.3 , the corresponding duals $\left\{F_{l, E^{c}}^{\vee}\right\}$ (where $\left.E^{c}=(P \cup \tilde{Q}) \backslash E\right)$ ) generate $L \pi^{*} D^{b}\left(\overline{\mathrm{M}}_{p, q}\right)$. As $F_{l, E}=F_{l, E^{c}}^{\vee} \otimes F_{0, \Sigma}$, it follows that the objects with $z \in E$ generate $L \pi^{*} D^{b}\left(\overline{\mathrm{M}}_{p, q}\right) \otimes F_{0, \Sigma}$. As in the proof of Lemma 8.9 the result now follows by Orlov's Theorem.

## 12. FULLNESS OF THE EXCEPTIONAL COLLECTION ON $\overline{\mathrm{M}}_{2 r, 2 s+2}$

Notation 12.1. We let $\mathcal{A}$ to be the subcategory in $D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ generated by the torsion sheaves $\mathcal{O}_{\delta_{T, T^{c}}}(-a,-b)$ of Notation 1.6. Let $\mathcal{C}$ be the collection of Theorem 1.8 (exceptional by $\S 10$ ) which consists of generators of $\mathcal{A}$, vector bundles $F_{l, E}$ in group $1 A$ (resp., $1 B$ ) combined with the complexes $\tilde{\mathcal{T}}_{l, E}$ in group $2 B$. We let $\mathcal{C}^{\prime}$, resp., $\mathcal{C}_{F}^{\prime}$, be the collection obtained from $\mathcal{C}$ by replacing complexes $\tilde{\mathcal{T}}_{l, E}$ in $\mathcal{C}$ with the torsion sheaves $\mathcal{T}_{l, E}$, resp., the vector bundles $F_{l, E}$. Note that the collections $\mathcal{C}^{\prime}$ and $\mathcal{C}_{F}^{\prime}$ are not, in general, exceptional. We follow Notation 8.2; $p=2 r, q=2 s+1, \tilde{Q}=Q \cup\{z\}$. Throughout the section, unless we specify otherwise, we assume $p \geq 4$, $q+1 \geq 0$.
Proposition 12.2. Let $S^{\prime}$ be the score of Notation 4.2. Let $l \geq 0, E \subseteq P \cup \tilde{Q}$, $l+e$ is even, $S^{\prime}(l, E) \leq r+s$. Then
(1) The bundle $F_{l, E}$ is generated by $\mathcal{A}$ together with $\left\{F_{l^{\prime}, E^{\prime}}\right\}$ with $\left(l^{\prime}, E^{\prime}\right)$ in group $1 A$ (resp., $1 B$ ) or $2 B$ and such that $S^{\prime}\left(l^{\prime}, E^{\prime}\right) \leq S^{\prime}(l, E)$.
(2) The bundle $F_{l, E}$ is generated by $\mathcal{C}_{F}^{\prime}$.

We prove Proposition 12.2 later in this section, after we first establish some auxiliary results. Recall that by Lemma 4.3, for $l \geq 0, E \subseteq P \cup \tilde{Q}$ and $(l, E)$ in any of the groups $1 A, 1 B, 2 A$ or $2 B$, we have $S^{\prime}(l, E) \leq r+s$.

Similarly, for $l \geq 0, E \subseteq P \cup Q$ and $(l, E)$ in any of the groups $1 A, 1 B$, or 2 , we have $S(l, E) \leq r+s-1$ (here $p \geq 4, q \geq 0$ ).

Notation 12.3. Recall the notation of Lemma 7.2, let $\alpha: W \rightarrow \overline{\mathrm{M}}_{p, q+1}$ be the universal family and $f: W \rightarrow \overline{\mathrm{M}}_{p, q+2}=\overline{\mathrm{M}}_{P, \tilde{Q} \cup\{y\}}$ the birational map that contracts the $T^{c}$ component of boundary divisors $\delta_{T \cup\{y\}, T^{c}}$ on $W$ (where $y$ is the new marking on $W$ ).

Remark 12.4. For $l \geq 0, E \subseteq \tilde{Q}$, the pair $(l, E)$ can be considered on both of $\overline{\mathrm{M}}_{p, q+1}$ and $\overline{\mathrm{M}}_{p, q+2}$. Consider such a pair $(l, E)$.
(i) Clearly, $(l, E)$ is in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q+1}$ if and only if it is in group $1 A$ (resp., $1 B$ ) on $\overline{\mathrm{M}}_{p, q+2}$. Furthermore, $(l, E)$ is in group $2 B$ on $\overline{\mathrm{M}}_{p, q+1}$ if and only if $(l, E)$ is in group 2 on $\overline{\mathrm{M}}_{p, q+2}$.
(ii) The score $S^{\prime}(l, E)$ on $\overline{\mathrm{M}}_{p, q+1}$ relates to the score

$$
\begin{aligned}
& S(l, E)=l+\min \left\{e_{p}, p-e_{p}\right\}+\min \left\{e_{q}, q+2-e_{q}\right\} \text { on } \overline{\mathrm{M}}_{p, q+2} \\
& \text { by } S^{\prime}(l, E)= \begin{cases}S(l, E) & \text { if } \quad e_{q} \leq s+1 \\
S(l, E)-1 & \text { if } \quad e_{q} \geq s+2\end{cases}
\end{aligned}
$$

(iii) As noted above, for $(l, E)$ on $\overline{\mathrm{M}}_{p, q+2}$ in groups $1 A, 1 B, 2$, we have $S(l, E) \leq r+s$. Furthermore, equality holds if and only if we are in one of the following cases:
(1A) $e_{p}=r-1-l, e_{q}=s+1$ or $s+2$;
(1B) $e_{p}=r+1+l, e_{q}=s+1$ or $s+2$;
(2) $e_{p}=r, e_{q}=s-l$ or $l+s+3$.

Lemma 12.5. Let $l \geq 0$ and $E \subseteq P \cup \tilde{Q}, l+e$ even. Then on $\overline{\mathrm{M}}_{p, q+1}$ the bundle $F_{l, E}$ and the complex $R \alpha_{*} f^{*} F_{l, E}$ are related by quotients of the form

$$
\begin{equation*}
Q=\mathcal{O}_{\delta_{T, T^{c}}}(-v, u), \quad 0 \leq u \leq m_{T^{c}}-1, \quad 0<v \leq m_{T^{c}}-1 \tag{12.1}
\end{equation*}
$$

where $m_{T^{c}}=\max \left(0, f_{T^{c}, E, l}\right) \leq S^{\prime}(l, E) / 2$. In particular, $F_{l, E}$ and $R \alpha_{*} f^{*} F_{l, E}$ are related by quotients in $\mathcal{A}$ if the score $S^{\prime}(l, E) \leq r+s$.

