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EQUATIONS FOR $\overline{M}_{0,n}$

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We show that the log canonical bundle, κ, of $\overline{M}_{0,n}$ is very ample, show the homogeneous coordinate ring is Koszul, and give a nice set of rank 4 quadratic generators for the homogeneous ideal: The embedding is equivariant for the symmetric group, and the image lies on many Segre embedded copies of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$, permuted by the symmetric group. The homogeneous ideal of $\overline{M}_{0,n}$ is the sum of the homogeneous ideals of these Segre embeddings.

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1. Introduction and Statement of Results

Let $\overline{M}_S = \overline{M}_{0,n}$ be the moduli space of stable rational n-pointed curves, with marked points labeled by the elements of the finite set S, with cardinality $|S| = n$, over an algebraically closed field k of characteristic zero. Our goal is to study the equations of $\overline{M}_S$ in its most natural embedding. For a finite set T, let W_T be the standard irreducible representation of the symmetric group $\mathrm{Aut}(T)$, i.e. T-tuples of integers that sum to zero. Let $B \subset \overline{M}_S$ be the boundary, i.e. $B := \partial\overline{M}_S := \overline{M}_S \backslash M_S$.

Theorem 1.1. *The log canonical line bundle*

$$\kappa := \mathcal{O}(K_{\overline{M}_S} + B)$$

is very ample on $\overline{M}_S$. *A flag of subsets*

$$S_3 \subset S_4 \subset \cdots \subset S_n = S$$

of S *with* $|S_k| = k$ *canonically induces an identification*

$$H^0(\overline{M}_S, \kappa) = W_{S_3} \otimes W_{S_5} \otimes \cdots \otimes W_{S_{n-1}},$$

and $\overline{M}_S \subset \mathbb{P}(H^0(\overline{M}_S, \kappa)^*)$ factors through the Segre embedding

$$\mathbb{P}(W^*_{S_3}) \times \mathbb{P}(W^*_{S_4}) \times \cdots \times \mathbb{P}(W^*_{S_{n-1}}) \subset \mathbb{P}(H^0(\overline{M}_S, \kappa)^*).$$

The homogeneous ideal of $\overline{M}_S$ is the sum of the homogeneous ideals for all Segre embeddings over all flags of subsets, i.e. we have the ideal theoretic equality

$$\overline{M}_S = \bigcap_{S_3 \subset S_4 \cdots \subset S} \mathbb{P}(W^*_{S_3}) \times \mathbb{P}(W^*_{S_4}) \times \cdots \times \mathbb{P}(W^*_{S_{n-1}}) \subset \mathbb{P}(H^0(\overline{M}_S, \kappa)^*).$$

The coordinate ring $R := \bigoplus_{n \geq 0} H^0(\overline{M}_S, \kappa^{\otimes n})$ has very nice properties:

Theorem 1.2. *The algebra R is Koszul, i.e. the trivial R-module k has a resolution of form*

$$\cdots \to R[-k]^{a_k} \to R[-(k-1)]^{a_{k-1}} \to \cdots \to R[-1]^{a_1} \to R \to k \to 0.$$

The variety $\overline{M}_S$ is projectively normal, i.e. the natural map

$$\mathrm{Sym}^\bullet(R_1) \to R$$

is surjective. Its kernel is generated by quadrics of rank at most 4.

The embedding

$$\overline{M}_S \subset \mathbb{P} := \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$$

has nice properties as well. Let

$$\mathcal{L} := \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n-3}}(1)$$

and let B be the coordinate ring

$$B := \bigoplus_{n \geq 0} H^0(\mathbb{P}, \mathcal{L}^{\otimes n}).$$

Theorem 1.3. *The ring R is the homogeneous coordinate ring of $\overline{M}_S \subset \mathbb{P}$. The embedding satisfies the analog of Green–Lazarsfeld's property N_p for all p, i.e. the minimal resolution of R over B is of form*

$$\cdots \to B[-k]^{a_k} \to \cdots \to B[-2]^{a_2} \to B \to R \to 0.$$

Remark 1.4. The analogous statement for

$$\overline{M}_S \subset \mathbb{P}(H^0(\kappa)^*)$$

fails. For example $\overline{M}_{0,5} \subset \mathbb{P}^5$ fails to satisfy N_3. In fact the only non-degenerate irreducible subvarieties X of projective space that satisfy N_p for all p are varieties of minimal degree $\deg X = 1 + \operatorname{codim} X$, i.e. quadric hypersurfaces, rational normal scrolls, or a cone over the Veronese surface in $\mathbb{P}^5$, see [6, 7].

We note that the compactification $M_S \subset \overline{M}_S$ is canonical, indeed the so called log canonical compactification of (the log minimal variety) M_S (see [20]), and the coordinate ring R is the log canonical ring of M_S.

The above results allow us to construct $\overline{M}_S$ from M_S in a purely topological and combinatorial way: Dropping the last point gives a tautological fibration

$$\pi : M_{0,n} \to M_{0,n-1}$$

with fiber $\mathbb{P}^1 \backslash \{p_1, \ldots, p_{n-1}\}$ for the distinct points $p_1, \ldots, p_{n-1}$.

Arnold [1] observed that the Leray spectral sequence degenerates at the E^2 term inducing a canonical isomorphism

$$H^{n-3}(M_{0,n}, \mathbb{Z}) = H^{n-4}(M_{0,n-1}, \mathbb{Z}) \otimes H^1(\mathbb{P}^1 \backslash \{p_1, \ldots, p_{n-1}\}, \mathbb{Z}).$$

See Theorem 2.5. By induction we have

$$H^{n-3}(M_{0,n}, \mathbb{Z}) = \bigotimes_{i=3}^{n-1} H^1(\mathbb{P}^1 \backslash \{p_1, \ldots, p_i\}, \mathbb{Z}).$$

Note that $H^1(\mathbb{P}^1 \backslash \{p_1, \ldots, p_i\}, \mathbb{Z})$ is naturally identified with W_i, the standard irreducible representation of the symmetric group $\mathrm{Aut}\{p_1, \ldots, p_i\}$. We have in particular a tautological identification

$$H_{n-3}(M_{0,n}, \mathbb{Z}) = W_3^* \otimes \cdots \otimes W_{n-1}^*.$$

The symmetric group S_n acts naturally on $M_{0,n}$, and so it acts on the left-hand side of the above equality. This representation was introduced by Kontsevich [14] in the context of (cyclic) Lie operads. The action does not preserve the tensor product, but by Theorem 1.1 (see Theorem 2.5) we have

Corollary 1.5. *Let X be the set of elements in*

$$H_{n-3}(M_{0,n}, \mathbb{Z}) = W_3^* \otimes \cdots \otimes W_{n-1}^*$$

all of whose translates by the symmetric group are totally decomposable (*i.e. of form* $x_3 \otimes x_2 \otimes \cdots \otimes x_{n-1}$). *$X$ is closed under scalar multiplication and so defines a subset*

$$\mathbb{P}(X) \subset \mathbb{P}(H_{n-3}(M_{0,n}, \mathbb{C})).$$

This subset is canonically identified with $\overline{M}_{0,n}$, embedded by the full log canonical series $|K_{\overline{M}_{0,n}} + \partial \overline{M}_{0,n}|$ of $M_{0,n}$.

Thus in particular we see that the compactification $M_{0,n} \subset \overline{M}_{0,n}$ can be canonically recovered from the homology of the spaces $M_{0,k}$ together with their symmetric group action, i.e. from the cyclic Lie operad [9].

The paper is organized as follows: Sec. 8 contains the proof of Theorem 1.1. Section 7 contains the proof of Theorems 1.2 and 1.3. The main tools are the syzygy bundles on $\overline{M}_S$, introduced in Sec. 4, and strong vanishing theorems they satisfy, proved in Sec. 6.

2. Preliminary Results

Throughout the paper we will make use of the following nonstandard notation:

Notation 2.1. We will say that a module (or sheaf) M is *extended from* modules (or sheaves) with a given property if there is a finite filtration

$$0 = M^r \subset M^{r-1} \cdots \subset M^1 \subset M^0 = M$$

such that each of the quotients M^i/M^{i+1} has the given property. Similarly, we say that M is *extended from* the collection of submodules (or sheaves) M^i/M^{i+1}.

For a subset $T \subset S$, let

$$\pi_T : \overline{M}_S \to \overline{M}_T$$

be the tautological fibration given by dropping the points of $S\backslash T$ (and stabilizing).

Let $s \in S$ and let $S' = S\backslash\{s\}$. $\pi_{S'}$ has $n-1$ tautological sections, which we indicate by the elements of S'. Let

$$\psi_s := \omega_{\pi_{S'}}(S')$$

(where $S' \subset \overline{M}_S$ here means the union of the graphs of the tautological sections).

For any subset $T \subset S$ with $|T|, |S\backslash T| \geq 2$, let δ_T be the corresponding boundary divisor of $\overline{M}_S$.

We refer to [17], [16], and [8] for background material on $\overline{M}_S$. We will make in particular frequent use of the formulae in [8] for pullbacks of tautological line bundles under the canonical fibrations π_T. In this section, k may have arbitrary characteristic. Throughout the paper, we will often abuse notation and use the same symbol for a sheaf and its pullback under some morphism.

Notation 2.2. For a subset $T \subset S$, let

$$S^T := S\backslash T.$$

In particular $S^s := S\backslash\{s\}$ for $s \in S$.

Lemma 2.3. *Formation of ${\pi_{T^t}}_*(\psi_t)$ commutes with all pullbacks, and for any vector bundle F on $\overline{M}_{T^t}$ there are canonical identifications*

$${\pi_{T^t}}_*(\pi_{T^t}^*(F) \otimes \psi_t) = F \otimes_k W_{T^t}$$

$$H^0(\overline{M}_T, \pi_{T^t}^*(F) \otimes \psi_t) = H^0(\overline{M}_{T^t}, F) \otimes_k W_{T^t}$$

induced by taking residues along the tautological sections T^t.

