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GENERIC ALGEBRAS

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Abstract. The paper studies generic commutative and anticommutative algebras of a fixed dimension, their invariants, covariants and algebraic properties (e.g., the structure of subalgebras). In the case of 4-dimensional anticommutative algebras a construction is given that links the associated cubic surface and the 27 lines on it with the structure of subalgebras of the algebra. The rationality of the corresponding moduli variety is proved. In the case of 3-dimensional commutative algebras a new proof of a recent theorem of Katsylo and Mikhailov about the 28 bitangents to the associated plane quartic is given.

Introduction

Let $V = \mathbb{C}^n$. Consider the vector space $V^* \otimes V^* \otimes V$ of bilinear multiplications in V. We identify points of this vector space with the corresponding algebras, i.e., $A \in V^* \otimes V^* \otimes V$ simultaneously denotes the underlying vector space V equipped with the corresponding multiplication.

One has the GL(V)-module decomposition

$$V^* \otimes V^* \otimes V = \mathcal{A} \oplus \mathcal{C} = (\mathcal{A}_0 \oplus \widetilde{\mathcal{A}}) \oplus (\mathcal{C}_0 \oplus \widetilde{\mathcal{C}}), \tag{0.1}$$

where the modules $\mathcal{A}_0$, $\widetilde{\mathcal{A}}$, $\mathcal{C}_0$, $\widetilde{\mathcal{C}}$ are irreducible; their highest weights (as of SL(V)-modules) are $\omega_1 + \omega_{n-2}$, ω_{n-1}, $\omega_1 + 2\omega_{n-1}$, ω_{n-1}, respectively (in the Bourbaki numbering of the fundamental weights).

The elements of $\mathcal{A}$ (resp. $\mathcal{C}$) are anticommutative (resp. commutative) algebras; the elements of $\mathcal{A}_0$ and $\mathcal{C}_0$ are algebras with *zero trace* (algebra A has zero trace if for any $v \in A$ the operator of the left multiplication by v has zero trace).

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We denote by $[\cdot,\cdot]$ (resp. $(\cdot,\cdot)$) the multiplication in anticommutative (resp. commutative) algebras.

Algebras in the summands $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ of the decomposition (0.1) have a very simple structure, namely

$$[v, w] = f(v)w - f(w)v \quad (f \in V^*) \tag{0.2}$$

for algebras in $\widetilde{\mathcal{A}}$ and

$$(v, w) = f(v)w + f(w)v \quad (f \in V^*) \tag{0.3}$$

for algebras in $\widetilde{\mathcal{C}}$. Any subspace of such an algebra is a subalgebra.

We say that *a generic algebra* in $\mathcal{A}$, $\mathcal{C}$, $\mathcal{A}_0$, or $\mathcal{C}_0$ satisfies some property if there exists a nonempty Zariski open subset in $\mathcal{A}$, $\mathcal{C}$, $\mathcal{A}_0$, or $\mathcal{C}_0$, respectively, such that all algebras in this subset share this property. Sometimes we do not specify this property and its meaning becomes clear from the context. For example, the phrase "Let $A \in \mathcal{A}_0$ be a generic algebra. Then A has ...." means "Consider all algebras in $\mathcal{A}_0$ that have .... Then this set contains a nonempty open subset of $\mathcal{A}_0$." The aim of this paper is to establish some nice properties of generic algebras.

In Section 1, we describe the structure of subalgebras of a generic algebra. We find all integers k such that a generic anticommutative (commutative) algebra contains k-dimensional subalgebras and describes the variety of subalgebras in these cases. In particular, we prove that a generic n-dimensional anticommutative algebra A has no k-dimensional subalgebras for $3 < k < n$; the number of 3-dimensional subalgebras is finite and can be precisely computed; the 2-dimensional subalgebras form a smooth irreducible subvariety in the Grassmannian $\mathrm{Gr}(2, A)$. Similar results are obtained in the commutative case.

In Section 2, we consider 3-dimensional anticommutative and 2-dimensional commutative algebras.

Section 3 is the main section of the paper. Here we consider generic anticommutative 4-dimensional algebras. We construct in the GL_4-module $\mathcal{A}_0$ two natural κ-sections (see the definition below) arising from the structure of subalgebras. As a consequence we find two actions of finite groups with the field of invariants isomorphic to $\mathbb{C}(\mathcal{A}_0)^{\mathrm{GL}_4}$ and prove that this field is rational. To each $A \in \mathcal{A}_0$ we assign a cubic surface $K \subset PA$. We show that if A is generic then K is also generic. This enables us to give a full description of the variety of 2-dimensional subalgebras. Finally we describe the 27 lines on K in terms of the algebra A.

In Section 4, we show that n-dimensional commutative algebras can be identified with some n-dimensional linear systems of quadrics in $\mathbb{P}^n$.

In Section 5, we assign to each commutative 3-dimensional algebra with zero trace A a plane quartic $Q \subset PV^*$. We give a new short proof of a recent theorem of Katsylo and Mikhailov describing the 28 bitangents to Q in terms of the idempotents of A.

We use the following notation and terminology:

$\mathbb{C}(X)$	– the field of rational functions on an irreducible algebraic variety X,
$\mathrm{T}_x(X)$	– Zariski tangent space of X at $x \in X$,
$\overline{S}$	– Zariski closure of a set $S \subset X$,
$\mathbb{C}(X)^G$	– the field of G-invariant rational functions on an irreducible G-variety X,
PL	– the projectivization of a vector space L,
$\langle v_1, \dots, v_k \rangle$	– the linear span of vectors $v_1, \dots, v_k \in L$,
$\{e_1, \dots, e_n\}$	– the fixed basis in V,
$\{x_1, \dots, x_n\}$	– the corresponding coordinates.

We say that a linear operator has at least 2 zero eigenvalues if it has the eigenvalue zero of algebraic multiplicity at least 2.