Recall that unless we will write $\mathcal{O}_{\delta}(a, b)=\mathcal{O}_{T}(a) \boxtimes \mathcal{O}_{T^{c}}(b)$ to emphasize that $\mathcal{O}(a)$ corresponds to the $T$ component, the notation $\mathcal{O}_{\delta}(a, b)$ will generally mean that $\mathcal{O}(a)$ could correspond to either the $T$ or $T^{c}$ component.

Proof of Lemma 12.5 By Proposition 7.7, the bundles $F_{l, E}$ and $f^{*} F_{l, E}$ on $W$ are related by quotients $-j H \boxtimes \mathcal{O}(u), 0<j \leq m_{T^{c}}, 0 \leq u \leq m_{T^{c}}-1$, supported on $\delta_{T \cup\{y\}, T^{c}} \cong \mathrm{Bl}_{1} \mathbb{P}^{r+s} \times \mathbb{P}^{r+s-1}$. By Lemma 5.9, $R \pi_{*}(-j H)=0$ if $j=1$, while if $j \geq 2$, it is generated by $\mathcal{O}(-v)$ with $1 \leq v \leq j-1$. It follows that $R \alpha_{*}(-j H \boxtimes \mathcal{O}(u))$ is generated by quotients (12.1). The rest follows from Lemma 4.4

Lemma 12.6. Let $l \geq 0, E \subseteq P \cup \tilde{Q}, l+e$ even, $S^{\prime}(l, E) \leq r+s$. On $\overline{\mathrm{M}}_{p, q+2}$, the bundle $F_{l, E}$ can be generated by $\left\{F_{l^{\prime}, E^{\prime}}\right\}$ for ( $l^{\prime}, E^{\prime}$ ) belonging to group $1 A$ (resp., $1 B$ ) and 2 on $\overline{\mathrm{M}}_{p, q+2}$ such that $y \notin E^{\prime}$, and such that $S^{\prime}\left(l^{\prime}, E^{\prime}\right) \leq S^{\prime}(l, E)$.

Proof. The proof is parallel to the proof of Theorem11.1, playing the same Koszul games as on $\overline{\mathrm{M}}_{p, q}$, except this time on $\overline{\mathrm{M}}_{p, q+2}$. We do an induction
on the score $S(l, E)$ on $\overline{\mathrm{M}}_{p, q+2}$ and for equal scores induction on $l$. By Remark 12.4(ii), $S(l, E)=S^{\prime}(l, E)$ if $e_{q} \leq s+1$ and $S(l, E)=S^{\prime}(l, E)+1$ if $e_{q} \geq s+2$. We only point out the main arguments for each case.

Case 1: $e_{p}<r$. Clearly, we may assume $(E, l)$ is not in group $1 A$ (resp., $1 B)$. This is parallel to Case 1 in the proof of Theorem 11.1. As in that case, using the same Koszul game 1 on $\overline{\mathrm{M}}_{p, q+2}$ (with $I=I_{p}$ with $\left|I_{p}\right|=r+1$, disjoint from $E$ ), we generate $F_{l, E}$ with $F_{\tilde{l}, \tilde{E}}$ with $\tilde{E}$ of the form $E \cup J$, with $J$ a set of heavy indices (in particular, $y \notin \tilde{E}$ ) and such that either $S(\tilde{l}, \tilde{E})<S(l, E)$ or $S(\tilde{l}, \tilde{E})=S(l, E), \tilde{l}<l$, to which one has to add, when working with group $1 B$, the possibility that $S(\tilde{l}, \tilde{E})=S(l, E)$ and $(\tilde{l}, \tilde{E})$ is in group $1 B$. Since $\tilde{E}_{q}=E_{q}$, we have that either that $S^{\prime}(\tilde{l}, \tilde{E})=S(\tilde{l}, \tilde{E})$, $S^{\prime}(l, E)=S(l, E)$, or $S^{\prime}(\tilde{l}, \tilde{E})=S(\tilde{l}, \tilde{E})-1, S^{\prime}(l, E)=S(l, E)-1$. In particular, $S^{\prime}(\tilde{l}, \tilde{E}) \leq S^{\prime}(l, E) \leq r+s$, i.e., the hypotheses in the Lemma are satisfied for $(\tilde{l}, \tilde{E})$.
Case 2: $e_{p}=r, e_{q} \leq s+1$. Clearly, we may assume $(E, l)$ is not in group 2 on $\overline{\mathrm{M}}_{p, q+2}$. By our assumption that $S^{\prime}(l, E) \leq r+s$, we cannot have $e_{q}=$ $s+1$. Hence, $e_{q} \leq s$. This is parallel to Case 2 in the proof of Theorem 11.1. As in that case, using the Koszul game 1 on $\overline{\mathrm{M}}_{p, q+2}$ for a set $I$ disjoint from $E$ with $\left|I_{p}\right|=r$ and $\left|I_{q}\right|=s+2$, and in addition with $y \notin I_{q}$ (possible since $e_{q} \leq s$ ), we generate $F_{l, E}$ with $F_{\tilde{l}, \tilde{E}}$ with $\tilde{E}$ of the form $E \cup J$, with $J \subseteq I$ (in particular, $y \notin \tilde{E})$ and such that either $S(\tilde{l}, \tilde{E})<S(l, E)$ or $S(\tilde{l}, \tilde{E})=S(l, E)$, $\tilde{l}<l$. Furthermore, we have $S^{\prime}(\tilde{l}, \tilde{E}) \leq S(\tilde{l}, \tilde{E}) \leq S(l, E)=S^{\prime}(l, E) \leq r+s$, (as $S^{\prime}(l, E)=S(l, E)$ because $\left.e_{q} \leq s\right)$ and if $S^{\prime}(\overline{\tilde{l}}, \tilde{E})=S^{\prime}(l, E)$, then $\tilde{l}<l$.

Case 3: $e_{p} \geq r+1$. Clearly, we may assume ( $E, l$ ) is not in group $1 A$ (resp., 1B). This is parallel to Case 3 in the proof of Theorem11.1. As in that case, using the same Koszul game 2 on $\overline{\mathrm{M}}_{p, q+2}\left(I=I_{p} \subseteq E_{p},|I|=r+1\right)$, we generate $F_{l, E}$ with $F_{\tilde{l}, \tilde{E}}$ with $\tilde{E}$ of the form $E \backslash J^{\prime}$, with $J^{\prime}$ a set of heavy indices (in particular, $y \notin \tilde{E}$ ) and such that either $S(\tilde{l}, \tilde{E})<S(l, E)$, or $S(\tilde{l}, \tilde{E})=S(l, E), \tilde{l}<l$, or, when working with group $1 A, S(\tilde{l}, \tilde{E})=S(l, E)$ and $(\tilde{l}, \tilde{E})$ is in group $1 A$. Since $\tilde{E}_{q}=E_{q}$, we have that either that $S^{\prime}(\tilde{l}, \tilde{E})=$ $S(\tilde{l}, \tilde{E}), S^{\prime}(l, E)=S(l, E)$, or $S^{\prime}(\tilde{l}, \tilde{E})=S(\tilde{l}, \tilde{E})-1, S^{\prime}(l, E)=S(l, E)-1$. In particular, $S^{\prime}(\tilde{l}, \tilde{E}) \leq S^{\prime}(l, E) \leq r+s$, i.e., the hypotheses in the Lemma are satisfied for $(\tilde{l}, \tilde{E})$.