Let $[C] \in \overline{M}_T$ be a stable T-pointed rational curve. Taking residues at the points $t \in T$ gives a canonical identification

$$H^0(C, \omega_C(T)) = W_T.$$

Proof. Let $\pi := \pi_{T^t}$. We have the natural exact sequence

$$0 \to \omega_\pi \to \omega_\pi(\Sigma) \to \omega_\pi(\Sigma)|_\Sigma \to 0$$

where Σ means the disjoint union of the sections T^t. Taking residues gives a canonical identification of the right-hand term with $\mathcal{O}_\Sigma$. Note that $\psi_t = \omega_\pi(\Sigma)$ by definition. $H^1(C, \omega_C(T^t)) = 0$ for a T^t-pointed stable curve of genus 0, so $R^1\pi_*(\psi_s)$ vanishes, and $R^i\pi_*(\psi_s)$ are vector bundles and their formation commutes with all base extensions, for all i, by the semi-continuity theorem.

Applying π_* and using the duality identification $R^1\pi_*(\omega_\pi) = \mathcal{O}$ we obtain a natural exact sequence

$$0 \to \pi_*(\psi_t) \to \bigoplus_{x \in T^t} \mathcal{O}_{\delta_{x,t}} \to \mathcal{O}_{\overline{M}_{T^t}} \to 0$$

where the map $\mathcal{O}_{x,t} \to \overline{M}_{T^t}$ is the identity (note that $\delta_{x,t}$ is the section corresponding to $x \in T^t$). The identification $\pi_*(\psi_t) = W_{T^t} \otimes \mathcal{O}$. The sheaf $R^1\pi_*(\psi_t)$ is zero. The rest follows from this and the projection formula. $\square$

Definition 2.4. For a subset $F \subset S$, let $\kappa_F := \pi_F^*(\kappa)$ and $L_F := \kappa \otimes \kappa_F^*$.

Lemma 2.5. *Let $F \subset T \subset S$ be subsets, with $|F| \geq 3$. Let*

$$T = T_{|T|} \subset T_{|T|+1} \cdots \subset T_{|S|} = S$$

be a flag of subsets, and define $t_i := T_i \backslash T_{i+1}$ for $i > |T|$. Then

$$L_F = \pi_T^*(L_F) \otimes \bigotimes_{|S| \geq i > |T|} \pi_{T_i}^*(\psi_{t_i}).$$

Proof. Note by definitions

$$L_F \otimes \pi_T^*(L_F) = \kappa \otimes \kappa_T^*.$$

This we compute by induction on $|S \backslash T|$. Its enough to consider the case $T = S^s$, and so prove the formula

$$\pi_{S^s}^*(\kappa) \otimes \psi_s = \kappa \tag{2.1}$$

which is given by wedge product of forms. $\square$

Corollary 2.6. *For $s \in S$ there is a canonical identification*

$$H^0(\overline{M}_S, \kappa) = H^0(\overline{M}_{S^s}, \kappa_{S^s}) \otimes H^0(\overline{M}_S, \psi_s).$$

A flag of subsets as in Theorem 1.1 *canonically induces identifications*

$$H^0(\overline{M}_S, \kappa) = \bigotimes_{n \geq i \geq 4} H^0(\overline{M}_{S_i}, \psi_{s_i}) = \bigotimes_{n \geq i \geq 4} W_{S_{i-1}}.$$

Proof. Immediate from Lemma 2.5 and Lemma 2.3. $\square$

Corollary 2.7. *The line bundle L_F is globally generated and big. κ is very ample.*

Proof. Consider first L_F. By (2.5) and induction it is enough to consider ψ_i. Global generation of ψ_i is due to Kapranov, the associated map is his birational contraction

$$\overline{M}_S \to \mathbb{P}^{|S|-3}.$$

See [16]. Now it follows from (2.6) and induction, that κ is globally generated, and the map given by global sections

$$\overline{M}_S \to \mathbb{P}(H^0(\overline{M}_S, \kappa)^*)$$

factors through the Segre embedding

$$\overline{M}_{S^s} \times \mathbb{P}(H^0(\overline{M}_S, \psi_s)^*) \subset \mathbb{P}(H^0(\overline{M}_{S^s}, \kappa)^*) \times \mathbb{P}(H^0(\overline{M}_S, \psi_s)^*).$$

To prove the map

$$\overline{M}_S \to \overline{M}_{S^s} \times \mathbb{P}(H^0(\overline{M}_S, \psi_s)^*)$$

is a closed embedding, it is enough to prove this for the restriction to each fiber $C = \pi_{S^s}^{-1}([C])$ of π_{S^s}. But by Lemma 2.3 restriction gives a canonical identification

$$H^0(\overline{M}_S, \psi_s) = H^0(C, \omega_C(S^s))$$

and one checks easily that on the stable S^s-pointed curve C, $\omega_C(S^s)$ is very ample. □

Next, we prove a topological analog of Corollary 2.6.

Corollary 2.8. *Given $a, b \in T$ there is a unique section*

$$\omega(ab) \in H^0(C, \omega_C(T))$$

which has residue 1 at a, -1 at b, and is regular everywhere else. Let $a, b \in F \subset T$ with $|F| \geq 3$ and let

$$\pi_F : C \to C'$$

be the stabilization of (C, F). Pullback induces canonical identifications

$$H^0(C', \omega_{C'}(F)) = H^0(C, \omega_C(F)) \subset H^0(C, \omega_C(T))$$

under which $\omega(ab)$ is sent to $\omega(ab)$.

Proof. This is immediate from Lemma 2.3 and the definition of stabilization. □

Lemma 2.9. *Given distinct $a, b \in S^s$ there is a a global 1-form*

$$\omega \in H^0(\overline{M}_S, \Omega^1(\log \partial \overline{M}_S))$$

whose restriction to $C \subset \overline{M}_S$ is $\omega(ab)$, for all $[C] \in \overline{M}_{S^s}$.

Proof. We have a commutative diagram

$$\begin{tikzcd} \overline{M}_S \arrow[r, "\pi_{F\cup s}"] \arrow[d, "\pi_{S^s}"'] & \overline{M}_{F\cup s} = C' = \mathbb{P}^1 \arrow[d, "\pi_F"] \\ \overline{M}_{S^s} \arrow[r, "\pi_F"] & \overline{M}_F = \mathrm{pt} \end{tikzcd}$$

where C' is as in Lemma 2.8. Now we take for ω the pullback of $\omega(ab)$ from C'. □

Theorem 2.10. *Let $p_1,\dots,p_{n-1}$ be distinct points in $\mathbb{P}^1$. Over the complex numbers there are vector space identifications*

$$H^\bullet(M_n,\mathbb{C}) = H^\bullet(M_{n-1},\mathbb{C}) \otimes H^\bullet(\mathbb{P}^1\backslash\{p_1,\dots,p_{n-1}\},\mathbb{C})$$

$$H^\bullet(M_n,\mathbb{C}) = \bigotimes_{i=3}^{n-1} H^\bullet(\mathbb{P}^1\backslash\{p_1,\dots,p_i\},\mathbb{C})$$

and canonical identifications

$$H^{n-3}(M_n,\mathbb{C}) = \bigotimes_{i=3}^{n-1} H^1(\mathbb{P}^1\backslash\{p_1,\dots,p_i\},\mathbb{C})$$

$$H^k(M_n,\mathbb{C}) = H^0(\Omega^k(\log\partial\overline{M}_n))$$

$$H^{n-3}(M_n,\mathbb{C}) = H^0(\overline{M}_n,\kappa).$$

Proof. For $[C]\in M_{S^s}$ and fixed $a\in S^s$ the differential forms $\omega(ab)$, $b\in S^{s,a}$ give a basis of

$$H^0(C,\omega_C(S^s)) = H^1(C\backslash S^s,\mathbb{C}).$$

By Lemma 2.9 these are restrictions of global log forms, and in particular global cohomology classes. Thus by induction and the Leray–Hirsch theorem, [3], there is an additive isomorphism

$$H^\bullet(\overline{M}_S,\mathbb{C}) = H^\bullet(\overline{M}_{S^s},\mathbb{C})\otimes H^\bullet(\mathbb{P}^1\backslash S^s,\mathbb{C})$$

and $H^\bullet(\overline{M}_S,\mathbb{C})$ is generated by meromorphic 1-forms with log poles on the boundary. By [5], such forms are never exact. Note the last formula is just the $k=n-3$ case of the preceding formula. The isomorphism given by the Leray–Hirsch theorem depends on choosing global lifts for the $\omega(ab)$. However in the top degree it is independent of choices. □

3. Filtrations

For any globally generated line bundle L on a projective variety X, we define a vector bundle V_L by the exact sequence

$$0\to V_L\to H^0(X,L)\otimes\mathcal{O}_X\to L\to 0.$$

Lemma 3.1. *There is a natural exact sequence*

$$0\to \pi^*_{S^t}(V_{\psi_s})\to V_{\psi_s}\to\mathcal{O}(-\delta_{s,t})\to 0.$$

Corollary 3.2. *Choose a flag of subsets*

$$S_3\subset S_4\subset\cdots\subset S_n = S$$

as in (1.1). *The bundle* V_{ψ_s} *is extended from the line bundles*

$$\pi^*_{S_i}(\mathcal{O}(-\delta_{s,s_i}))$$

for $|S| \geq i \geq 4$.

The bundle $\wedge^q V_{\psi_s}$ *is extended from line bundles of form* $\mathcal{O}(-E)$, *with* E *a sum of distinct boundary divisors with* $a \equiv b \not\equiv s$, *where* $S_3 = \{a, b, s\}$ *and we think of boundary divisors as a partition (or equivalence relation) on* S.