Let X be an irreducible G-variety, S an irreducible subvariety. Then S is called *a section* of X if $\overline{G \cdot S} = X$. The section S is called a κ-section if the following condition holds: there exists a nonempty open subset $U \subset S$ such that if $x \in U$ and $gx \in S$ then $g \in H$, where $H = N_G(S) = \{g \in G \mid gS \subset S\}$ is the normalizer of S in G (see [PV]). In this case for any $f \in \mathbb{C}(X)^G$ the restriction $f|_S$ is well-defined and the map

$$\mathbb{C}(X)^G \to \mathbb{C}(S)^H, \quad f \mapsto f|_S,$$

is an isomorphism. Any κ-section defines a G-equivariant rational map $\psi : X \to G/H$: if $g^{-1}x \in S$, then $x \mapsto gH$. Conversely, any G-equivariant rational map $\psi : X \to G/H$ with irreducible fibers defines a κ-section $\overline{\psi^{-1}(eH)}$.

Let X be a G-variety, S a subvariety, $P \subset G$ a subgroup such that $P \cdot S \subset S$. Then one has the following easy formula:

$$\operatorname{codim}_X G \cdot S \geq \operatorname{codim}_X S - \operatorname{codim}_G P. \tag{0.4}$$

Therefore if the right hand side of (0.4) is positive, then S is not a section. We use this argument ("dimension count") very often to show that generic points do not satisfy some property.

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1. Subalgebras in generic algebras

Theorem 1.1.
(1) *A generic algebra in $\mathcal{C}$ or $\mathcal{C}_0$ has no k-dimensional subalgebras for* $1 < k < n$;
(2) *A generic algebra in $\mathcal{A}$ or $\mathcal{A}_0$ has no k-dimensional subalgebras for* $3 < k < n$.

To prove this theorem we need the following lemma:

Lemma 1.1. *Suppose G is a connected reductive group, $P \subset G$ a parabolic subgroup, $\mathcal{L}$ the homogeneous vector bundle with base G/P and fiber U defined by an irreducible representation $R : P \to \mathrm{GL}(U)$, where $\dim U > \dim G/P$. Suppose $\mathrm{H}^0(G/P, \mathcal{L}) \neq 0$. Then the scheme of zeroes of a generic global section of $\mathcal{L}$ is empty.*

Proof of Lemma 1.1. Let $M = \mathrm{H}^0(G/P, \mathcal{L})$. Consider the P-equivariant map

$$\psi : M \to U, \quad s \mapsto s(P).$$

Since R is irreducible, $M \neq 0$, and G/P is a homogeneous space, ψ is surjective. Let $M_P = \ker \psi$.

We have $\mathrm{codim}_M M_P = \dim U > \dim G/P = \mathrm{codim}_G P$. Hence,

$$\mathrm{codim}_M G \cdot M_P \geq \mathrm{codim}_M M_P - \mathrm{codim}_G P > 0.$$

On the other hand, homogeneity of G/P implies that $G \cdot M_P$ is exactly the set of global sections that have zeroes. $\square$

Proof of Theorem 1.1. It follows from (0.1)–(0.3) that it suffices to consider only algebras with zero trace.

The Grassmannian $\mathrm{Gr}(k, V)$ is the homogeneous space $\mathrm{GL}(V)/P$, where P is the group of matrices $p = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$, where $A \in \mathrm{GL}_k$ and $B \in \mathrm{GL}_{n-k}$. To prove (1) and (2) we apply Lemma 1.1 to the vector bundles

$$\mathcal{L}_c = \mathrm{S}^2\mathcal{E}^* \otimes \mathcal{V}/\mathcal{E} \quad \text{and} \quad \mathcal{L}_a = \Lambda^2\mathcal{E}^* \otimes \mathcal{V}/\mathcal{E}$$

on the Grassmannian $\mathrm{Gr}(k, V)$, where $\mathcal{E}$ is the tautological vector bundle and $\mathcal{V}$ is the homogeneous vector bundle with fiber V. It is easily seen that for each of the bundles $\mathcal{L}_c$ and $\mathcal{L}_a$, the fiber U is a simple P-module.

For $\mathcal{L}_c$ we have

$$\dim U = \frac{k+1}{2} k(n-k) > k(n-k) = \dim \mathrm{Gr}(k, V) \quad \text{as} \quad k > 1.$$

For $\mathcal{L}_a$ we have

$$\dim U = \frac{k-1}{2}k(n-k) > k(n-k) = \dim \mathrm{Gr}(k,V) \quad \text{as} \quad k > 3.$$

Using the Bott theorem (see [Bo]) one can check that $\mathrm{H}^0(\mathrm{Gr}(k,V),\mathcal{L}_a) \simeq \mathcal{A}_0$ and $\mathrm{H}^0(\mathrm{Gr}(k,V),\mathcal{L}_c) \simeq \mathcal{C}_0$.

Any algebra A and any linear subspace L of V define in a natural way an element $A_L \in \mathrm{Hom}(L \otimes L, V/L)$. For any $A \in \mathcal{A}$ (resp. $A \in \mathcal{C}$) the map $L \mapsto A_L$ is a global section s_A of $\mathcal{L}_a$ (resp., $\mathcal{L}_c$).

Thus we obtain the maps of $\mathrm{GL}(V)$-modules $\mathcal{A} \to \mathrm{H}^0(\mathrm{Gr}(k,V),\mathcal{L}_a)$ and $\mathcal{C} \to \mathrm{H}^0(\mathrm{Gr}(k,V),\mathcal{L}_c)$ taking A to s_A. Note that $s_A(L) = 0$ iff L is a subalgebra of A. Thus any section of $\mathcal{L}_a$ (resp. $\mathcal{L}_c$) has the form s_A for some $A \in \mathcal{A}_0$ (resp. $A \in \mathcal{C}_0$) and zeroes of this section are identified with subalgebras of A. $\square$

We shall use the ideas of this proof in the proof of the next theorem.