Case 4: $e_{p}=r, e_{q} \geq s+2$. Clearly, we may assume $(E, l)$ is not in group 2 on $\overline{\mathrm{M}}_{p, q+2}$. This is parallel to Case 4 in the proof of Theorem 11.1. As in that case, using the Koszul game 2 on $\overline{\mathrm{M}}_{p, q+2}(I=E)$, we generate $F_{l, E}$ with $F_{\tilde{l}, \tilde{E}}$ with $\tilde{E}$ of the form $\tilde{E} \subseteq E$, (in particular, $y \notin \tilde{E}$ ) and such that either $S(\tilde{l}, \tilde{E})<S(l, E)$ or $S(\tilde{l}, \tilde{E})=S(l, E), \tilde{l}<l$. Note, if $S(\tilde{l}, \tilde{E})<S(l, E)$ then $S^{\prime}(\tilde{l}, \tilde{E}) \leq S(\tilde{l}, \tilde{E})<S(l, E)=S^{\prime}(l, E)+1$ (since $e_{q} \geq s+2$ ). In particular, it follows that $S^{\prime}(\tilde{l}, \tilde{E}) \leq S^{\prime}(l, E) \leq r+s$.

Assume now $S(\tilde{l}, \tilde{E})=S(l, E), \tilde{l}<l$. We still need to prove that in this case $S^{\prime}(\tilde{l}, \tilde{E}) \leq S^{\prime}(l, E) \leq r+s$. As in Case 4 of the proof of Theorem11.1, the only way to have $S(\tilde{l}, \tilde{E})=S(l, E)$ is when we are in case b) $(\tilde{l}, \tilde{E})=$ $(j-e+l, J)(j<e$, so $\tilde{l}<l)$ and $j_{q} \geq s+2(s$ is replaced by $s+1$, since we
are on $\overline{\mathrm{M}}_{p, q+2}$ instead of $\overline{\mathrm{M}}_{p, q}$. Since $\left|\tilde{E}_{q}\right|=j_{q} \geq s+2$, we have $S^{\prime}(\tilde{l}, \tilde{E})=$ $S(\tilde{l}, \tilde{E})-1$. Hence, $S^{\prime}(\tilde{l}, \tilde{E})=S(\tilde{l}, \tilde{E})-1=S(l, E)-1=S^{\prime}(l, E) \leq r+s$.
Proof of Proposition 12.2 Let $l \geq 0, E \subseteq P \cup \tilde{Q}, l+e$ even, $S^{\prime}(l, E) \leq r+s$. By Lemma 12.6 , the bundle $F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+2}$ can be generated by bundles $F_{l^{\prime}, E^{\prime}}$ for $\left(l^{\prime}, E^{\prime}\right)$ belonging to group $1 A$ (resp., $1 B$ ) and 2, with $y \notin E^{\prime}$, and such that $S^{\prime}\left(l^{\prime}, E^{\prime}\right) \leq S^{\prime}(l, E)$. By Remark $12.4(\mathrm{i})$, such $\left(l^{\prime}, E^{\prime}\right)$ can be considered also on $\overline{\mathrm{M}}_{p, q+1}$ and they correspond to groups $1 A$ (resp., $1 B$ ) and group $2 B$ on $\overline{\mathrm{M}}_{p, q+1}$. It follows that the object $R \alpha_{*} f^{*} F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+1}$ can be generated by the objects $R \alpha_{*} f^{*} F_{l^{\prime}, E^{\prime}}$. By Lemma 4.3, for all such ( $l^{\prime}, E^{\prime}$ ) on $\overline{\mathrm{M}}_{p, q+1}$ we have $S^{\prime}\left(l^{\prime}, E^{\prime}\right) \leq r+s$. Lemma 12.5 now implies part (1) of Proposition 12.2. Part (2) follows immediately from (1).
Definition 12.7. Let $l \in \mathbb{Z}$ (positive or negative). Recall that objects $F_{l, E}$ on the stack $\mathcal{M}_{n}$, and therefore also on all GIT quotients without strictly semistable points, were defined in Proposition 3.9. It was proved there that $F_{l, E}=0$ for $l=-1$ and $F_{l, E} \simeq F_{-l-2, E}[-1]$ for $l \leq-2$. We would like to define analogous objects on $\overline{\mathrm{M}}_{p, q+1}$. Let $\alpha: W \rightarrow \overline{\mathrm{M}}_{p, q+1}$ be the universal family. For any $l \in \mathbb{Z}$ (positive or negative) and $E \subseteq P \cup \tilde{Q}$, let

$$
\begin{equation*}
N_{l, E}^{\prime}=\omega_{\alpha}^{\frac{e-l}{2}}(E), \quad F_{l, E}^{\prime}=R \alpha_{*}\left(N_{l, E}^{\prime}\right) \tag{12.2}
\end{equation*}
$$

Proposition 12.8. Let $l \geq-1, E \subseteq P \cup \tilde{Q}, l+e$ even. The bundle $F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+1}$ as defined in Definition [5.2 and the object $F_{l, E}^{\prime}$ of (12.2) are related by quotients of the form $\mathcal{O}_{\delta}(-v, u), v>0, u \geq 0, u, v \leq \frac{S^{\prime}(l, E)}{2}-l-1$. In particular, if $l \geq 0$, $S^{\prime}(l, E) \leq r+s, F_{l, E}^{\prime}, F_{l, E}$ are related by quotients in $\mathcal{A}$.
Proof. By Definition 5.2, $F_{l, E}=R \alpha_{*}\left(N_{l, E}\right)$. We compare $F_{l, E}^{\prime}$ with $F_{l, E}$ by comparing on $W$ the line bundles $N_{l, E}^{\prime}$ and $N_{l, E}$ :

$$
N_{l, E}^{\prime}=N_{l, E}+\sum_{T \sqcup T^{c}=P \cup \tilde{Q}, f_{T, E, l}<0}\left(-f_{T, E, l}\right) \delta_{T \cup\{y\}} .
$$