Proof. Immediate from the lemma, and the formula for pulling back boundary divisors under maps π_T. □

Proof of Lemma 3.1. We have that

$$\psi_s = \pi^*_{S^t}(\psi_s) \otimes \mathcal{O}(\delta_{s,t})$$

and that $\psi_s|_{\delta_{s,t}}$ is canonically trivial (by taking residues). This induces a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*_{S^t}(V_{\psi_s}) & \longrightarrow & V_{\psi_s} & \longrightarrow & \mathcal{O}(-\delta_{s,t}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*_{S^t}(H^0(\psi_s)) \otimes_k \mathcal{O}_{\overline{M}_S} & \longrightarrow & H^0(\psi_s) \otimes_k \mathcal{O}_{\overline{M}_S} & \longrightarrow & H^0(\mathcal{O}_{\delta_{s,t}}) \otimes_k \mathcal{O}_{\overline{M}_S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*_{S^t}(\psi_s) & \longrightarrow & \psi_s & \longrightarrow & \mathcal{O}_{\delta_{s,t}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here, the second row is obtained from the third by taking global sections — which gives a short exact sequence of vector spaces as the H^1 term vanishes — and then tensoring with the structure sheaf $\mathcal{O}_{\overline{M}_S}$. Now, the first row is given by taking kernels of the vertical maps.

Now, the result follows from the snake lemma. □

Lemma 3.3. *Let* $a, b, s \in S$ *be distinct. Let* E *be the* (*reduced*) *union of all boundary divisors of* $\overline{M}_S$ *with* $a \equiv b \not\equiv s$. *Then there is member of* $|\kappa|$ *which is an effective combination of irreducible components of* $B - E$.

Proof. We induct on $|S|$. For $|S| = 4$ we can take $\delta_{a,s}$. By induction it is enough to find a member of $|\psi_t|$, $t \in S^{a,b,s}$, supported on components of $B - E$. This is clear from Lemma 3.4. □

Lemma 3.4. *The Weil divisor*

$$\sum_{a,s\in F,t\in F^c} \delta_F$$

is linearly equivalent to ψ_t.

Proof. The sum above is $\Psi_t^*(\Psi_t(\delta_{a,s}))$, where

$$\Psi_t : \overline{M}_S \to \mathbb{P}^{|S|-3}$$

is the Kapranov model. $\Psi_t(\delta_{a,s})$ is a hyperplane, and so the above pullback represents ψ_t. □

Lemma 3.5. *The bundle* $\wedge^q V_{\psi_s} \otimes \kappa$ *is extended from globally generated line bundles. It is also extended from line bundles of form* $\mathcal{O}(E)$ *where* E *is a divisor linearly equivalent to a* $\mathbb{Q}$*-divisor of form* $K_{\overline{M}_S} + \Delta + A$, *for* Δ *an effective combination of boundary divisors with coefficients strictly less than one, and* A *an ample divisor.*

Proof. By Corollary 3.2, $\wedge^q V_{\psi_s} \otimes \kappa$ is extended from line bundles associated to divisors

$$\kappa + \sum_{t=1}^{r} \pi^*_{S_{i_t}}(\delta_{s,s_{i_t}})$$

for some sequence of integers

$$n \geq r > r-1 \cdots > r_1 \geq 4.$$

Global generation of such a divisor follows by induction, the formulae

$$\kappa_S = \kappa_{S^t} + \psi_t$$

$$\psi_t = \pi^*_{S^s}(\psi_t) + \delta_{s,t},$$

and the global generation of κ and ψ_i.

The second statement is immediate from Lemmas 3.2 and 3.3. □

Corollary 3.6. $H^i(\wedge^q V_{\psi_s} \otimes \kappa \otimes M) = 0$ *for any* $i > 0$ *and any nef line bundle* M.

Proof. This is immediate from the preceding lemma and the Kawamata–Viehweg vanishing theorem. □

Corollary 3.7. *Let* (C, S^s) *be an* S^s*-pointed stable curve and* $\psi := \omega_C(S^s)$. *Then* $H^i(C, \wedge^q(V_\psi) \otimes \psi) = 0$ *for* $i > 0$, *and* $H^0(\wedge^q(V_\psi)) = 0$ *for all* $q > 0$. *The log canonical embedding* $C \subset \mathbb{P}(H^0(C, \omega_C(S^s))^*)$ *satisfies Green–Lazarsfeld's condition* N_p *for all* $p \geq 0$.

Proof. We have $C = \pi^{-1}_{S^s}[C] \subset \overline{M}_S$, $\psi = \psi_s|_C = \kappa|_C$, and $V_\psi = V_{\psi_s}|_C$. The bundle $\wedge^q V_{\psi_s}$ is extended from line bundles satisfying the Kawamata–Viehweg vanishing theorem by Lemma 3.5, and restriction. This gives the vanishing result, which implies N_p, for all p (see [10]). By Lemma 3.2 and restriction, $\wedge^q(V_\psi)$ is extended from line bundles with no global sections, and thus has itself no global sections. □

4. The Syzygy Bundles

We begin by giving a canonical resolution of the structure sheaf of $\overline{M}_S$ by natural vector bundles over $\overline{M}_{S^s} \times \mathbb{P}^{n-3}$, using a special case of the Beilinson spectral sequences as in [11].

The diagonal embedding

$$\mathbb{P}^r \subset \mathbb{P}^r \times \mathbb{P}^r$$

is the zero locus of a regular tautological section of $V_{\mathcal{O}(1)} \boxtimes \mathcal{O}(1)$. For any morphism $f : X \to \mathbb{P}^r$ the section pulls back to a regular section of $V_{\mathcal{L}} \boxtimes \mathcal{O}(1)$, $\mathcal{L} := f^*(\mathcal{O}(1))$, on $X \times \mathbb{P}^r$ with zero locus the graph $X = \Gamma_f \subset X \times \mathbb{P}^r$. The Koszul complex then gives an exact sequence

$$0 \to \wedge^r V_{\mathcal{L}} \boxtimes \mathcal{O}(-r) \to \cdots \to V_{\mathcal{L}} \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{X\times\mathbb{P}^r} \to \mathcal{O}_X \to 0. \tag{4.1}$$

We have a closed embedding $\Phi : \overline{M}_S \subset \overline{M}_{S'} \times \mathbb{P}^{n-3}$ given by $\pi := \pi_{S^s}$ and the linear series $|\psi_s|$, see the proof of Corollary 2.7.

Lemma 4.1. *The sheaf* $\mathcal{M}^q := R^1\pi_*(\wedge^{q+1}V_{\psi_s})$ *is a vector bundle on* $\overline{M}_{S^s}$ *for* $q \geq 0$. $\mathcal{M}^0 = 0$. *One has an exact sequence of sheaves on* $\overline{M}_{S'} \times \mathbb{P}^{N-3}$:

$$0 \to \mathcal{M}^{N-4} \boxtimes \mathcal{O}(3-N) \to \cdots \to \mathcal{M}^1 \boxtimes \mathcal{O}(-2) \to \mathcal{O}_{\overline{M}_{S'}\times\mathbb{P}^{N-3}} \to \Phi_*\mathcal{O}_{\overline{M}_S} \to 0. \tag{4.2}$$

The fiber of $\mathcal{M}^q$ *at the point* $[C] \in \overline{M}_{S^s}$ *is canonically identified with the q-th syzygy for the* S^s*-pointed stable curve* $(C, S^s) \subset \mathbb{P}^{n-3}$, *i.e.*

$$\mathcal{M}^q|_{[C]} = \mathrm{Tor}_q^A(k, B)$$

where $A = R(\mathbb{P}^{n-3}, \mathcal{O}(1))$ *is the homogeneous coordinate ring of* $\mathbb{P}^{n-3}$, *and* $B = R(C, \omega_C(S^s))$ *is the homogeneous coordinate ring of* $C \subset \mathbb{P}^{n-3}$ *for* $q > 0$.

Proof. We apply the above construction in the case of $X = \overline{M}_S \to \mathbb{P}^{n-3}$, and push forward the sequence (4.1) along

$$p = \pi \times id : \overline{M}_S \times \mathbb{P}^{n-3}$$

where the fibers of p (or equivalently π) are S^s-pointed stable curves. So the formation of $R^i\pi_*(\wedge^{q+1}V_{\psi_s})$ commutes with all base extensions, vanishes for $i = 0$ or $i = q = 1$, and are vector bundles for $i = 1$, by Lemma 3.7.

Exactness of (4.2) follows by analyzing the spectral sequence for the hyper-derived pushforward:

$$E^1_{i,j} = R^j p_*(\mathcal{F}^i), \quad i \leq 1, \;\; j \geq 0$$

with $\mathcal{F}^{-i} = \wedge^i(V_{\psi_s}) \boxtimes \mathcal{O}(-i)$, $i \leq 0$, $\mathcal{F}^1 = \mathcal{O}_{\overline{M}_S}$. Since (4.1) is exact, the spectral sequence abuts to zero. By Corollary 3.7, the sequence has only two nonzero rows:

$$E^1_{*,1} : \cdots \to R^1p_*(\wedge^3 V_{\psi_s}) \boxtimes \mathcal{O}(-3) \to R^1p_*(\wedge^2 V_{\psi_s}) \boxtimes \mathcal{O}(-2) \to 0 \to 0 \cdots$$

and

$$E^1_{*,0} : \cdots 0 \to 0 \to \mathcal{O}_{\overline{M}_{S^s}\times\mathbb{P}^{n-3}} \to \Phi_*(\mathcal{O}_{\overline{M}_S}) \to 0 \to 0 \cdots.$$

The exactness of (4.2) follows easily.