Theorem 1.2.
(1) *The number of* 1*-dimensional subalgebras of a generic* n*-dimensional commutative algebra equals* $2^n - 1$.
(2) *The number of* 3*-dimensional subalgebras of a generic anticommutative* n*-dimensional algebra equals the top Chern class of the vector bundle* $\mathcal{L} = \Lambda^2\mathcal{E}^* \otimes \mathcal{V}/\mathcal{E}$ *on the Grassmannian* $\mathrm{Gr}(3,n)$. *The following table contains these numbers for small* n:

the dimension of the algebra	4	5	6	7	8
the number of 3*-dimensional subalgebras*	5	22	98	465	2349

(3) *The* 2*-dimensional subalgebras of a generic anticommutative* n*-dimensional algebra form a smooth irreducible* $(n-2)$*-dimensional subvariety in* $\mathrm{Gr}(2,n)$.

Proof. To prove (1) one can use the same ideas as for (2), but we shall give a more natural argument in §4.

To prove (2) we need to check that generic global sections of the bundle $\mathcal{L}$ intersect the zero section transversally. Then (2) will follow from standard intersection theory [F]. Unfortunately, I do not know any explicit formula for the number of 3-dimensional subalgebras of a generic n-dimensional anticommutative algebra. On the other hand, for any n this number can be algorithmically computed by means of the well-known procedure of calculating Chern classes of homogeneous vector bundles on the Grassmannian [F].

The Grassmannian $\mathrm{Gr}(3,n)$ is the homogeneous space G/P, where $G = \mathrm{GL}_n$ and P is the group of matrices $p = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$, where $A \in \mathrm{GL}_3$ and $B \in \mathrm{GL}_{n-3}$.

Suppose $M = \mathrm{H}^0(G/P, \mathcal{L})$, M_P is the linear subspace of all sections vanishing at eP, $N \subset M$ the set of all sections that intersect the zero section nontransversally, $N_P \subset L$ the set of all sections that intersect the zero section nontransversally at eP. We should prove that $\operatorname{codim}_M N > 0$.

It is clear that $N = G \cdot N_P$. Each section $s \in M_P$ defines the linear map

$$ds : \mathrm{T}_P(G/P) \to \mathrm{T}_P(G/P) \oplus U,$$

where U is the fiber of $\mathcal{L}$ over eP. It follows from $\dim G/P = \dim U$ that $s \in N_P$ iff the map $\varphi = p_U \circ ds$ is not injective, where p_U is the projection on U.

We can naturally identify the representation of P in $\mathrm{T}_P(G/P)$ with the isotropy representation of P and, hence, with the representation of P in $\mathrm{Mat}(n-3,3)$ by $p \cdot X = BXA^{-1}$, for $X \in \mathrm{Mat}(n-3,3)$. If $n \geq 6$, it has 3 nonzero orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ with the representatives

$$v_1 = E_{11},\ v_2 = E_{11} + E_{22},\ \text{and } v_3 = E_{11} + E_{22} + E_{33}.$$

If $n = 4$ or $n = 5$, it has 1 or 2 nonzero orbits with the representatives v_1 or v_1 and v_2, respectively (here E_{ij} are the matrix units).

Denote by $P_{\langle x \rangle}$ the stabilizer in P of the line $\langle x \rangle$ spanned by x. Then an easy computation shows that the codimensions of $P_{\langle v_1 \rangle}$, $P_{\langle v_2 \rangle}$, $P_{\langle v_3 \rangle}$ in P are equal to $n-2$, $2n-5$, and $3n-10$, respectively.

Consider the linear subspaces $N_{v_1}, N_{v_2}, N_{v_3} \subset M_P$ of all algebras that share the following property: if s is the corresponding section of $\mathcal{L}$ then $\varphi(v_i) = 0$. To prove (2) it suffices to check that $\operatorname{codim}_M G \cdot N_{v_i} > 0$ for $i = 1, 2, 3$. Since

$$\begin{aligned} \operatorname{codim}_M G \cdot N_{v_i} &\geq \operatorname{codim}_M N_{v_i} - \operatorname{codim}_G P_{\langle v_i \rangle} \\ &= \operatorname{codim}_{M_P} N_{v_i} + \operatorname{codim}_M M_P - \operatorname{codim}_P P_{\langle v_i \rangle} - \operatorname{codim}_G P \\ &= \operatorname{codim}_{M_P} N_{v_i} - \operatorname{codim}_P P_{\langle v_i \rangle}, \end{aligned}$$

this inequality will follow immediately from

$$\operatorname{codim}_{M_P} N_{v_i} > \operatorname{codim}_P P_{\langle v_i \rangle}. \tag{1.1}$$

We have already computed the right hand side of (1.1), the left hand side is computed in the following lemma.

Lemma 1.2. $\operatorname{codim}_{M_P} N_{v_i} = 3(n-3)$ *for* $i = 1, 2, 3$.

Proof of Lemma 1.2. Consider the following open immersion

$$\mathrm{Mat}(n-3,3) \subset \mathrm{Gr}(3,n),$$

given by the formula $X \mapsto T(S)$, where $S = \langle e_1, e_2, e_3\rangle$, $T = I + \widetilde{X}$, I is the identity operator on V and $\widetilde{X}$ is the operator with the matrix

$$\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}.$$

Using this map it is easy to check that the map $\varphi : \mathrm{Mat}(n-3,3) \to \Lambda^2 S^* \otimes V/S$ has the form

$$\varphi(X)(s_1 \wedge s_2) = \big([\widetilde{X}s_1, s_2] + [s_1, \widetilde{X}s_2] - \widetilde{X}[s_1, s_2]\big) + S \tag{1.2}$$

for any $s_1, s_2 \in S$.