Since $l \geq-1$ and $e+l$ is even, we cannot have both $f_{T, E, l}<0, f_{T^{c}, E, l}<0$. Hence, for every partition $T \sqcup T^{c}=P \cup \tilde{Q}$ at most one of $\delta_{T \cup\{y\}, T^{c}}, \delta_{T^{c} \cup\{y\}, T}$ appears on the right hand side of the equality. The quotients on $W$ relating $N_{l, E}^{\prime}$ and $N_{l, E}$ have the form $Q=\left(N_{l, E}^{\prime}+\left(-\alpha_{T}+i\right) \delta\right)_{\mid \delta}=(-i H) \boxtimes \mathcal{O}\left(\alpha_{T}-i\right)$ (use Lemma 5.8), where $\delta=\delta_{T \cup\{y\}, T^{c}}, \quad 0<i \leq \alpha_{T}:=-f_{T, E, l}$. Since by Lemma [5.9, $R \pi_{*}(-i H)$ is 0 if $i=1$ and generated by $\mathcal{O}(-v)$, with $v=$ $1, \ldots, i-1$ if $i \geq 2$, it follows that the non-zero quotients relating $F_{l, E}^{\prime}$ and $F_{l, E}$ have the form $\mathcal{O}_{T}(-v) \boxtimes \mathcal{O}_{T^{c}}(u), 0 \leq u=\alpha_{T}-i \leq \alpha_{T}-1,0<v<i \leq$ $\alpha_{T}$. By Lemma 4.4, we have $\alpha_{T} \leq \frac{S^{\prime}(l, E)}{2}-l$, hence, $u, v \leq \frac{S^{\prime}(l, E)}{2}-l-1$.
Corollary 12.9. Suppose $r+s$ is even. Fix a boundary $\delta=\delta_{T_{0}, T_{0}^{c}} \subset \overline{\mathrm{M}}_{p, q+1}$ and let $J \subseteq T_{0}, j=|J|(0 \leq j \leq r+s)$. Then $F_{j-1, T_{0}^{c} \cup J}$ and $F_{j-1, T_{0}^{c} \cup J}^{\prime}(i f j>0)$ and $F_{-1, T_{0}^{c}}^{\prime}$ and $\mathcal{O}_{\delta}\left(-\frac{r+s}{2}, 0\right)=\mathcal{O}_{T}\left(-\frac{r+s}{2}\right) \boxtimes \mathcal{O}_{T^{c}}$ are related by objects in $\mathcal{A}$.
Proof. This is a particular case of Proposition 12.8 (and its proof) for pairs $(l, E)=\left(j-1, T_{0}^{c} \cup J\right),|J|=j$. Note that $S^{\prime}(l, E)=r+s$. If $j>0$, the first statement follows from Proposition 12.8. Consider now the case $j=0$. Set
$(l, E)=\left(-1, T_{0}^{c}\right)$. Recall from the proof of Proposition 12.8 that the nonzero quotients relating $F_{l, E}^{\prime}$ and $F_{l, E}=0$ have the form $\mathcal{O}_{T}(-v) \boxtimes \mathcal{O}_{T^{c}}(u)$ (for some boundary $\delta_{T, T^{c}}$ ) $0 \leq u=\alpha_{T}-i \leq \alpha_{T}-1,0<v<i \leq \alpha_{T}$, and $\alpha_{T} \leq \frac{S^{\prime}(l, E)}{2}-l=\frac{r+s}{2}+1$. Such quotients belong to the generators of $\mathcal{A}$ when $v<\frac{r+s}{2}$. So the only possible exception is when $v=\frac{r+s}{2}$, which implies that $i=\alpha_{T}=\frac{S^{\prime}(l, E)}{2}-l$. In particular, $u=0$. By Lemma 4.4, the equality $\alpha_{T}=\frac{S^{\prime}(l, E)}{2}-l$ happens if and only if $T=T_{0}$ and the second statement follows.
Corollary 12.10. Suppose $r+s$ is even. Consider a partition $T \sqcup T^{c}=P \cup \tilde{Q}$. The sheaves $\mathcal{O}_{\delta}\left(-\frac{r+s}{2}, 0\right), \mathcal{O}_{\delta}\left(0,-\frac{r+s}{2}\right), \delta=\delta_{T, T^{c}} \subseteq \overline{\mathrm{M}}_{p, q+1}$ are generated by $\mathcal{C}_{F}^{\prime}$.

In Corollary 12.10 we write both $\mathcal{O}_{\delta}(a, b), \mathcal{O}_{\delta}(b, a)$ to emphasize that $\mathcal{O}(a)$ could correspond to either the $T$ or $T^{c}$ component. We will do the same in Proposition 12.12
Proof of Corollary 12.10 We prove that $\mathcal{O}_{\delta}\left(-\frac{r+s}{2}, 0\right)$ (hence, by symmetry, also $\left.\mathcal{O}_{\delta}\left(0,-\frac{r+s}{2}\right)\right)$ is generated by the bundles $\left\{F_{l, E}\right\}$ on $\overline{\mathrm{M}}_{p, q+1}$ with score $S^{\prime}(l, E)=r+s$ and objects in $\mathcal{A}$. In particular, by Proposition 12.2(2), it is generated by the collection $\mathcal{C}_{F}^{\prime}$.

Consider the Koszul resolution of $\bigcap_{i \in T} \delta_{i y}=\emptyset$ in $W$ :

$$
0 \leftarrow \mathcal{O} \leftarrow \ldots \leftarrow \ldots \leftarrow \oplus_{J \subseteq T,|J|=j} \mathcal{O}\left(-\sum_{i \in J} \delta_{i y}\right) \leftarrow \ldots \leftarrow \mathcal{O}\left(-\sum_{i \in T} \delta_{i y}\right) \leftarrow 0 .
$$

Dualizing and tensoring with $\omega_{\alpha}^{\frac{r+s}{2}+1}\left(\sum_{i \in T^{c}} \delta_{i y}\right)$ and using the notation $N_{l, E}^{\prime}:=\omega_{\alpha}^{\frac{e-l}{2}}(E)$ (see Definition 12.7), we have the following long exact sequence of line bundles on $W$ :

$$
0 \rightarrow N_{-1, T^{c}}^{\prime} \rightarrow \cdots \rightarrow \bigoplus_{J \subseteq T,|J|=j} N_{j-1, T^{c} \cup J}^{\prime} \rightarrow \cdots \rightarrow N_{r+s, T^{c} \cup T}^{\prime} \rightarrow 0 .
$$

By applying $R \alpha_{*}(-)$, we obtain the following objects on $\overline{\mathrm{M}}_{p, q+1}$ :

$$
\begin{array}{cccccc}
F_{-1, T^{c}}^{\prime} \quad \oplus_{i \in T} F_{0, T^{c} \cup\{i\}}^{\prime} \quad \ldots & \oplus_{J \subseteq T,|J|=j} F_{j-1, T^{c} \cup J}^{\prime} & \ldots & F_{r+s, T^{c} \cup T}^{\prime}
\end{array}
$$

All the vector bundles $F_{j-1, T^{c} \cup J}$ have score $r+s$ and the statement follows by Proposition 12.2 and Corollary 12.9 .
Notation 12.11. As in Notn. 8.2, let $\beta=\beta_{z}: \overline{\mathrm{M}}_{p, q+1} \rightarrow \mathcal{U}_{p, q}$ be the morphism that contracts the $T$ component of any boundary divisor $\delta_{T, T^{c}}$ if $z \in T$. When $q+1=0$, we choose $z \in P$ and $\beta_{z}$ is the map $\overline{\mathrm{M}}_{p} \rightarrow \overline{\mathrm{M}}_{p-1,1}$ which lowers the weight of a heavy index $z$.
Proposition 12.12. Let $l \in \mathbb{Z}$ (positive or negative), $E \subseteq P \cup \tilde{Q}, z \in \tilde{Q}$. Then $F_{l, E}^{\prime}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients of the form $\mathcal{O}_{\delta}(-v, u), \mathcal{O}_{\delta}(u,-v)$, where
$0 \leq u, 0<v \leq \max \left\{\frac{S^{\prime}(l, E)}{2}-l-1, \frac{S^{\prime}(l, E)}{2}\right\}$.
The proof of Proposition 12.12 will show in fact that $F_{l, E}^{\prime}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients $\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}(-v), 0 \leq u, 0<v \leq \frac{S^{\prime}(l, E)}{2}$ and $\mathcal{O}_{T}(-v) \boxtimes$ $\mathcal{O}_{T^{c}}(u), 0 \leq u, 0<v \leq \frac{S^{\prime}(l, E)}{2}-l-1$, but we will not use this fact.