If we restrict (4.2) to $[C] \times \mathbb{P}^{n-3}$ it remains exact, (for example, by the flatness of π) and we obtain the exact sequence of sheaves on $\mathbb{P}^{n-3}$:

$$\begin{aligned} 0 &\to H^1(C, \wedge^{n-3}V_\psi) \otimes \mathcal{O}_{\mathbb{P}^{n-3}}(-(n-3)) \\ &\to \cdots \to H^1(C, \wedge^2 V_\psi) \otimes \mathcal{O}(-2) \to \mathcal{O}_{\mathbb{P}^{n-3}} \to \mathcal{O}_C \to 0 \end{aligned}$$

where $\psi := \psi_s|_C$ as in Lemma 3.7. Tensoring with $\bigoplus_{n\geq 0} \mathcal{O}(n)$ and taking global sections, a simple spectral sequence analysis shows the resulting sequence of $R(\mathbb{P}^{n-3}, \mathcal{O}(1))$ modules is exact. This yields a resolution of the homogeneous coordinate ring of $R(C, \psi)$. Note that each of the maps is given by linear forms, and thus when we tensor with k, the maps in the resulting complex are all zero, and thus the terms of the sequence are identified with the relevant Tors. $\square$

Let $T = S^s$. Observe that the vector bundle, $\mathcal{M}^i$, on $\overline{M}_T$ is intrinsic to $\overline{M}_T$ (it does not depend on S). When there is the possibility of confusion we refer to it as $\mathcal{M}^i_T$.

Lemma 4.2. *Let $s, t \in S$. Let E be a vector bundle on $\overline{M}_{S^t}$. There are canonical identifications*

$$R^p \pi_{S^s *}(\pi^*_{S^t}(E)) = \pi^*_{S^{s,t}}(R^p \pi_{S^{s,t} *}(E)).$$

for all $p \geq 0$.

Proof. Consider first the commutative pullback diagram:

$$\begin{array}{ccc} \overline{M}_{S^s} \times_{\overline{M}_{S^{s,t}}} \overline{M}_{S^t} & \xrightarrow{\pi_2} & \overline{M}_{S^t} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_{S^{s,t}}} \\ \overline{M}_{S^s} & \xrightarrow{\pi_{S^{t,s}}} & \overline{M}_{S^{s,t}}. \end{array}$$

We have the equality

$$R^p \pi_{1*}(\pi^*_2(E)) = \pi^*_{S^{t,s}}(R^p \pi_{S^{s,t} *}(E)),$$

since $\pi_{S^{s,t}}$ is flat (see [12]). Now by the Leray spectral sequence and the projection formula it is enough to show that

$$R^q f_*(\mathcal{O}) = 0$$

for $q > 0$ where

$$f : \overline{M}_S \to \overline{M}_{S^s} \times_{\overline{M}_{S^{s,t}}} \overline{M}_{S^t}$$

is the natural map. This holds as the map is birational and the image has rational singularities (see e.g. [17]). □

Theorem 4.3. *For $t \in A$ there is a vector bundle Q on $\overline{M}_A$ and exact sequences*

$$0 \to \mathcal{M}^p_{A^t} \to \mathcal{M}^p_A \to Q \to 0 \tag{4.3}$$

$$0 \to \wedge^p(V_{\psi_t}) \to Q \to \mathcal{M}^{p-1}_{A^t} \to 0 \tag{4.4}$$

Proof. Let $A = S^s$. The exact sequence in Lemma 3.1 yields

$$0 \to \pi^*_{S^t}(\wedge^{p+1}(V_{\psi_s})) \to \wedge^{p+1}(V_{\psi_s}) \to \pi^*_{S^t}(\wedge^p(V_{\psi_s}) \otimes \mathcal{O}(-\delta_{s,t}) \to 0.$$

We have

$$\pi_{S^s *}(\mathcal{O}(-\delta_{s,t})) = 0,$$

since $\delta_{s,t}$ is a section of π_{S^s}. So by Lemma 4.2 we obtain the exact sequence

$$0 \to \pi^*_{S^{s,t}}(\mathcal{M}^p) \to \mathcal{M}^p_{S^s} \to Q \to 0,$$

where

$$Q := R^1\pi_{S^s *}(\pi^*_{S^t}(\wedge^p(V_{\psi_s})) \otimes \mathcal{O}(-\delta_{s,t})).$$

We study Q beginning with the exact sequence:

$$0 \to \pi^*_{S^t}(\wedge^p(V_{\psi_s})) \otimes \mathcal{O}(-\delta_{s,t}) \to \pi^*_{S^t}(\wedge^p(V_{\psi_s})) \to \pi^*_{S^t}(\wedge^p(V_{\psi_s}))|_{\delta_{s,t}} \to 0.$$

Since $\delta_{s,t}$ is a section of π_{S^s}, $R^1\pi_{S^s *}$ vanishes on the right term. $\pi_{S^s *}$ applied to the middle term gives

$$\pi_{S^s *}(\pi^*_{S^t}(\wedge^p(V_{\psi_s}))) = \pi^*_{S^{t,s}}(\pi_{S^t *}(\wedge^q(V_{\psi_s}))) = 0$$

by Lemmas 4.2 and 3.7. So we obtain the exact sequence:

$$0 \to \wedge^p(V_{\psi_t}) \to Q \to \pi^*_{S^{s,t}}(\mathcal{M}^{p-1}_{S^{s,t}}) \to 0.$$ □

We have the immediate:

Corollary 4.4. *A flag of subsets*

$$S_3 \subset S_4 \subset \cdots \subset S_n = S$$

as in Theorem 1.1 *extends $\mathcal{M}^p_S$ from vector bundles of form $\pi^*_{S_i}(\wedge^q V_{\psi_{s_i}})$ for $p \geq q$.*

Corollary 4.5. *We have*

$$c_1(\mathcal{M}^p_S) = \binom{|S|-4}{p-1}\kappa.$$

Proof. An easy induction argument using Theorem 4.3 and Lemma 2.6. □

Remark 4.6. As $\mathcal{M}_T^k$ is the bundle with fiber at $[C]$ the k-th syzygy module, it is naturally a subbundle of the trivial bundle $\mathrm{Sym}_2(\mathbb{V}) \otimes \mathbb{V}^{\otimes(k-1)}$ where $\mathbb{V}$ is the trivial bundle $\mathbb{V} := \pi_*(V_{\psi_s})$, e.g. $\mathcal{M}^1 \subset \mathrm{Sym}_2(\mathbb{V}) \otimes \mathcal{O}$ has fiber at $[C]$ the space of conics vanishing on $C \subset \mathbb{P}^{|T|-2}$, or $\mathcal{M}^2 \subset \mathcal{M}^1 \otimes \mathbb{V}$ has fiber the linear relations among these quadrics, etc. As a subbundle of a trivial bundle, $\mathcal{M}^k$ induces a map of $\overline{M}_T$ to a Grassmannian. By Corollary 4.5, the first Chern class of $\mathcal{M}^k$ is ample, thus the map is finite, and one can check (with a bit of work) that it is in each case a closed embedding. (At least) two of these embeddings have been previously studied: As noted $\mathcal{M}^1 \subset \mathrm{Sym}_2(\mathbb{V})$ gives fiberwise the space of conics vanishing on $C \subset \mathbb{P}^{|T|-2}$. By Lemma 3.7, C is cut out by such quadrics. The induced closed embedding is Kapranov's realization of $\overline{M}_{S^s}$ as the closure in the Hilbert scheme of the locus of rational normal curves in $\mathbb{P}^{|T|-2}$ through $|T|$ fixed points. See [16]. Let $T := S^s$, $\pi := \pi_T$. We have the equality

$$\mathcal{M}^{|T|-3} = R^1\pi_*(\wedge^{|T|-2}V_{\psi_s}) = R^1\pi_*(\omega_\pi(\Sigma))$$

where Σ is the union of the $|T|$ tautological sections. It follows then by the deformation theory of pairs that $\mathcal{M}^{|T|-3}$ is the log tangent bundle $T_{\overline{M}_T}(-\log B)$, and one can check that the map to the Grassmannian is given by the space of global log 1-forms. The induced map to projective space (composing with the Plücker embedding of the Grassmannian) is the log canonical embedding of M_T, and is an instance of a general construction that holds for any complement to a hyperplane arrangement (see [13]).

5. Koszulness and p-Linearity

In this section, $A = \bigoplus_{n\geq 0} A_n$ is a commutative graded algebra over a field $A_0 = k$, and M is a graded A-module. All our modules are concentrated in nonnegative degrees.

Definition 5.1. The module M is called *p-linear* if $\mathrm{Tor}_i^A(M,k)$ is concentrated in degrees $i, i+1, \ldots, i+p$ for all i. The algebra A is called *Koszul* if the trivial A-module, $A_0 = k$ is 0-linear.

Example 5.2. A polynomial algebra S over k (graded by total degree) is Koszul: the minimal resolution of k over S is given by the standard Koszul complex.

Among the many nice properties of Koszul algebras we note the result of [2]:

Theorem 5.3. *Let A be a Koszul algebra. Then the natural map*

$$\mathrm{Sym}_{A_0}(A_1) \to A$$

is surjective, and its kernel is generated by the degree two elements.

Lemma 5.4. *Consider a short exact sequence of graded A-modules*:

$$0 \to M' \to M \to M'' \to 0. \tag{5.1}$$

If M' and M'' are k-linear then M is k-linear. If M is k-linear and M'' is $(k-1)$-linear, then M' is k-linear. If M is k-linear and M' is $(k+1)$-linear, then M'' is k-linear.