Let $[e_i, e_j] = \sum\limits_k c_{ij}^k e_k$. We take $X = v_i$, $i = 1, 2, 3$, write the right hand side of (1.2) in coordinates and set it equal to zero. In the case $i = 3$ (other cases are similar) we get the system of $3(n-3)$ linear equations

$$\begin{gathered} c_{15}^i = c_{24}^i,\ c_{16}^i = c_{34}^i,\ c_{26}^i = c_{35}^i, \quad i > 6, \\ c_{12}^i = c_{15}^{i+3} - c_{24}^{i+3},\ c_{13}^i = c_{16}^{i+3} - c_{34}^{i+3},\ c_{23}^i = c_{26}^{i+3} - c_{35}^{i+3}, \quad i = 1,2,3. \end{gathered} \tag{1.3}$$

The condition on the trace of the algebra has the form

$$\sum_{j=1}^{n} c_{ij}^j = 0, \quad i = 1, \dots, n. \tag{1.4}$$

The conditions for the algebra to have S as a subalgebra have the form

$$c_{ij}^k = 0, \quad 1 \le i, j \le 3,\ k \ge 4. \tag{1.5}$$

One can easily check that equations (1.3)–(1.5) are linearly independent. This completes the proof of Lemma 1.2. $\square$

Let us prove (3). We keep the notation used in proof of (2), substituting $\mathrm{Gr}(3,n)$ for $\mathrm{Gr}(2,n)$, etc. First we prove transversality.

This can be obtained as above. Let us point out only the differences. In this case the transversality of a global section s at eP is equivalent to the surjectivity of φ. The action of P on $\mathrm{Gr}(n-3, U)$ is transitive. Fix $\pi \in \mathrm{Gr}(n-3, U)$. Let $N_\pi \subset N_P$ be the linear subspace of all sections s such that $\mathrm{Im}\big(\varphi(T_P(G/P))\big) \subset \pi$. A simple computation shows that $\operatorname{codim}_G G_{N_\pi} \le$

$3(n-2)-1$. On the other hand, arguing as in the proof of Lemma 1.2, we get $\operatorname{codim}_M N_\pi = 3(n-2)$. Dimension count completes the proof of transversality.

We proved that the scheme of zeroes $Z(s)$ of a generic global section s of the vector bundle $\mathcal{L} = \Lambda^2 E^* \otimes V/E$ over $\mathrm{Gr}(2,n)$ is a smooth unmixed $(n-2)$-dimensional subvariety. To prove the irreducibility of $Z(s)$, let us consider the Koszul complex.

In our case it is exact since s is regular (see [F]):

$$0 \to \Lambda^{n-2}\mathcal{J}^* \xrightarrow{s} \dots \xrightarrow{s} \Lambda^2\mathcal{J}^* \xrightarrow{s} \mathcal{J}^* \xrightarrow{s} \mathcal{O} \to \mathcal{O}_{Z(s)} \to 0,$$

where $\mathcal{J}$ is the sheaf corresponding to $\mathcal{L}$. An application of the Bott theorem (the "singular" case) yields $\mathrm{H}^*(\mathrm{Gr}(2,n), \Lambda^p\mathcal{J}^*) = 0$ for $1 \le p \le n-2$. Hence,

$$\mathrm{H}^0(Z(s), \mathcal{O}_{Z(s)}) = \mathrm{H}^0(\mathrm{Gr}(2,n), \mathcal{O}) = \mathbb{C}.$$

This completes the proof of Theorem 1.2. □

Corollary 1.1. *A generic anticommutative algebra is generated by two elements; a generic commutative algebra is generated by one element.* □

Theorem 1.3. *Generic algebras in $\mathcal{A}$, $\mathcal{C}$, $\mathcal{A}_0$ and $\mathcal{C}_0$ are simple, i.e., have no proper ideals (if $n > 2$ in the anticommutative case).*

Proof. The proof is by dimension count. Consider, for example, the case of $\mathcal{C}_0$.

According to Theorem 1.2, a generic algebra contains only 1-dimensional subalgebras. Let $N \subset \mathcal{C}_0$ be the subset of all algebras having a 1-dimensional ideal. Since $G = \mathrm{GL}_n$ acts transitively on PV, $N = G \cdot N_1$, where N_1 is the linear subspace of all algebras having the ideal $\langle e_1 \rangle$. Let $P = G_{\langle e_1 \rangle}$. Then $\operatorname{codim}_G P = n-1$, $P \cdot N_1 \subset N_1$, $\operatorname{codim}_{\mathcal{C}_0} N_1 = n(n-1)$. Now apply dimension count. □

2. Generic algebras of small dimensions

Using the classification of irreducible linear actions of connected simple groups with nontrivial generic stabilizer due to A. Elashvili and A. Popov (see [PV]), one can easily prove that generic n-dimensional anticommutative algebras with zero trace have no automorphisms if $n \ge 4$; the same is true for commutative algebras if $n \ge 3$.

In this section we consider the cases when generic algebras have a nontrivial group of automorphisms, i.e., 3-dimensional anticommutative and 2-dimensional commutative algebras with zero trace.

Generic 3-dimensional anticommutative algebras with zero trace

We have the isomorphism of SL_3-modules

$$\mathcal{A}_0 \simeq \mathrm{S}^2 V.$$

Therefore each nonzero algebra defines a conic in PV^*. If $A \in \mathcal{A}_0$, then the corresponding quadratic function $V^* = \Lambda^2 V \to \mathbb{C} = \Lambda^3 V$ has the form

$$\Lambda^2 V \ni x \mapsto x \wedge m(x),$$

where m is the multiplication in A, hence the corresponding conic can be identified with the set of 2-dimensional subalgebras.

The isomorphism defined above takes each nonsingular quadratic function to an algebra isomorphic to $\mathfrak{sl}_2$. Hence, these algebras are generic. Since the Jacobi identity is polynomial, all algebras in $\mathcal{A}_0$ are Lie algebras. Quadratic functions of rank 2 correspond to algebras with the following multiplication:

$$[u\,,v] = v,\ \ [u\,,w] = -w,\ \ [v\,,w] = 0.$$

The conic consists of two lines in PV^* that intersect at the point corresponding to the derived algebra $\langle v, w\rangle$. Quadratic functions of rank 1 correspond to Heisenberg algebras

$$[u\,,v] = 0,\ \ [u\,,w] = 0,\ \ [v\,,w] = u.$$

The conic is the double line corresponding to the derived algebra $\langle u\rangle$.

Let $A \in \mathcal{A}_0$. For any $v \in A$ consider the characteristic polynomial of the induced operator $[v\,,\cdot]$ of the left multiplication by v. Since A is anticommutative with zero trace, only one coefficient of this polynomial, apart from the leading one, may be not equal to zero. This coefficient defines a quadratic covariant $\mathcal{A}_0 \to \mathrm{S}^2 V^*$. Therefore each algebra defines a conic in PV (that may coincide with PV). Evidently, this conic is the projectivization of the set of nilpotents of A.