Proof of Proposition 12.12 Let $N_{1}=N_{l, E}^{\prime}$. Recall from (12.2) that $F_{l, E}^{\prime}=$ $R \alpha_{*}\left(N_{1}\right)$. We have by (8.3) that $\beta_{z}^{*} F_{l, E}=R \alpha_{*}\left(N_{2}\right)$, where $N_{2}=N_{1}+$ $\sum_{z \in T} f_{T, E, l} \delta_{T \cup\{y\}}$. We let

$$
N_{3}:=N_{1}+\sum_{z \in T, f_{T, E, l}>0} f_{T, E, l} \delta_{T \cup\{y\}}=N_{2}+\sum_{z \in T, f_{T, E, l}<0}\left(-f_{T, E, l}\right) \delta_{T \cup\{y\}} .
$$

The line bundles $N_{1}$ and $N_{3}$ are related by quotients $\left(N_{1}+i \delta\right)_{\mid \delta}=\left(f_{T, E, l}-\right.$ i) $H \boxtimes \mathcal{O}_{T^{c}}(-i), \delta=\delta_{T \cup\{y\}, T^{c}}(z \in T), 0<i \leq f_{T, E, l}$ (use Lemma [5.8). It follows by Lemma 5.9 that $F_{l, E}^{\prime}=R \alpha_{*}\left(N_{1}\right)$ and $R \alpha_{*}\left(N_{3}\right)$ are related by quotients of the form $Q_{1}=\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}(-i), u \geq 0,0<i \leq f_{T, E, l}$. By Lemma 4.4, $f_{T, E, l} \leq \frac{S^{\prime}(l, E)}{2}$.

The line bundles $N_{2}$ and $N_{3}$ are related by quotients $\left(N_{2}+i \delta\right)_{\mid \delta}=$ $\left(N_{1}-\left(-f_{T, E, l}\right) \delta+i \delta\right)_{\mid \delta}=(-i H) \boxtimes \mathcal{O}_{T^{c}}\left(-f_{T, E, l}-i\right), \delta=\delta_{T \cup\{y\}, T^{c}}(z \in T)$, $0<i \leq-f_{T, E, l}$ (use Lemma 5.8). Since by Lemma 5.9 that $R \pi_{*}(-i H)$ is either 0 or generated by $\mathcal{O}(-v)$ with $v=1, \ldots, i-1$, it follows that $F_{l, E}^{\prime}=R \alpha_{*}\left(N_{2}\right)$ and $R \alpha_{*}\left(N_{3}\right)$ are related by quotients of the form $\mathcal{O}_{T}(-v) \boxtimes$ $\mathcal{O}_{T^{c}}(u), u \geq 0,0<v \leq-f_{T, E, l}-1$. Since by Lemma 4.4- $-f_{T, E, l} \leq \frac{S^{\prime}(l, E)}{2}-l$, the result follows.

Corollary 12.13. Let $l \geq 0, E \subseteq P \cup \tilde{Q}, \tilde{Q}=Q \cup\{z\}$. Assume $S^{\prime}(l, E) \leq r+s$. Then $F_{l, E}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients in $\mathcal{A}$ and sheaves of type $\mathcal{O}_{\delta}\left(0,-\frac{r+s}{2}\right)=\mathcal{O}_{T} \boxtimes \mathcal{O}_{T^{c}}\left(-\frac{r+s}{2}\right)$. In fact the quotients in $\mathcal{A}$ suffice if any of the following holds:
(i) $r+s$ is odd or $S^{\prime}(l, E)<r+s$;
(ii) $r+s$ is even, $S^{\prime}(l, E)=r+s, e_{q} \geq s+1$ and $z \notin E$;

Proof. By Lemma 4.4, for all partitions $T \sqcup T^{c}=P \cup \tilde{Q}$ we have $m_{T, E, l} \leq$ $\frac{S^{\prime}(l, E)}{2}$, with equality if and only if ( $T_{p} \subseteq E_{p}$ or $E_{p} \subseteq T_{p}$ ) and ( $T_{q} \subseteq E_{q}$ or $E_{q} \subseteq T_{q}$ ). By Proposition 8.5(i), $F_{l, E}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients in $\mathcal{A}$ and sheaves of type $\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}(-j)$ where $0<j \leq m, 0<j \leq m$, $0 \leq u<m, m:=m_{T, E, l}$. The assumption $S^{\prime}(l, E) \leq r+s$ implies that $m \leq \frac{r+s}{2}$. It follows that $F_{l, E}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients in $\mathcal{A}$ and sheaves of type $\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}\left(-\frac{r+s}{2}\right), 0 \leq u<\frac{r+s}{2}$. But as the generators of $\mathcal{A}$ and the sheaves $\mathcal{O}_{T} \boxtimes \mathcal{O}_{T^{c}}\left(-\frac{r+s}{2}\right)$ generate all $\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}\left(-\frac{r+s}{2}\right)$ with $u$ positive or negative, the first statement follows. Clearly, if $r+s$ is odd or if $S^{\prime}(l, E)<r+s$, then the sheaves $\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}\left(-\frac{r+s}{2}\right)$ do not appear, which implies (i). To see (ii), note that for the sheaves $\mathcal{O}_{T}(u) \boxtimes \mathcal{O}_{T^{c}}\left(-\frac{r+s}{2}\right)$ to appear, we must have $S^{\prime}(l, E)=r+s$ is even and $m_{T, E, l}=\frac{S^{\prime}(l, E)}{2}$. Hence, by Lemma 4.4, $T_{q} \subseteq E_{q}$ or $E_{q} \subseteq T_{q}$. But if $e_{q} \geq s+1$, we must have $T_{q} \subseteq E_{q}$. In particular, $z \in E$. This implies (ii).
Corollary 12.14. Let $l \geq 0, E \subseteq P \cup \tilde{Q}, l+e$ even, $S^{\prime}(l, E) \leq r+s$. For any $z \in \tilde{Q}$, the bundle $\beta_{z}^{*} F_{l, E}$ on $\overline{\mathrm{M}}_{p, q+1}$ is generated by $\mathcal{C}_{F}^{\prime}$.
Proof. By Corollary 12.13, $F_{l, E}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients in $\mathcal{A}$ and sheaves of type $\mathcal{O}_{\delta}\left(0,-\frac{r+s}{2}\right)$. It follows by Corollary 12.10 that $F_{l, E}$ and $\beta_{z}^{*} F_{l, E}$ are related by quotients generated by $\mathcal{C}_{F}^{\prime}$. But by Proposition 12.2, $F_{l, E}$ is generated by $\mathcal{C}_{F}^{\prime}$. Hence, $\beta_{z}^{*} F_{l, E}$ is generated by $\mathcal{C}_{F}^{\prime}$.