Proof. All statements immediately follow from the long exact sequence for Tor. □

Corollary 5.5. *If M is extended from k-linear modules then M is k-linear.*

Proof. Induction on the depth of the filtration using Lemma 5.4. □

Lemma 5.6. *If A is Koszul and M is concentrated in degrees at most p then M is p-linear.*

Proof. M has the natural filtration

$$M = M_{\geq 0} \supset M_{\geq 1} \supset \cdots \supset M_{\geq p+1} = 0$$

where $M_{\geq i} := \bigoplus_{n \geq i} M_n$. Thus M is extended from the modules $M_{\geq i}/M_{\geq i+1}$. $M_{\geq i}/M_{\geq i+1}$ is a k-module (i.e. annihilated by A_n, $n > 0$) concentrated in degree i, and thus i-linear by the definition of a Koszul algebra. Thus M is p-linear by Corollary 5.5. □

Lemma 5.7. *If A is Koszul then M is k-linear if and only if $M' := \bigoplus_{n \geq k} M_n$ is k-linear.*

Proof. Consider the short exact sequence (5.1), where $M'' = \oplus_{n<k} M_n$ has a natural structure of a nilpotent A-module. M'' is $(k-1)$-linear by Lemma 5.6. Therefore M is k-linear by Lemma 5.4. □

Lemma 5.8. *Consider the complex of graded A-modules*

$$0 \to M^r \to \cdots \to M^3 \to M^2 \to M^1 \to M^0 \to 0. \tag{5.2}$$

Assume that M^k is $p+k$-linear, for $k \geq 1$, and that the cohomology module H^k is $p+k+1$-linear for $k \geq 0$. Then M^0 is $p+1$-linear.

Remark 5.9. We will use this Lemma in the situation when A is Koszul and H^k is concentrated in degrees at most $p+k+1$, and thus $(p+k+1)$-linear by Lemma 5.6.

Proof. Let $d^k : M^k \to M^{k-1}$ be the differential with kernel Ker^k and image Im^k.

We argue by induction, the case $r = 0$ being obvious. Consider the complex

$$0 \to M^r \to \cdots \to M^3 \to M^2 \to \mathrm{Ker}^1 \to 0$$

that obviously has the same cohomology as the original complex. It follows by induction that Ker^1 is $(p+2)$-linear.

By Lemma 5.4 and the sequence

$$0 \to \mathrm{Ker}^1 \to M^1 \to \mathrm{Im}^1 \to 0,$$

Im^1 is $(p+1)$-linear.

By Lemma 5.4 and the sequence

$$0 \to \mathrm{Im}^1 \to M^0 \to H^1 \to 0,$$

M^0 is $(p+1)$-linear. □

Definition 5.10 (Segre Product). For graded k-modules N and M let

$$N \mathbin{\widehat{\otimes}} M := \bigoplus_{n \geq 0} N_n \otimes_k M_n.$$

Proposition 5.11. *Let A and B be Koszul algebras. Then $A \mathbin{\widehat{\otimes}} B$ is a Koszul algebra. Moreover, if M is p-linear over A, and N is p-linear over B, then $M \mathbin{\widehat{\otimes}} N$ is p-linear over $A \mathbin{\widehat{\otimes}} B$.*

Proof. For $p = 0$, this is the content of [4]. In particular (taking $M = N = k$), $A \mathbin{\widehat{\otimes}} B$ is Koszul.

For a graded vector space V let

$$V\langle p\rangle := \bigoplus_{n \geq 0} V_{n+p}.$$

Observe $(M \mathbin{\widehat{\otimes}} N)\langle p\rangle = M\langle p\rangle \mathbin{\widehat{\otimes}} N\langle p\rangle$, and that by Lemma 5.7, M is p-linear if and only if $M\langle p\rangle$ is 0-linear. Thus the p-linear case is reduced to the zero linear case. □

The following is a slight generalization of [22].

Lemma 5.12. *Let $A \to B$ be a homomorphism of graded rings, with $A_0 = B_0 = k$. Assume B is 1-linear over A. Let M be a graded B-module. If M is p-linear over A, then M is p-linear over B.*

Proof. The case of $M = k$ is [22]. The same proof works for any M. □

Definition 5.13. For any sheaves $\mathcal{L}$, $\mathcal{M}$ on an algebraic variety X, we define

$$\mathcal{G}^X(\mathcal{M}, \mathcal{L}) = \bigoplus_{n \geq 0} \mathcal{G}^X_n(\mathcal{M}, \mathcal{L}), \quad \text{where } \mathcal{G}^X_n(\mathcal{M}, \mathcal{L}) = H^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes n}).$$

We drop X from the notation if it is clear from the context. We let $\mathcal{G}(\mathcal{L}) := \mathcal{G}(\mathcal{O}_X, \mathcal{L})$. Notice that $\mathcal{G}(\mathcal{L})$ is a graded k-algebra and $\mathcal{G}(\mathcal{M}, \mathcal{L})$ is a graded $\mathcal{G}(\mathcal{L})$-module. We call $\mathcal{L}$ a *Koszul sheaf* if $\mathcal{G}(\mathcal{L})$ is a Koszul algebra. We call $\mathcal{M}$, *p-linear over $\mathcal{L}$*, if $\mathcal{G}(\mathcal{M}, \mathcal{L})$ is p-linear over $\mathcal{G}(\mathcal{L})$.

Lemma 5.14. *Let $\mathcal{L}$ on X be a very ample line bundle and assume the coordinate ring $B := \mathcal{G}^X(\mathcal{L})$ is Koszul. Assume the embedding $f : Y \hookrightarrow X$ is nondegenerate, i.e.*

$$H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L}|_Y)$$

is an isomorphism. Then $f_*(\mathcal{O}_Y)$ is 1-linear over $\mathcal{L}$ if and only if the minimal resolution of $R := \mathcal{G}^Y(\mathcal{L}|_Y)$ over B is of form

$$\cdots \to B[-k]^{a_k} \to \cdots \to B[-2]^{a_2} \to B \to R \to 0$$

Proof. Clearly, if we have such a resolution, then R is 1-linear. So assume R is 1-linear over B. By Lemma 5.12, R is Koszul, thus is generated by degree 1 elements, by Theorem 5.3. Thus $B \to R$ is surjective and the kernel, K, is generated by elements of degree at least 2. Consider now a minimal free resolution of K:

$$\cdots \to F_k \to \cdots \to F_2 \to K \to 0.$$

Clearly, minimal generators of F_k have degree at least k. But K is 2-linear by Lemma 5.4, thus minimal generators of F_k are of degree exactly k, see e.g. the second paragraph of [21]. $\square$

Lemma 5.15. *Let $\mathcal{L}_i$, $\mathcal{M}_i$ be sheaves on X_i, $i = 1, 2$. Assume that $\mathcal{L}_i$ is Koszul and $\mathcal{M}_i$ is p-linear over $\mathcal{L}_i$, $i = 1, 2$. Then $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a Koszul sheaf on $X_1 \times X_2$ and $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is p-linear over $\mathcal{L}_1 \boxtimes \mathcal{L}_2$.*

Proof. Immediate from Lemma 5.11. $\square$

Lemma 5.16. *Let $f : Y \hookrightarrow X$ be a closed embedding. Let $\mathcal{L}$ be the Koszul sheaf on X. Assume that $f_*(\mathcal{O}_Y)$ is 1-linear over $\mathcal{L}$. Then $\mathcal{L}|_Y$ is Koszul. Moreover, if $\mathcal{M}$ is the sheaf on Y and $f_*(\mathcal{M})$ is p-linear over $\mathcal{L}$ then $\mathcal{M}$ is p-linear over $\mathcal{L}|_Y$.*

Proof. Immediately follows from Lemma 5.12. $\square$

6. Fundamental Vector Bundles

Definition 6.1. By a fundamental vector bundle on $\mathbb{P}^m$ we mean a bundle of form $\Lambda^q V$ for $q \geq 0$ where V is the universal rank m subbundle.

By a *fundamenal* vector bundle on $\overline{M}_S$ we mean the pullback of a fundamental vector bundle on projective space under a composition $\Psi_t \circ \pi_T$ for $t \in T \subset S$, where $\Psi_t : \overline{M}_T \to \mathbb{P}^{|T|-3}$ is Kapranov's map.

For any Young diagram λ with at most $N-3$ rows, let S_λ be the corresponding Schur functor, and let $S_\lambda(V)$ be the corresponding vector bundle on $\mathbb{P}^{N-3}$. For example, vector bundles $\Lambda^k V$ correspond to k-box diagrams with k rows (of one box each).

Lemma 6.2. *Let D be the tensor product of at most p fundamental vector bundles on $\mathbb{P}^m$. Then D is the direct sum of vector bundles of the form $S_\lambda(V)$, where all rows of λ have at most p boxes.*

Proof. Follows from the Littlewood–Richardson rule. □

Proposition 6.3. *If λ has at most p boxes in each row then*

$$H^j(S_\lambda(i)) = 0$$

for $j > 0$, $i \geq p - j$.