When our algebra is generic (isomorphic to $\mathfrak{sl}_2$), we have a natural bijection between the conic of 2-dimensional subalgebras (i.e. the conic of Borel subalgebras) and the conic of nilpotents, which assigns to each 2-dimensional subalgebra its (1-dimensional) derived algebra. This is the canonical bijection from a conic to its dual conic.

We shall use the following lemma in the proof of Theorem 3.6.

Lemma 2.1. *Let A be a 3-dimensional anticommutative algebra (not necessarily with zero trace). Consider in PV the conic Q (that may coincide with PV):*

$$P\langle v\rangle \in Q \quad \text{iff the operator} \quad [v\,,\cdot] \text{ has at least 2 zero eigenvalues.}$$

Then Q is not a smooth conic if and only if A satisfies one of the following conditions:

(1) *A has a two-dimensional commutative subalgebra;*

(2) *A has a two-dimensional ideal;*

(3) *there exists $v \in A$, $v \neq 0$, such that any two-dimensional subspace containing v is a subalgebra.*

Proof. Suppose Q is a singular curve or coincides with PV. Then there exists a 2-dimensional subspace $L \subset A$ such that for any $x \in L$ the operator $[x\,,\cdot]$ has at least 2 zero eigenvalues. If L is a commutative subalgebra, we have (1).

Let L be a noncommutative subalgebra. Then there exists $e \in L$, $e \neq 0$, such that $[e\,,L] \subset \langle e\rangle$. Take any $f \in L$, $f \notin \langle e\rangle$, and $h \in A$, $h \notin L$. In the basis $\{e, f, h\}$ the operator $[f\,,\cdot]$ has a matrix of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

Since $[e\,,f] \neq 0$ and $[f\,,\cdot]$ should have at least 2 zero eigenvalues, we have $[f\,,A] \subset L$. Therefore $[L\,,A] \subset L$, i.e. L is an ideal.

Suppose L is not a subalgebra. Then $[L\,,L] = \langle v\rangle$, $v \notin L$. Take any basis $\{e, f\}$ of L. In the basis $\{e, f, v\}$ the operator $[e\,,\cdot]$ has a matrix of the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix}.$$

Since $[e\,,f] \neq 0$ and $[e\,,\cdot]$ should have at least 2 zero eigenvalues, we have $[e\,,v] \in \langle e, v\rangle$. Therefore we have (3).

Conversely, suppose $A \supset L$, where L is a 2-dimensional commutative subalgebra or a 2-dimensional ideal. Then clearly $PL \subset Q$ and Q is not a smooth curve.

Suppose we have (3). Then any 2-dimensional subspace L, $L \ni v$, is stable under $[v\,,\cdot]$. Therefore $[v\,,\cdot]$ induces a scalar operator λI on $A/\langle v\rangle$. If $\lambda = 0$ then $\langle v\rangle$ is an ideal (and, therefore, A has a 2-dimensional commutative subalgebra); in the opposite case $A = L \oplus \langle v\rangle$, where $[v\,,e] = \lambda e$ for any $e \in L$. Therefore $PL \subset Q$ and Q is not a smooth curve. □

Generic 2-dimensional commutative algebras with zero trace

We have the isomorphism of SL_2-modules

$$\mathcal{C}_0 \simeq \mathrm{S}^3\, V^*$$

that takes any algebra to the cubic function

$$v \mapsto v \wedge (v, v) \in \Lambda^2 V = \mathbb{C}.$$

Zeroes of this cubic function correspond to 1-dimensional subalgebras.

3. Generic 4-dimensional anticommutative algebras

Let $\mathcal{A}_0$ be the GL_4-module of 4-dimensional anticommutative algebras with zero trace. We shall construct in $\mathcal{A}_0$ two natural κ-sections arising from the structure of subalgebras.

3-dimensional subalgebras

By Theorem 1.2, a generic 4-dimensional anticommutative algebra A contains exactly five 3-dimensional subalgebras. Their projectivizations form in PV a configuration of 5 planes.

Lemma 3.1. *This configuration consists of* 5 *planes in general position.*

Proof. Let Π_1 be the set of all triples of distinct planes in $\mathbb{P}^3$ that intersect at a line; let Π_2 be the set of all quadruples of distinct planes in $\mathbb{P}^3$ that intersect at one point and such that any three of them intersect at one point. It is easily seen that the actions of GL_4 on Π_1 and Π_2 are transitive. Denote by $N_i \subset \mathcal{A}_0$ $(i = 1, 2)$ the set of all algebras that contain a configuration of subalgebras whose projectivization lies in Π_i. Denote by $N_{\pi_i} \subset N_i$ $(i = 1, 2)$ the linear subspace of all algebras that contain subalgebras whose projectivizations form some "standard" configuration $\pi_i \in \Pi_i$, e.g.

$$\pi_1 = \{x_1 = 0,\ x_2 = 0,\ x_1 + x_2 = 0\},$$
$$\pi_2 = \{x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_1 + x_2 + x_3 = 0\}.$$

Now we can use dimension count to show that $\operatorname{codim}_{\mathcal{A}_0} N_i > 0$, $i = 1, 2$ (see the proof of Theorem 1.3). □

The action of SL_4 on the set of configurations of 5 planes in general position in PV is transitive with a finite stabilizer. Let us fix a standard configuration (*the Sylvester pentahedron*) in PV formed by the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_1 + x_2 + x_3 + x_4 = 0$. It follows from Lemma 3.1 that the set of all algebras, such that the planes of the Sylvester pentahedron are the projectivizations of their subalgebras, is a 5-dimensional linear κ-section S of the SL_4-module $\mathcal{A}_0$.