Lemma 12.15. Assume $G \in D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ is such that $R \operatorname{Hom}(C, G)=0$, for any $C \in \mathcal{C}_{F}^{\prime}$. Then $G=0$.
Proof. By Corollary 12.14 , for any $z \in \tilde{Q}$, we have $R \operatorname{Hom}\left(\beta_{z}^{*} F_{l, E}, G\right)=0$ if $S^{\prime}(l, E) \leq r+s$. It follows that $R \operatorname{Hom}\left(F_{l, E}, R \beta_{z_{*}} G\right)=0$ whenever $S^{\prime}(l, E) \leq$ $r+s$. Since by Theorem 11.4 and Lemma 4.3 the collection $\left\{F_{l, E}\right\}$ on $\mathcal{U}_{p, q}$ with $S^{\prime}(l, E) \leq r+s$ contains as a subcollection a full exceptional collection of $D^{b}\left(\mathcal{U}_{p, q}\right)$, it follows that $R \beta_{z_{*}} G=0$ for every $z \in \tilde{Q}$. In particular, $G$ has support on the boundary. Since the boundary is a disconnected union of components, $G$ is isomorphic to a direct sum of complexes supported on irreducible boundary divisors. The result follows from Lemma 12.16 ,
Lemma 12.16. Let $X$ be a smooth variety and let $X \rightarrow X_{0}$ be a contraction of a divisor $E \simeq \mathbb{P}^{l} \times \mathbb{P}^{l}$ with normal bundle $\mathcal{O}(-1,-1)$. Let $f^{ \pm}: X \rightarrow X^{ \pm}$be two small resolutions of $X_{0}$ (contracting $E$ to $\mathbb{P}^{l}$ in two directions). Let $G \in D^{b}(X)$ and suppose $R f_{*}^{-}(G)=R f_{*}^{+}(G)=R \operatorname{Hom}\left(\mathcal{O}_{E}(-a,-b), G\right)=0$ for every $a, b=1, \ldots, l$. Then $G=0$.
Proof. By Orlov's blow-up theorem applied to $f^{+}, G$ belongs to a subcategory generated by the exceptional collection $\mathcal{O}_{E}(-a,-b)$ for every $a=$ $1, \ldots, l+1$ and $b=1, \ldots, l$, which has a s.o.d. $\langle\mathcal{B}, \mathcal{A}\rangle$ with $\mathcal{A}$ generated by sheaves with $a=1, \ldots, l$ and $\mathcal{B}$ by sheaves with $a=l+1$. Since $R \operatorname{Hom}(\mathcal{A}, G)=0$ by assumption, we conclude that $G \in \mathcal{B}$. We prove that $G=0$ by proving, by induction on $i$, that $G$ belongs to a subcategory $\mathcal{B}_{i}$ generated by $\left\{\mathcal{O}_{E}(-(l+1),-l), \ldots, \mathcal{O}_{E}(-(l+1),-i)\right\}$ for every $i>1$. Applying $R f_{*}^{-}$to the triangle $X \rightarrow Y \rightarrow G \rightarrow X[1]$ with $Y \simeq \mathcal{O}_{E}(-(l+$ 1), $-i$ ) $\otimes K^{\bullet}$ and $X \in \mathcal{B}_{i+1}$ (and $K^{\bullet}$ a complex of vector spaces with trivial differentials) gives $R f_{*}^{-} X \simeq R f_{*}^{-} \mathcal{O}_{E}(-(l+1),-i) \otimes K^{\bullet}=\mathcal{O}_{\mathbb{P}^{l}}(-i)[-l] \otimes K^{\bullet}$. On the other hand, $R f_{*}^{-} X$ belongs to a triangulated subcategory generated by $\mathcal{O}_{\mathbb{P} l}(-l)[-l], \ldots, \mathcal{O}_{\mathbb{P}^{l}}(-(i+1))[-l]$. By shrinking $X_{0}$ we can assume that the restriction map $\operatorname{Pic} X^{-} \rightarrow \operatorname{Pic} \mathbb{P}^{l}$ is surjective. Tensoring with an appropriate line bundle shows that $\mathcal{O}_{\mathbb{P}^{l}} \otimes K^{\bullet}$ belongs to a triangulated subcategory generated by $\mathcal{O}_{\mathbb{P} l}(-l+i), \ldots, \mathcal{O}_{\mathbb{P} l}(-1)$. Computing $R \Gamma$ shows that $K^{\bullet}=0$. It follows that $R f_{*}^{-} X=0$ and therefore $G \simeq X \in \mathcal{B}_{i+1}$.

In the remaining part of this section we prove the following Theorem (whose consequence is then the fullness part of Theorem 1.8).

Theorem 12.17. The exceptional collection $\mathcal{C}$ of Theorem 1.8 generates the collection $\mathcal{C}_{F}^{\prime}$ (see Notation 12.1).
Proof. By Lemma 10.6, $\mathcal{T}_{l, E}$ and $\tilde{\mathcal{T}}_{l, E}$ differ by quotients in $\mathcal{A}$. Hence, it suffices to prove that $\mathcal{C}^{\prime}$ generates $\mathcal{C}_{F}^{\prime}$, i.e., that we can "replace" $\mathcal{T}_{l, E}$ with $F_{l, E}$ when $(l, E)$ is in group $2 B$. By Lemma 4.3, it suffices to prove
Claim 12.18. Let $l \geq 0, E \subseteq P \cup \tilde{Q}, l+e$ even, $S^{\prime}(l, E) \leq r+s$. Then the bundle $F_{l, E}$ is generated by the collection $\mathcal{C}^{\prime}$.