Proof. By the Borel–Bott–Weyl theorem (see [23]) the following holds: Suppose

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$$

with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. If for some i, $r = \lambda_i - i$, then all the cohomologies vanish (the so called singular case of the theorem). Otherwise there is a unique j such that

$$H^j(S_\lambda(r)) \neq 0$$

and it is described as follows: If $r > \lambda_1 - 1$ then $j = 0$. If $\lambda_m - m > r$ then $j = m$. Otherwise j is the unique index i so that

$$\lambda_i - i > r > \lambda_{i+1} - (i + 1).$$

In particular, if $r \geq \lambda_i - i$ then $H^i(S_\lambda(r)) = 0$. Since $p \geq \lambda_i$, the result follows. □

Lemma 6.4. *Let D be the tensor product of at most p fundamental vector bundles on $\mathbb{P}^m$. Then $D(n)$ has no higher cohomology for $n \geq p$.*

Proof. Follows from Lemma 6.2 and Proposition 6.3. □

Lemma 6.5. *If $\tilde{D}$ is a product of at most $p + 1$ fundamental vector bundles on $\overline{M}_{S'}$ then $H^i(\tilde{D} \otimes (\kappa')^{\otimes n}) = 0$ for any $i > 0$ and $n \geq p + 1$.*

Proof. Immediate from the Kawamata–Viehweg vanishing theorem and Lemma 3.5. □

Corollary 6.6. *If D is a product of at most p fundamental vector bundles on $\overline{M}_S$, then $H^i(\mathcal{M}^k \otimes D \otimes \kappa^{\otimes n}) = 0$ for any $i > 0$, $k > 0$, and $n \geq p + 1$.*

Proof. Follows from Lemma 6.5 and Proposition 4.4. □

Proposition 6.7. *Let D be the tensor product of at most p fundamental vector bundles on $\mathbb{P}^m$, $p \geq 0$ (when $p = 0$, $D = \mathcal{O}$). Then D is p-linear over $\mathcal{O}_{\mathbb{P}^m}(1)$.*

Proof. By Lemma 5.7, it suffices to prove that

$$M' := \bigoplus_{n \geq p} H^0(\mathbb{P}^{N-3}, D(n))$$

is p-linear over $\mathcal{O}(1)$. The bundle D is the tensor product of at most p fundamental vector bundles of the form $\Lambda^{n_i} V$. We argue by induction on $\sum n_i$.

If all $n_i = 0$ then $p = 0$ and $D = \mathcal{O}$, which is obviously 0-linear. Otherwise suppose we have a tensor factor $\Lambda^q V$ with $q > 0$. Wedging the defining sequence for V gives an exact sequence

$$0 \to \Lambda^q V \to E \to (\Lambda^{q-1} V) \otimes \mathcal{O}(1) \to 0$$

with E a trivial bundle. Thus by Lemma 6.4 the module M' sits in an exact sequence

$$0 \to M' \to M \to M''(1) \to 0,$$

where M and M'' are products of trivial vector bundles with products of at most p fundamental bundles with smaller $\sum n_i$. Now the result follows from Lemma 5.4 and induction. $\square$

7. Koszulness of $\boldsymbol{\kappa}$

We fix a filtration

$$S_4 \subset S_5 \subset \cdots \subset S_N = S$$

as in Theorem 1.1. Let $S' = S_{N-1}$.

Let $\pi : \overline{M}_S \to \overline{M}_{S'}$ be the projection map, let $\Psi : \overline{M}_S \to \mathbb{P}^{N-3}$ be Kapranov's map. Let κ' be the log canonical line bundle on $\overline{M}_{S'}$. We have a closed embedding

$$\Phi : \overline{M}_S \subset \mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^2 \cdots \times \mathbb{P}^{n-3}.$$

Define

$$\mathcal{L} := \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2) \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n-3}}(1).$$

Theorem 7.1. *The sheaf κ is Koszul. The sheaf $\mathcal{O}_{\overline{M}_S}$ is 1-linear over $\mathcal{L}$.*

Remark 7.2. Thus $\overline{M}_S \subset \mathbb{P}$ satisfies the analog of the Green–Lazarsfeld condition N_p for all p (see Lemma 5.14).

Proof. We argue by induction on N, the case of $\overline{M}_{0,4} = \mathbb{P}^1$ being obvious. $\mathcal{L}$ is a Koszul sheaf by Example 5.2 and Lemma 5.11. So by Lemma 5.12 it is enough to show $\mathcal{O}_{\overline{M}_S}$ is 1-linear over $\mathcal{L}$. For this, we prove simultaneously by induction:

Theorem 7.3. *Let D be the tensor product of at most p fundamental vector bundles (if $p = 0$ then let $D = \mathcal{O}$) on $\overline{M}_S$. Then $\Phi_*(D)$ is $(p+1)$-linear over $\mathcal{L}$.*

It is clear that D can be written uniquely as

$$D = \pi^*(D') \otimes \Psi^*(D''),$$

where D' is a tensor product of at most p fundamental vector bundles on $\overline{M}_{S'}$ and D'' is a tensor product of at most p fundamental vector bundles on $\mathbb{P}^{N-3}$ of the form $\Lambda^k V$, where V is the tautological quotient bundle.

For any sheaf $\mathcal{M}$ on $\mathbb{P}$, let

$$\mathcal{D}(\mathcal{M}) = \bigoplus_{n \geq p+1} \mathcal{G}_n(\mathcal{M}, \mathcal{L}).$$

The exact sequence of sheaves (4.2) induces the complex of $\mathcal{G}(\mathcal{L})$-modules

$$\cdots \to \mathcal{D}([D' \otimes \mathcal{M}^1] \boxtimes D''(-2)) \to \mathcal{D}(D' \boxtimes D'') \to \mathcal{D}(\Phi_*(D)) \to 0 \tag{7.1}$$

Let $\mathbb{P}' := \mathbb{P}^1 \times \mathbb{P}^2 \cdots \times \mathbb{P}^{n-4}$ and let $\mathcal{L}'$ be the analog of $\mathcal{L}$ on $\mathbb{P}'$.

Lemma 7.4. *The sheaf $D' \otimes \mathcal{M}^q$ is $(p+2)$-linear over $\mathcal{L}'$ for all $q \geq 1$. The sheaf D' is $(p+1)$-linear over $\mathcal{L}'$.*

Proof. We treat the case of $D' \otimes \mathcal{M}^q$, the argument for D' is entirely analogous. By Lemma 5.7, it suffices to prove that $M := \bigoplus_{n \geq p+2} \mathcal{G}_n(D' \otimes \mathcal{M}^q, \kappa')$ is $(p+2)$-linear over $\mathcal{G}(\mathcal{L}')$. By Proposition 4.4 and Corollary 6.6, M is extended from modules of the form $\bigoplus_{n \geq p+2} \mathcal{G}_n(\tilde{D}, \kappa')$, where $\tilde{D}$ is a product of at most $p+1$ fundamental vector bundles. So the statement follows from Corollary 5.5 and the inductive assumption about $\overline{M}_{S'}$. ◻

Proposition 7.5. *The i-th cohomology module of* (7.1) *lives in degrees $\leq p+i$.*

Proof. Tensoring the exact sequence (4.2) with $[D' \boxtimes D''] \otimes \bigoplus_{n \geq p+1} \mathcal{L}^{\otimes n}$ induces an exact sequence of graded sheaves

$$\cdots \to \mathcal{F}^2 \to \mathcal{F}^1 \to \mathcal{F}^0 \to 0, \tag{7.2}$$

where (for $k \geq 2$) $\mathcal{F}^k = [D' \otimes \mathcal{M}^{k-1} \boxtimes D''(-k)] \otimes \bigoplus_{n \geq p+1} \mathcal{L}^{\otimes n}$. Notice that the complex (7.1) is the complex of global sections of (7.2). Since (7.2) is exact, the corresponding hypercohomology spectral sequence with

$$E^1_{ij} = H^j(\mathcal{F}^{-i}), \quad i \leq 0, \quad j \geq 0$$

abuts to zero. Therefore, it suffices to prove that $H^j(\mathcal{F}^i)$ lives in degrees less than $i - j + p$ for $0 < j < i$.

By the Künneth formula and Lemma 6.6, it suffices to prove (using Lemma 6.2) that $H^j(S_\lambda(n-i)) = 0$ for $n \geq i - j + p$, $0 < j < i$. Here λ is any Young diagram with at most p boxes in each row. This follows from Proposition 6.3. ◻

Corollary 7.6. *The sheaf $\Phi_*(D)$ is $(p+1)$-linear over $\mathcal{L}$.*

Proof. We check that the complex (7.1) satisfies conditions of Lemma 5.8. This follows from Proposition 6.7 and Lemma 7.4 (using Lemma 5.15), and from Proposition 7.5. ◻

This concludes the proof of Theorem 7.3.

8. Quadrics

We make use of Kapranov's Hilbert scheme realization of $\overline{M}_{S^s}$ as the subscheme of $\mathrm{Hilb}(\mathbb{P}^{|S|-3})$ of Veronese curves (i.e. stable S^s-pointed rational curves embedded by global sections of $\omega_C(S^s)$) through $|S^s|$ fixed general points in $\mathbb{P}^{|S|-3}$, with

$$\overline{M}_S \subset \overline{M}_{S^s} \times \mathbb{P}^{|S|-3}$$

the universal family. See [16].

We let $\pi = \pi_{S^s}$.

We consider the bundle I_2 on $\overline{M}_{S^s}$ whose fiber at $[C]$ is the vector space of quadrics in $\mathbb{P}^{n-3}$ vanishing on C, i.e.

$$0 \to I_2 \to \mathrm{Sym}_2(\mathbb{V}) \to \pi_*(\mathcal{O}_{\overline{M}_S} \otimes \psi_s^{\otimes 2}) \to 0,$$

where $\mathbb{V}$ is the trivial bundle $H^0(\overline{M}_S, \psi_s)$. Note $I_2 = \mathcal{M}^1$ of Lemma 4.1.

Definition 8.1 (Segre Quadrics). Let V, W be vector spaces, $X, Y \in V$, $\sigma, \gamma \in W$

$$Q(X, Y \in V, \sigma, \gamma \in W) := (X \otimes \sigma) \otimes (Y \otimes \gamma) - (X \otimes \gamma) \otimes (Y \otimes \sigma) \in (V \otimes W)^{\otimes 2}.$$

We will abuse notation and use the same symbol for the image of Q in $\mathrm{Sym}_2(V \otimes W)$.

Remark 8.2. It is well known that the homogeneous ideal of the Segre embedding

$$\mathbb{P}(V) \times \mathbb{P}(W) \subset \mathbb{P}(V \otimes W)$$

is generated by Segre quadrics.