One can easily show that the multiplication in algebras of S is given by the formulas $[e_i, e_j] = a_{ij} e_i + b_{ij} e_j$ $(1 \le i < j \le 4)$, where a_{ij} and b_{ij} satisfy some linear conditions. Consider 6 algebras $A_1, \dots, A_6$ with the following structure constants:

	A_1	A_2	A_3	A_4	A_5	A_6
a_{12}	0	1	1	-1	0	-1
b_{12}	1	0	-1	1	-1	0
a_{13}	1	1	0	-1	-1	0
b_{13}	0	-1	1	0	1	-1
a_{14}	1	0	1	0	-1	-1
b_{14}	-1	1	0	-1	0	1
a_{23}	-1	-1	1	0	0	1
b_{23}	1	0	0	-1	1	-1
a_{24}	0	-1	0	-1	1	1
b_{24}	-1	1	1	0	-1	0
a_{34}	-1	1	-1	1	0	0
b_{34}	0	0	1	-1	-1	1

Any $A \in S$ can be written as $\alpha_1 A_1 + \ldots + \alpha_6 A_6$, where $\alpha_1 + \ldots + \alpha_6 = 0$.

The stabilizer of the Sylvester pentahedron in the group PSL_4 is the group $\mathbb{S}_5$, represented by the permutations of its planes. Its preimage in SL_4 is a group H containing 480 elements. This group is generated by $\mathbb{A}_5$ and the preimage of a transposition (see [Be] for the details). The latter is isomorphic to $\mathbb{Z}_8$ and is generated by

$$\sigma = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}, \quad \varepsilon^4 = -1.$$

The action of σ on S is given by

$$\begin{pmatrix} \sigma A_1 \\ \vdots \\ \sigma A_6 \end{pmatrix} = M \begin{pmatrix} A_1 \\ \vdots \\ A_6 \end{pmatrix}, \text{ where } M = -\varepsilon^{-1} \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The group $\mathbb{A}_5$ can be represented as the group of rotations of the dodecahedron. Let $\Gamma_1, \ldots, \Gamma_6$ be the pairs of opposite faces of the dodecahedron. Consider the vector space of functions

$$f : \{\Gamma_1, \ldots, \Gamma_6\} \to \mathbb{C}, \quad \sum_{i=1}^{6} f(\Gamma_i) = 0.$$

This vector space has a natural structure of $\mathbb{A}_5$-module. The corresponding representation R is irreducible and coincides with the representation of $\mathbb{A}_5$ in S after the identification $A_i \mapsto f_i$, where $f_i(\Gamma_i) = 5$, $f_i(\Gamma_j) = -1$, $j \neq i$.

Consider any of two 5-dimensional irreducible representations of $\mathbb{S}_5$ that extend the representation R. It is not difficult to prove that the action of $\mathbb{S}_5$ in the projectivization of S coincides with the projectivization of this representation. Therefore one has

Theorem 3.1. *The restriction of invariants on the κ-section S induces an isomorphism*

$$\mathbb{C}(\mathcal{A}_0)^{GL_4} \simeq \mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^* \times \mathbb{S}_5},$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^5$ by scalar multiplications and $\mathbb{S}_5$ by an irreducible 5-dimensional representation (any of two). $\square$

Remark. The Sylvester pentahedron appears also in the theory of cubic surfaces. The SL_4-module of cubic forms $S^3(\mathbb{C}^4)^*$ admits a κ-section (so-called "*Sylvester section*") that has the same normalizer as our S. It was proved (see [Be]) that the restriction of invariants induces an isomorphism

$$\mathbb{C}(S^3(\mathbb{C}^4)^*)^{GL_4} \simeq \mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^* \times \mathbb{S}_5},$$

where $\mathbb{C}^*$ acts by scalar multiplications and $\mathbb{S}_5$ by permutations of coordinates.

Commutative subalgebras

Theorem 3.2. *A generic 4-dimensional anticommutative algebra A (with zero trace or without this condition) contains exactly two commutative 2-dimensional subalgebras L_1 and L_2. The lines PL_1 and PL_2 are skew.*

Proof. First consider the case of $\mathcal{A}$. The algebra A can be identified with a linear map $\Lambda^2 V \to V$. Suppose it is generic; then the projectivization of the kernel of this map is a generic line in $P\Lambda^2 V$. Two-dimensional commutative subalgebras are identified with the intersection of this line with $\mathrm{Gr}(2,4) \subset P\Lambda^2 V$, which is the Plücker quadric.

Now consider the case of $\mathcal{A}_0$. Let Π_1 be the set of all pairs of intersecting lines in $\mathbb{P}_3$, Π_2 the set of all triples of mutually skew lines in $\mathbb{P}_3$. It is easily seen that the actions of GL_4 on Π_1 and Π_2 are transitive. Denote by $N_i \subset \mathcal{A}_0$ $(i = 1, 2)$ the set of all algebras that contain a configuration of commutative subalgebras such that its projectivization lies in Π_i. Denote by $N_{\pi_i} \subset N_i$ $(i = 1, 2)$ the linear subspace of all algebras that contain the "standard" configuration of commutative subalgebras formed by 2-dimensional subspaces

$$\pi_1 = \{\langle e_1, e_2\rangle,\ \langle e_2, e_3\rangle\},$$
$$\pi_2 = \{\langle e_1, e_2\rangle,\ \langle e_3, e_4\rangle,\ \langle e_1 + e_3, e_2 + e_4\rangle\}.$$

Now we can use dimension count to show that $\mathrm{codim}_{\mathcal{A}_0} N_i > 0$, $i = 1, 2$ (see the proof of Theorem 1.3). $\square$

Let us fix two 2-dimensional subspaces in $\mathbb{C}^4$:

$$U_1 = \langle e_1, e_2\rangle \text{ and } U_2 = \langle e_3, e_4\rangle.$$

Their projectivizations are skew lines in $\mathbb{P}^3$. Let S be the linear subspace of all algebras which contain U_1 and U_2 as commutative subalgebras. Since GL_4 acts transitively on the set of pairs of skew lines in $\mathbb{P}^3$, it follows from Theorem 3.2 that S is a κ-section. The normalizer of this κ-section is the subgroup $H \subset \mathrm{GL}_4$, generated by τ and $\widetilde{H}$, where

$$\tau = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\widetilde{H} \simeq \mathrm{GL}_2 \times \mathrm{GL}_2$ is the group of all matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A, B \in \mathrm{GL}_2$.