We prove the statement by induction on the score $S^{\prime}(l, E)$ and for equal score, by induction on $l$. If $(l, E)$ is in group $1 A$ (resp. $1 B$ ), there is nothing to prove, so we assume this is not the case. We consider two cases, depending on whether $(l, E)$ is or is not in group $2 B$. Assume $(l, E)$ is not in
group 2B. By Proposition 12.2(1), the bundle $F_{l, E}$ is generated by $\left\{F_{l^{\prime}, E^{\prime}}\right\}$ with pairs $\left(l^{\prime}, E^{\prime}\right)$ either in group $1 A$ (resp. $1 B$ ) or $2 B$ and in addition with $S^{\prime}\left(l^{\prime}, E^{\prime}\right) \leq S^{\prime}(l, E)$, i.e., we are reduced to prove the statement for $(l, E)$ in group $2 B$. Consider now the case when $(l, E)$ is in group $2 B$. We need to prove that $F_{l, E}$ is generated by $\mathcal{C}^{\prime}$. By Proposition 12.8, it suffices to prove that $F_{l, E}^{\prime}$ is generated by $\mathcal{C}^{\prime}$.

Lemma 12.19. $F_{l, E}^{\prime}$ and $\mathcal{T}_{l, E}\left(E_{p}=R\right)$, are related by objects $\left\{F_{\tilde{l}, \tilde{E}}^{\prime}\right\}$ for $(\tilde{l}, \tilde{E})=\left(j+l-r, E_{q} \cup J\right), J \subseteq R, j=|J|, 0 \leq j<r$. (Note that $\tilde{l}=j+l-r$ could be negative.)

Proof. Let $W \rightarrow \overline{\mathrm{M}}_{p, q+1}$ be as usual the universal family ( $y$ new marking on $W$ ). Consider the Koszul complex for $\Delta=\cap_{i \in R} \delta_{i y} \subset W$ :

$$
0 \leftarrow \mathcal{O}_{\Delta} \leftarrow \mathcal{O} \leftarrow \ldots \leftarrow \bigoplus_{J \subseteq R, j=|J|} \mathcal{O}\left(-\sum_{k \in J} \delta_{k y}\right) \leftarrow \ldots \leftarrow \mathcal{O}\left(-\sum_{k \in R} \delta_{k y}\right) \leftarrow 0
$$

We claim that after tensoring with $\omega_{\alpha}^{1-\frac{e_{q+r-l}}{2}}\left(-E_{q}\right)$ and applying $R \alpha_{*}(-)$, we obtain objects

$$
\left(\begin{array}{llllll}
\left.\mathcal{T}_{l, E}\right)^{\vee} & \left(F_{l-r, E_{q}}^{\prime}\right)^{\vee} & \ldots & \bigoplus_{J \subseteq R, j=|J|}\left(F_{l+j-r, E_{q} \cup J}^{\prime}\right)^{\vee} & \ldots & \left(F_{l, E}^{\prime}\right)^{\vee},
\end{array}\right.
$$

(up to a shift), i.e., that we have

$$
\begin{gather*}
R \alpha_{*}\left(\omega_{\alpha}^{1-\frac{e_{q}+r-l}{2}}\left(-E_{q}\right)_{\mid \Delta}\right)=\left(\mathcal{T}_{l, E}\right)^{\vee},  \tag{12.3}\\
R \alpha_{*}\left(\omega_{\alpha}^{1-\frac{e_{q}+r-l}{2}}\left(-E_{q}-J\right)\right)=\left(F_{j+l-r, E_{q} \cup J}^{\prime}\right)^{\vee} \text { for all } J \subseteq R, \tag{12.4}
\end{gather*}
$$

(up to a shift). To see (12.3), consider the universal family $W_{R} \rightarrow Z_{R}$ over $Z_{R}$ with the section $\sigma_{u}$ corresponding to the indices in $R$. Note that $\Delta=$ $\sigma_{u}\left(Z_{R}\right)$. Then
$R \alpha_{*}\left(\omega_{\alpha}^{1-\frac{e_{q}+r-l}{2}}\left(-E_{q}\right)_{\mid U}\right)=\sigma_{u}^{*}\left(\omega_{\alpha}^{1-\frac{e_{q}+r-l}{2}}\left(-E_{q}\right)\right)=\left(1-\frac{e_{q}+r-l}{2}\right) \psi_{u}-\sum_{k \in E_{q}} \delta_{k u}$,
and this equals $\left(\mathcal{T}_{l, E}\right)^{\vee}$ by Claim 11.2 (up to a shift). To see (12.4): for any $\left(l^{\prime}, E^{\prime}\right)$ Grothendieck-Verdier duality (Remark 3.10) gives $\left(F_{l^{\prime}, E^{\prime}}^{\prime}\right)^{\vee}=$ $R \alpha_{*}\left(\omega_{\alpha}^{1-\frac{e^{\prime}-l^{\prime}}{2}}\left(-E^{\prime}\right)\right)$ (up to a shift). It follows that $\left(F_{l, E}^{\prime}\right)^{\vee}$ and $\left(\mathcal{T}_{l, E}\right)^{\vee}$ are related by $\left\{\left(F_{\tilde{l}, \tilde{E}}^{\prime}\right)^{\vee}\right\}$. By dualizing, the Lemma follows.

Lemma 12.20. The objects $F_{\tilde{l}, \tilde{E}}^{\prime}$ from Lemma 12.19 i.e., for $(\tilde{l}, \tilde{E})=(j+l-$ $\left.r, E_{q} \cup J\right), J \subseteq R, j=|J|, \quad 0 \leq j<r,($ where $\tilde{l}=j+l-r$ could be negative) are generated by $\mathcal{C}^{\prime}$.

Proof. The score $S^{\prime}(\tilde{l}, \tilde{E})=(j+l-r)+j+\min \left\{e_{q}, q+1-e_{q}\right\}<$ $l+r+\min \left\{e_{q}, q+1-e_{q}\right\}=S^{\prime}(l, E)$. If $\tilde{l} \geq 0$, by Proposition 12.8, $F_{\tilde{l}, \tilde{E}}^{\prime}$ and $F_{\tilde{l}, \tilde{E}}$ are related by quotients in $\mathcal{A}$ since $S^{\prime}(\tilde{l}, \tilde{E})<S^{\prime}(l, E) \leq r+s$. As $S^{\prime}(\tilde{l}, \tilde{E})<S^{\prime}(l, E), F_{\tilde{\imath}, \tilde{E}}$ is generated by $\mathcal{C}^{\prime}$ by induction. It follows that $F_{\tilde{l}, \tilde{E}}^{\prime}$ is generated by $\mathcal{C}^{\prime}$.