The following is obvious from the definitions:

Lemma 8.3. *For vector spaces V, W, Z, and elements $X, Y \in V$, $\sigma, \gamma \in W$, $a, b \in Z$,*

$$\begin{aligned} &Q(X, Y \in V, \sigma \otimes a, \gamma \otimes b \in W \otimes Z) + Q(X, Y \in V, \sigma \otimes b, \gamma \otimes a \in W \otimes Z) \\ &\quad = Q(X, Y \in V, \sigma, \gamma \in W) \otimes (a \otimes b + b \otimes a) \end{aligned}$$

in

$$(X \otimes (Y \otimes Z))^{\otimes 2} = (X \otimes Y)^{\otimes 2} \otimes Z^{\otimes 2}.$$

Corollary 8.4. *Let*

$$G' \subset \mathrm{Sym}_2(V \otimes W), \quad G \subset \mathrm{Sym}_2(V \otimes (W \otimes Z))$$

be the subspaces generated by Segre quadrics. Then $G' \otimes \mathrm{Sym}_2(Z)$ is contained in the image of G under the natural map

$$\mathrm{Sym}_2(V \otimes (W \otimes Z)) \to \mathrm{Sym}_2(V \otimes W) \otimes \mathrm{Sym}_2(Z).$$

Lemma 8.5. *For any $F \subset S^s$,*

$$H^0(\overline{M}_{S^s}, L_F) \otimes H^0(\overline{M}_S, \psi_s) = H^0(\overline{M}_S, L_F).$$

Proof. Follows from Lemmas 2.3 and 2.5. □

Theorem 8.6. *The Segre quadrics*

$$Q(X, Y \in H^0(\kappa_F), \sigma, \gamma \in H^0(L_F))$$

for subsets $F \subset S$ generate the homogeneous ideal of

$$\overline{M}_S \subset \mathbb{P}(H^0(\kappa)^*).$$

Proof. We write $\mathcal{S}^k$ for Sym_k.

By Theorem 1.2 and Theorem 5.3 the homogeneous ideal is generated by quadrics so it is enough to show Segre quadrics generate the kernel of

$$\mathcal{S}^2(H^0(\kappa)) \to H^0(\kappa^{\otimes 2}). \tag{8.1}$$

The map (8.1) factors through

$$\mathcal{S}^2(H^0(\kappa)) = \mathcal{S}^2(H^0(\kappa_{S^s}) \otimes H^0(\psi_s)) \to \mathcal{S}^2(H^0(\kappa_{S^s})) \otimes \mathcal{S}^2(H^0(\psi_s)) \tag{8.2}$$

Simple linear algebra shows the kernel of (8.2) is generated by Segre quadrics, so it is enough to show that images of Segre quadrics generate the kernel of

$$\mathcal{S}^2(H^0(\kappa_{S^s})) \otimes \mathcal{S}^2(H^0(\psi_s)) \to H^0(\kappa_{S^s}^{\otimes 2} \otimes \psi_s^{\otimes 2}). \tag{8.3}$$

The map (8.3) factors through

$$\mathcal{S}^2(H^0(\kappa_{S^s})) \otimes \mathcal{S}^2(H^0(\psi_s)) \to H^0(\kappa_{S^s}^{\otimes 2}) \otimes \mathcal{S}^2(H^0(\psi_s)). \tag{8.4}$$

By induction, Corollary 8.4, and Lemma 8.5, the images of Segre quadrics generate the kernel of (8.4), thus it is enough to show that images of Segre quadrics generate the kernel of

$$H^0(\kappa_{S^s}^{\otimes 2}) \otimes \mathcal{S}^2(H^0(\psi_s)) \to H^0(\kappa^{\otimes 2}). \tag{8.5}$$

The map (8.5) is induced by pushforward and taking global sections from

$$0 \to I_{\overline{M}_S} \otimes \kappa_{S^s}^{\otimes 2} \otimes \psi_s^{\otimes 2} \to \mathcal{O}_{\overline{M}_{S^s} \times \mathbb{P}(H^0(\psi_s)^*)} \otimes \kappa_{S^s}^{\otimes 2} \otimes \psi_s^{\otimes 2} \to \kappa_{\overline{M}_S}^{\otimes 2} \to 0. \tag{8.6}$$

Pushing forward by π_{S^s} we obtain

$$0 \to I_2 \otimes \kappa_{S^s}^{\otimes 2} \to \kappa_{S^s}^{\otimes 2} \otimes \mathcal{S}^2(\mathbb{V}) \to \pi_{S^s *}(\kappa_{\overline{M}_S}^{\otimes 2}) \to 0 \tag{8.7}$$

where the right hand zero (which we will not use) is implied by Corollary 3.7.

Now to prove this theorem it is enough to show that

$$H^0(I_2 \otimes \kappa_{S^s}^{\otimes 2}) \subset H^0(\kappa_{S^s}^{\otimes 2}) \otimes \mathcal{S}^2(H^0(\psi_s))$$

is generated by images of Segre quadrics. We prove this by induction on $|S|$. Suppose first $|S| \geq 6$.

For $t \in S^s$ consider the commutative diagram.

$$\begin{array}{ccc} \overline{M}_S & \longrightarrow & \overline{M}_{S^t} \\ \Big\downarrow & & \Big\downarrow \\ \overline{M}_{S^s} & \longrightarrow & \overline{M}_{S^{s,t}} \end{array}.$$

There is a natural inclusion $\pi^*_{S^{s,t}}(I_2) \subset I_2$ (indeed a subbundle) which induces a natural inclusion

$$H^0(\overline{M}_{S^{s,t}}, I_2 \otimes \kappa^{\otimes 2}_{S^{s,t}}) \subset H^0(\overline{M}_{S^s}, I_2 \otimes \kappa^{\otimes 2}_{S^{s,t}}).$$

This then gives a natural map

$$H^0(\overline{M}_{S^{s,t}}, I_2 \otimes \kappa^{\otimes 2}_{S^{s,t}}) \otimes \mathcal{S}^2(H^0(\overline{M}_{S^s}, \psi_t)) \to H^0(\overline{M}_{S^s}, I_2 \otimes \kappa^{\otimes 2}_{S^s}). \tag{8.8}$$

Consider the following diagram:

$$\begin{array}{ccc}
(H^0(\kappa_{S^t}) \otimes H^0(\pi^*_{S^s}(\psi_t)))^{\otimes 2} & \xrightarrow{e} & H^0(\kappa_S)^{\otimes 2} \\
\| & & \| \\
\begin{array}{c} H^0(\overline{M}_{S^t}, \kappa_{S^{s,t}})^{\otimes 2} \otimes H^0(\overline{M}_{S^s}, \psi_t)^{\otimes 2} \\ \otimes H^0(\overline{M}_{S^t}, \psi_s)^{\otimes 2} \end{array} & & H^0(\kappa_{S^s})^{\otimes 2} \otimes H^0(\psi_s)^{\otimes 2} \\
\downarrow & & \downarrow \\
\begin{array}{c} H^0(\overline{M}_{S^s}, \kappa^{\otimes 2}_{S^{s,t}}) \otimes \mathcal{S}^2(H^0(\overline{M}_{S^s}, \psi_t)) \\ \otimes \mathcal{S}^2(H^0(\overline{M}_{S^t}, \psi_s)) \end{array} & \xrightarrow{f} & H^0(\kappa^{\otimes 2}_{S^s}) \otimes \mathcal{S}^2(H^0(\overline{M}_S, \psi_s))
\end{array}$$

where the maps are as follows:

We have natural identifications and inclusions

$$\pi^*_{A^a}(\psi_b) = \psi_b(-\delta_{a,b}),\ a \neq b \in A$$

$$H^0(\overline{M}_{A^a}, \psi_b) = H^0(\overline{M}_A, \pi^*_{S^a}(\psi_b)) \subset H^0(\overline{M}_A, \psi_b)$$

$$H^0(\overline{M}_A, \kappa_{A^a}) \otimes H^0(\overline{M}_A, \psi_a) = H^0(\overline{M}_A, \kappa_A)$$

$$\mathcal{S}^k(H^0(\overline{M}_A, \psi_a)) = \mathcal{S}^k(\mathbb{P}^{|A|-3}, \mathcal{O}(1)) = H^0(\overline{M}_A, \psi_a^{\otimes k})$$

$$H^0(\overline{M}_S, \kappa^{\otimes k}_{S^{s,t}}) = H^0(\overline{M}_{S^s}, \kappa^{\otimes k}_{S^{s,t}}) = H^0(\overline{M}_{S^t}, \kappa^{\otimes k}_{S^{s,t}}).$$

The map e is the composition

$$(H^0(\kappa_{S^t}) \otimes H^0(\pi^*_{S^s}(\psi_t)))^{\otimes 2} \subset (H^0(\kappa_{S^t}) \otimes H^0(\overline{M}_S, \psi_t))^{\otimes 2} = (H^0(\overline{M}_S, \kappa))^{\otimes 2}$$

and the other maps are given in the obvious way by multiplication of sections. One checks immediately that the diagram is commutative.