Now we use this κ-section to prove the following theorem:

Theorem 3.3. *The field of invariants $\mathbb{C}(\mathcal{A}_0)^{GL_4}$ is rational.*

Proof. Consider the linear space

$$S' = \mathrm{Hom}(U_1, \mathfrak{sl}(U_2)) \oplus \mathrm{Hom}(U_2, \mathfrak{sl}(U_1)),$$

where $\mathfrak{sl}(U)$ denotes the vector space of linear operators on U with zero trace. Then $\widetilde{H}$ acts on S' in a natural way: the first GL_2 acts on U_1, the second one acts on U_2; the action of τ on S' is defined by the isomorphism $e_1 \to e_3$, $e_2 \to e_4$ between U_1 and U_2. Therefore the linear action $H : S'$ is well-defined.

Lemma 3.2. *The H-modules S and S' are isomorphic.*

Proof of Lemma 3.2. An H-isomorphism between S' and S is defined as follows: a pair $(F, G) \in S'$ corresponds to the algebra with the multiplication

$$[u_1, u_2] = F(u_1)u_2 - G(u_2)u_1, \quad (u_1 \in U_1, u_2 \in U_2),$$

U_1 and U_2 being commutative subalgebras. The proof is obtained by a direct calculation. □

Now consider the rational map

$$\varphi : S' \to \mathrm{Gr}(2, \mathfrak{sl}(U_2)) \times \mathrm{Gr}(2, \mathfrak{sl}(U_1)), \quad (F, G) \mapsto (\mathrm{Im}(F), \mathrm{Im}(G)).$$

The H-variety

$$\mathrm{Gr}(2, \mathfrak{sl}(U_2)) \times \mathrm{Gr}(2, \mathfrak{sl}(U_1))$$

contains an open orbit, namely, the orbit of the point (T_2, T_1), where $T_i \subset \mathfrak{sl}(U_i)$ consists of all matrices of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Hence, the H-variety S' admits a κ-section

$$S_0 = \mathrm{Hom}(U_1, T_2) \oplus \mathrm{Hom}(U_2, T_1).$$

Its normalizer is isomorphic to $(\mathbb{C}^*)^4 \leftthreetimes \mathbb{D}_4$, where $\mathbb{D}_4$ is the dihedral group. It is embedded in GL_4 as the group of matrices generated by the diagonal torus and matrices

$$\tau = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}, \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

(notice that $\mathbb{D}_4 = \langle \tau, \rho \mid \tau^2 = \rho^2 = (\tau\rho)^4 = 1 \rangle$).

We have $T_i = \langle \xi_i, \eta_i \rangle$, where

$$\xi_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \eta_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So S_0 has the basis

$$x_1 \otimes \xi_2, x_1 \otimes \eta_2, x_2 \otimes \xi_2, x_2 \otimes \eta_2, x_3 \otimes \xi_1, x_3 \otimes \eta_1, x_4 \otimes \xi_1, x_4 \otimes \eta_1.$$

Let $(a_1, \dots, a_4, a'_1, \dots, a'_4)$ be the corresponding coordinates. In these coordinates the action of $(\mathbb{C}^*)^4$ is diagonal. One can show that $\mathbb{C}(S_0)^{(\mathbb{C}^*)^4} = \mathbb{C}(p_1, p_2, p_3, p_4)$, where

$$p_1 = \frac{a_1 a'_1 a'_2}{a_2 a'_3 a'_4}, \quad p_2 = \frac{a_3 a_4 a'_2}{a_1 a_2 a'_1}, \quad p_3 = \frac{a_4 a'_3 a'_4}{a_3 a'_1 a'_2}, \quad p_4 = \frac{a_1 a_2 a'_3}{a_3 a_4 a'_4}.$$

The group $\mathbb{D}_4$ acts on p_1, p_2, p_3, p_4 according to the following table

	τ	ρ
p_1	p_3^{-1}	p_2^{-1}
p_2	p_2^{-1}	p_1^{-1}
p_3	p_1^{-1}	p_4^{-1}
p_4	p_4^{-1}	p_3^{-1}

The final trick is to write this action in the coordinates

$$q_i = \frac{p_i - 1}{p_i + 1}, \quad i = 1, \dots, 4.$$

In these coordinates our action becomes linear! Namely, $\mathbb{D}_4$ acts on q_1, q_2, q_3, q_4 according to the following table

	τ	ρ
q_1	$-q_3$	$-q_2$
q_2	$-q_2$	$-q_1$
q_3	$-q_1$	$-q_4$
q_4	$-q_4$	$-q_3$

A short and straightforward calculation shows that the field of invariants of this action is rational. □

Remark. Combining Theorem 3.1 and Theorem 3.3 we get the rationality of the field

$$\mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^*\times\mathbb{S}_5},$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^5$ by homotheties and $\mathbb{S}_5$ acts by an irreducible 5-dimensional representation (any of two). Then well-known standard arguments (based on Hilbert's Theorem 90) show that the field

$$\mathbb{C}(\mathbb{C}^5)^{\mathbb{S}_5}$$

is rational as well. The same arguments also show that the field $\mathbb{C}(\mathcal{A}_0)^{SL_4}$ is rational.