Consider now the case $\tilde{l} \leq-1$. We claim that $F_{\tilde{l}, \tilde{E}}^{\prime}$ and $\beta_{z}^{*} F_{\bar{l}, \tilde{E}}$ (for an arbitrary $z \in \tilde{Q}$ ) are related by quotients in $\mathcal{A}$. Indeed, in this case $\max \left\{\frac{S^{\prime}(\tilde{l}, \tilde{E})}{2}-\tilde{l}-1, \frac{S^{\prime}(\tilde{l}, \tilde{E})}{2}\right\}=\frac{S^{\prime}(\tilde{l}, \tilde{E})}{2}-\tilde{l}-1=\frac{-l-2+r+\min \left\{e_{q}, q+1-e_{q}\right\}}{2} \leq$ $\frac{r+s-1}{2}$ (since $l \geq 0$ ) and this fact follows from Proposition 12.12. So it suffices to prove that $\beta_{z}^{*} F_{\tilde{l}, \tilde{E}}$ is generated by $\mathcal{C}^{\prime}$ when $\tilde{l} \leq-1$. Since $F_{-1, E}=0$ (see Prop. 3.9) on $\mathcal{U}_{p, q}$ for any $E$, we may assume $\tilde{l} \leq-2$. By Prop. 3.9, on $\mathcal{U}_{p, q}$ we have $F_{\tilde{l}, \tilde{E}} \cong F_{-\tilde{l}-2, \tilde{E}}[-1]$. We have $S^{\prime}(-\tilde{l}-2, \tilde{E})=(-\tilde{l}-2)+j+$ $\min \left\{e_{q}, q+1-e_{q}\right\}=-l-2+r+\min \left\{e_{q}, q+1-e_{q}\right\}=S^{\prime}(l, E)-2 l-2 \leq r+s-1$, since $l \geq 0$. Therefore, by Cor. 12.13(i), $\beta_{z}^{*} F_{-\tilde{l}-2, \tilde{E}}$ and $F_{-\tilde{l}-2, \tilde{E}}$ are related by quotients in $\mathcal{A}$. Since $S^{\prime}(-\tilde{l}-2, \tilde{E})=S^{\prime}(l, E)-2 l-2<S^{\prime}(l, E) \leq r+s$, by induction, $F_{-\tilde{l}-2, \tilde{E}}$ (and hence, $\beta_{z}^{*} F_{-\tilde{l}-2, \tilde{E}}$ ) are generated by $\mathcal{C}^{\prime}$.

This finishes the proof of the Theorem.
Proof of fullness in Theorem 1.8, It suffices to prove that if $G \in D^{b}\left(\overline{\mathrm{M}}_{p, q+1}\right)$ is such that $R \operatorname{Hom}(C, G)=0$ for every $C$ in the exceptional collection of Theorem 1.8 then $G=0$. By Thm. 12.17, we have $R \operatorname{Hom}(C, G)=0$ for every $C \in \mathcal{C}_{F}^{\prime}$. The result follows from Lemma 12.15

## References

[Beĭ78] A. A. Beĭlinson, Coherent sheaves on $\mathbf{P}^{n}$ and problems in linear algebra, Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68-69.
[BFK19] M. Ballard, D. Favero, and L. Katzarkov, Variation of geometric invariant theory quotients and derived categories, J. Reine Angew. Math. 746 (2019), 235-303.
[BM13] J. Bergström and S. Minabe, On the cohomology of moduli spaces of (weighted) stable rational curves, Math. Z. 275 (2013), no. 3-4, 1095-1108.
[CT20a] A.-M. Castravet and J. Tevelev, Derived category of moduli of pointed curves - I, Algebraic Geometry 7 (2020), no. 6, 722-757.
[CT20b]_, Exceptional collections on certain Hassett spaces, Épijournal de Géométrie Algébrique 4 (2020), 1-34.
[Dol03] I. Dolgachev, Lectures on invariant theory 296 (2003), xvi+220.
[DH98] I. Dolgachev and Y. Hu, Variation of Geometric Invariant Theory Quotients, Inst. Hautes Études Sci. Publ. Math. 87 (1998), 5-56.
[Dub98] B. Dubrovin, Geometry and analytic theory of Frobenius manifolds, Proceedings of the ICM, Vol. II (Berlin, 1998), 1998, pp. 315-326.
[Ela09] A. D. Elagin, Semi-orthogonal decompositions for derived categories of equivariant coherent sheaves, Izv. Ross. Akad. Nauk Ser. Mat. 73 (2009), no. 5, 37-66.
[FP13] C. Faber and R. Pandharipande, Tautological and non-tautological cohomology of the moduli space of curves, Handbook of moduli. Vol. I, 2013, pp. 293-330.
[Get95] E. Getzler, Operads and moduli spaces of genus 0 Riemann surfaces, The moduli space of curves (Texel Island, 1994), 1995, pp. 199-230.
[GK98] E. Getzler and M. M. Kapranov, Modular operads, Compositio Math. 110 (1998), no. 1, 65-126.
[GKM02] A. Gibney, S. Keel, and I. Morrison, Towards the ample cone of $\bar{M}_{g, n}$, J. Amer. Math. Soc. 15 (2002), no. 2, 273-294.
[Has03] B. Hassett, Moduli spaces of weighted pointed stable curves, Adv. Math. 173 (2003), no. 2, 316-352.
[HK98] J.-C. Hausmann and A. Knutson, The cohomology ring of polygon spaces, Ann. Inst. Fourier (Grenoble) 48 (1998), no. 1, 281-321.
[HL15] D. Halpern-Leistner, The derived category of a GIT quotient, J. Amer. Math. Soc. 28 (2015), no. 3, 871-912.
[Huy06] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.
[Kap93] M. M. Kapranov, Veronese curves and Grothendieck-Knudsen moduli space $\bar{M}_{0, n}$, J. Algebraic Geom. 2 (1993), no. 2, 239-262.
[Kee92] S. Keel, Intersection theory of moduli space of stable n-pointed curves of genus zero, Trans. Amer. Math. Soc. 330 (1992), no. 2, 545-574.
[KT09] S. Keel and J. Tevelev, Equations for $\bar{M}_{0, n}$, International Journal of Mathematics 20 (2009), no. 09, 1159-1184.
[KL14] A. Kuznetsov and V. A. Lunts, Categorical Resolutions of Irrational Singularities, International Mathematics Research Notices 2015 (2014), no. 13, 4536-4625.
[KS20] A. Kuznetsov and M. Smirnov, On residual categories for Grassmannians, Proc. Lond. Math. Soc. (3) 120 (2020), no. 5, 617-641.
[MS13] Yu. I. Manin and M. N. Smirnov, On the derived category of $\bar{M}_{0, n}$, Izv. Ross. Akad. Nauk Ser. Mat. 77 (2013), no. 3, 93-108.
[MP97] A. S. Merkurjev and I. A. Panin, K-theory of algebraic tori and toric varieties, KTheory 12 (1997), no. 2, 101-143.
[Or192] D. Orlov, Projective bundles, monoidal transformations, and derived categories of coherent sheaves, Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 4, 852-862.
[Tel00] C. Teleman, The quantization conjecture revisited, Ann. of Math. 152 (2000), 1-43.
[Tho87] R. W. Thomason, Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes, Adv. in Math. 65 (1987), no. 1, 16-34.

Université Paris-Saclay, UVSQ, CNRS, Laboratoire de Mathématiques de Versailles, 78000, Versailles, France

Email address: ana-maria.castravet@uvsq.fr
Department of Mathematics and Statistics, University of Massachusetts Amherst, 710 North Pleasant Street, Amherst, MA 01003 and Laboratory of Algebraic Geometry and its Applications, HSE, Moscow, Russia

Email address: tevelev@math.umass.edu