By Lemmas 8.9 and 8.10, the images of maps (8.8) over all $t \in S^s$ generate $H^0(\overline{M}_{S^s}, I_2 \otimes \kappa^{\otimes 2}_{S^s})$. By induction the image of

$$H^0(\overline{M}_{S^{s,t}}, I_2 \otimes \kappa^{\otimes 2}_{S^{s,t}}) \subset H^0(\kappa^{\otimes 2}_{S^{s,t}}) \otimes \mathcal{S}^2(H^0(\overline{M}_{S^t}, \psi_s))$$

is generated by images of Segre quadrics under the natural map

$$\begin{aligned} H^0(\overline{M}_{S^t}, \kappa_{S^t})^{\otimes 2} &= (H^0(\overline{M}_{S^{s,t}}, \kappa_{S^{s,t}}) \otimes H^0(\overline{M}_{S^t}, \psi_s))^{\otimes 2} \\ &\to H^0(\kappa^{\otimes 2}_{S^{s,t}}) \otimes \mathcal{S}^2(H^0(\overline{M}_{S^t}, \psi_s)). \end{aligned}$$

Thus

$$H^0(\overline{M}_{S^s}, I_2 \otimes \kappa^{\otimes 2}_{S^s}) \subset H^0(\kappa^{\otimes 2}_{S^s}) \otimes \mathcal{S}^2(H^0(\overline{M}_S, \psi_s))$$

is generated by elements of form

$$f(\overline{Q}\otimes\overline{(a\otimes b+b\otimes a)})$$

where

$$\overline{Q}\in H^0(\kappa_{S^{s,t}}^{\otimes 2})\otimes\mathcal{S}^2(H^0(\overline{M}_{S^t},\psi_s))$$

is the image of a Segre quadric $Q\in H^0(\kappa_{S^t})^{\otimes 2}$ and

$$\overline{(a\otimes b+b\otimes a)}\in\mathcal{S}^2(H^0(\overline{M}_{S^s},\psi_t))$$

is the image of the symmetric tensor $a\otimes b+b\otimes a\in H^0(\overline{M}_{S^s},\psi_t)^{\otimes 2}$. Now by the commutativity of the above diagram, Lemma 8.3, and Lemma 8.5, $f(\overline{Q}\otimes\overline{(a\otimes b+b\otimes a)})$ is in the span of images of Segre quadrics. This completes the induction step.

So now suppose $|S|=5$. In this case $H^0(I_2\otimes\kappa_{S^s}^{\otimes 2})$ is two dimensional. So it is enough to show there are two Segre quadrics whose images in $H^0(\kappa_{S^s}^{\otimes 2})\otimes\mathcal{S}^2(H^0(\psi_s))$ are linearly independent.

Let $S=\{x,y,z,g,s\}$. Let $R=\{x,y,s,g\}$. We will find a quadric

$$Q=Q(X,Y\in H^0(\kappa_R),\sigma,\gamma\in H^0(\psi_z)))$$

whose restriction to the fiber $[C]=\delta_{x,y}\in\overline{M}_{S^s}$ is nontrivial, but whose restriction to the fiber $[E]=\delta_{x,z}$ is identically zero. The result will then follow by symmetry.

By the relations

$$\kappa_R=\mathcal{O}(\delta_{x,s}+\delta_{y,g})=\mathcal{O}(\delta_{y,s}+\delta_{x,g})$$

$$\psi_z=\pi^*_{\{s,y,z,g\}}(\psi_z)+\mathcal{O}(\delta_{x,z})=\mathcal{O}(\delta_{z,s}+\delta_{y,g}+\delta_{x,z})=\mathcal{O}(\delta_{g,s}+\delta_{y,z}+\delta_{x,z})$$

we may choose sections $X,Y\in H^0(\kappa_R)$, $\sigma,\gamma\in H^0(\psi_z)$ with zero divisors

$$Z(X)=\delta_{x,s}+\delta_{y,g},\quad Z(Y)=\delta_{y,s}+\delta_{x,g},$$

$$Z(\sigma)=\delta_{z,s}+\delta_{y,g}+\delta_{x,z},\quad Z(\gamma)=\delta_{g,s}+\delta_{y,z}+\delta_{x,z}.$$

The restrictions of the sections

$$X\otimes\sigma,Y\otimes\gamma,X\otimes\gamma$$

to $H^0(C,\omega_C(S^s))$ have zero schemes $x+z$, $y+g$, $x+g$, respectively. In particular the restrictions of the three sections are linearly independent and so they give a basis of (the three dimensional vector space) $H^0(C,\omega_C(S^s))$. Now it is clear that the restriction of the quadric Q to $\mathcal{S}^2(H^0(C,\omega_C(S^s))$ is nontrivial. Observe that the restriction of Q to

$$\mathcal{S}^2(H^0(E,\omega_E(S^s))$$

is identically zero. Indeed the section σ itself vanishes identically along E. $\square$

Lemma 8.7. *The map*

$$\bigoplus_{a\in T^{s,t}}H^0(\pi^*_{T^a}(\mathcal{O}(-\delta_{s,t})\otimes\kappa^{\otimes 2})\to H^0(\mathcal{O}(-\delta_{s,t})\otimes\kappa^{\otimes 2})\tag{8.9}$$

is surjective for distinct $s,t\in T$, $|T|\geq 5$.

Proof. Choose $x, y \in T^{s,t}$. Since $\delta_{x,s,t}$ and $\delta_{y,s,t}$ are disjoint, we have an exact sequence:

$$0 \to \mathcal{O}(-\delta_{s,t} - \delta_{s,t,x} - \delta_{s,t,y}) \to \mathcal{O}(\pi^*_{S^x}(-\delta_{s,t})) \oplus \mathcal{O}(\pi^*_{S^y}(-\delta_{s,t})) \to \mathcal{O}(-\delta_{s,t}) \to 0.$$

Let $E := \delta_{s,t} + \delta_{s,t,x} + \delta_{s,t,y}$. Observe that E satisfies the conditions of Lemma 3.3 (for $s \equiv t \not\equiv a$, $a \in T^{s,t,x,y}$). So the sequence remains exact after tensoring by $\kappa^{\otimes 2}$ and taking global sections. □

Lemma 8.8. *The map*

$$\bigoplus_{x \in T^s} H^0(\pi^*_{T^x}(V_{\psi_s}) \otimes \kappa^{\otimes 2}) \to H^0(V_{\psi_s} \otimes \kappa^{\otimes 2}) \tag{8.10}$$

is surjective, for $s \in T$, $|T| \geq 5$.

Proof. Choose $t \in T^s$. We have by Lemmas 3.1 and 6.6, a commutative diagram with short exact rows (where we have omitted the left and right zeros for reasons of space):

$$\begin{array}{ccccc}
\displaystyle\bigoplus_{x \in T^s} H^0(\pi^*_{T^{x,t}}(V_{\psi_s}) \otimes \kappa^{\otimes 2}) & \longrightarrow & \displaystyle\bigoplus_{x \in T^s} H^0(\pi^*_{T^x}(V_{\psi_s}) \otimes \kappa^{\otimes 2}) & \longrightarrow & \displaystyle\bigoplus_{x \in T^{s,t}} H^0(\mathcal{O}(\pi^*_{T^x}(-\delta_{s,t})) \otimes \kappa^{\otimes 2}) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\pi^*_{T^t}(V_{\psi_s}) \otimes \kappa^{\otimes 2}) & \longrightarrow & H^0(V_{\psi_s} \otimes \kappa^{\otimes 2}) & \longrightarrow & H^0(\mathcal{O}(-\delta_{s,t}) \otimes \kappa^{\otimes 2}).
\end{array}$$

Note that the first column is surjective, since the sum on the upper left includes the case $x = t$, on which the map is the identity. The right column is surjective by the previous lemma. Thus the center column is surjective. □

Lemma 8.9. *Assume that $|T| \geq 5$. The map*

$$\bigoplus_{x \in T} H^0(\pi^*_{T^x}(I_2) \otimes \kappa^{\otimes 2}) \to H^0(I_2 \otimes \kappa^{\otimes 2})$$

is surjective.

Proof. Note that $I_2 = \mathcal{M}^1$ of Lemma 4.1. Choose $s \in T$. We have by Theorem 4.3 and Lemma 6.6, a commutative diagram with short exact rows (with left and right zeros omitted):

$$\begin{array}{ccccc}
\displaystyle\bigoplus_{x \in T} H^0(\pi^*_{T^{x,s}}(I_2) \otimes \kappa^{\otimes 2}) & \longrightarrow & \displaystyle\bigoplus_{x \in T} H^0(\pi^*_{T^x}(I_2) \otimes \kappa^{\otimes 2}) & \longrightarrow & \displaystyle\bigoplus_{x \in T^s} H^0(V_{\psi_s} \otimes \kappa^{\otimes 2}) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\pi^*_{T^s}(I_2) \otimes \kappa^{\otimes 2}) & \longrightarrow & H^0(I_2 \otimes \kappa^{\otimes 2}) & \longrightarrow & H^0(V_{\psi_s} \otimes \kappa^{\otimes 2}).
\end{array}$$

Note that the first column is surjective since the upper left term includes the case $x = s$ on which the map is the identity. The right column is surjective by the previous lemma. The result follows. □

Lemma 8.10. *For* $t \in T$, $|T \geq 5|$, *the map*

$$H^0(\pi^*_{S^t}(I_2) \otimes \kappa^{\otimes 2}_{S^t}) \otimes \mathrm{Sym}_2(H^0(\psi_t)) \to H^0(\pi^*_{S^t}(I_2) \otimes \kappa^{\otimes 2})$$

is surjective.

Proof. Since $(\pi_{S^t})_*(\psi_t)$ is trivial, we have by the projection formula

$$H^0(\pi^*_{S^t}(F) \otimes \psi_t) = H^0(\pi^*_{S^t}(F)) \otimes H^0(\psi_t)$$

for any vector bundle F on $\overline{M}_{T^t}$. So it is enough to show that

$$H^0(\pi^*_{S^t}(I_2) \otimes \kappa_{S^t} \otimes \kappa) \otimes H^0(\psi_t) \to H^0(\pi^*_{S^t}(I_2) \otimes \kappa^{\otimes 2})$$

is surjective. Note that for any vector bundle W,

$$H^0(W) \otimes H^0(\psi_t) \to H^0(W \otimes \psi_t)$$

is surjective so long as $H^1(W \otimes V_{\psi_t}) = 0$. $I_2 \otimes \kappa$ is extended from globally generated line bundles by Theorem 4.3 and Lemma 3.5. So we have the necessary vanishing by Corollary 3.6. □

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