Associated cubic surface

Suppose A is a 4-dimensional anticommutative algebra with zero trace. To each vector $v \in A$ assign the characteristic polynomial of the induced operator $[v, \cdot]$. Clearly, this polynomial may have only two nonzero coefficients (apart from the leading one). These coefficients define homogeneous forms of degrees 2 and 3 on A. The zeroes of these forms in PV are called *the associated quadric and cubic.* The main object of the rest of this section is the associated cubic K and its relation to the corresponding algebra. It is clear that $P\langle v\rangle \in K$ iff the operator $[v, \cdot]$ has at least two zero eigenvalues. Therefore if $L \subset A$ is a 2-dimensional commutative subalgebra, then $PL \subset K$. The coefficients of the cubic form are homogeneous polynomials of degree 3 in the structure constants; therefore this construction defines a cubical GL_4-covariant

$$\mathcal{A}_0 \to \mathrm{S}^3V^*.$$

Notice that

$$\dim \mathcal{A}_0 = \dim \mathrm{S}^3 V^* = 20.$$

Lemma 3.3. *This covariant is dominant, i.e., it is a finite rational covering.*

Proof. To prove the lemma, consider the restriction of this covariant to the κ-section S defined in the previous subsection. Recall that S consists of all algebras which contain $U_1 = \langle e_1, e_2\rangle$ and $U_2 = \langle e_3, e_4\rangle$ as commutative subalgebras, so

$$S \simeq \mathrm{Hom}(U_1, \mathfrak{sl}(U_2)) \oplus \mathrm{Hom}(U_2, \mathfrak{sl}(U_1)).$$

The image of S under our covariant lies in the linear subspace of S^3V^* that consists of all cubic forms vanishing on U_1 and U_2. This linear subspace is a section of the GL_4-module S^3V^* (a generic cubic surface contains skew lines) and can be identified with

$$\widetilde{S} = (U_1^* \otimes \mathrm{S}^2U_2^*) \oplus (U_2^* \otimes \mathrm{S}^2U_1^*).$$

To prove the lemma it suffices to check that the corresponding map $S \to \widetilde{S}$ is dominant. Consider the differential of this map at the point of S with the following multiplication table

$$[e_1\,, e_3] = -e_2, \quad [e_2\,, e_3] = e_4, \quad [e_1\,, e_4] = e_3, \quad [e_2\,, e_4] = -e_1,$$

and $[e_i\,, e_j] = 0$ in all other cases. A direct calculation shows that this differential is an isomorphism. Thus the lemma is proved. $\square$

Remark. Note that this covariant is not a birational morphism and does not induce a birational equivalence of projectivizations. The explicit description of its generic fibers is an interesting problem.

Corollary 3.1. *The associated cubic of a generic 4-dimensional anticommutative algebra with zero trace is a generic cubic surface.* $\square$

Now we use the concept of the associated cubic surface K to describe the variety of 2-dimensional subalgebras. Recall that

$$PL_1, PL_2 \subset K.$$

Theorem 3.4. *There exists a natural isomorphism between the variety of noncommutative* 2*-dimensional subalgebras of a generic anticommutative* 4*-dimensional algebra with zero trace and the variety*

$$K \setminus \{PL_1, PL_2\}.$$

Proof. The isomorphism is constructed as follows: we assign to each noncommutative subalgebra $L = \langle v, w\rangle$ the line spanned by $[v\,, w]$. Clearly, the corresponding point of PV lies in K. Let us prove that the above commutators do not belong to the commutative subalgebras. We use dimension count. Consider the linear subspace N of all algebras that satisfy the following condition:

$$[e_1, e_2] = 0, \ [e_1, e_3] \in \langle e_1 \rangle.$$

Clearly,

$$\operatorname{codim}_{\mathcal{A}_0} N = 7.$$

The normalizer $N_{GL_4}(N)$ contains the subgroup of upper triangular matrices; therefore its codimension in GL_4 does not exceed 6. Thus,

$$\operatorname{codim}_{\mathcal{A}_0} \mathrm{GL}_4 \cdot N \geq 1.$$

But this set contains all algebras such that there exists a noncommutative subalgebra $L = \langle v, w \rangle$, where $[v, w]$ lies in a commutative subalgebra.

Now let

$$P\langle v \rangle \in K \setminus \{PL_1, PL_2\}.$$

The induced operator $[v, \cdot]$ has at least 2 zero eigenvalues; on the other hand, it has a unique (up to a scalar multiple) eigenvector with zero eigenvalue, namely, v. Therefore, there exists a vector w such that $[v, w] = v$. This proves the surjectivity of our map.

To prove the injectivity it suffices to note that the vector w defined above is determined uniquely up to addition of a multiple of v. $\square$

Combining Theorem 3.4 with Theorem 1.2, one can easily show that the variety of 2-dimensional subalgebras of a generic 4-dimensional anticommutative algebra is a Del Pezzo surface of degree 5 (the blow-up of $\mathbb{P}^2$ at 4 generic points). Therefore the varieties of 2-dimensional subalgebras of generic 4-dimensional anticommutative algebras are isomorphic.

Seven lines on the associated cubic surface

We have already mentioned that the projectivizations PL_1 and PL_2 of the 2-dimensional commutative subalgebras are two lines lying on the associated cubic surface K. In this subsection we describe in algebraic terms five lines on K that are uniquely determined as the lines on K intersecting both PL_1 and PL_2.

Lemma 3.4. *Suppose A is a generic anticommutative 4-dimensional algebra with zero trace, L a two-dimensional commutative subalgebra, C_1 and C_2 distinct 3-dimensional subalgebras. Then $PL \cap (PC_1 \cap PC_2)$ is empty.*

Proof. Dimension count. $\square$

Theorem 3.5. *Let C be one of five 3-dimensional subalgebras of a generic anticommutative 4-dimensional algebra A with zero trace. Consider the subspace L spanned by $L_1 \cap C$ and $L_2 \cap C$. Then $\dim L = 2$ and PL lies on K. This construction gives five distinct lines on K.*

Proof. Let

$$L' = \{v \in C \mid [v, A] \subset C\}.$$

Since A is simple (Theorem 1.3), L' is a proper subspace of C. Since $L_1, L_2 \not\subset C$ (Lemma 3.4), $L_1 \cap C, L_2 \cap C \subset L'$; since $L_1 \cap L_2 = \{0\}$ (Theorem 3.2), $\dim L = 2$; therefore, $L = L'$. It is clear that for any $v \in L'$ the operator $[v, \cdot]$ has at least two zero eigenvalues. Thus, $L \subset K$. Applying Lemma 3.4 we see that this construction gives five distinct lines on the cubic surface. □

Fans and 10 new lines on the associated cubic surface

Definition. Suppose A is a 4-dimensional anticommutative algebra, $p \in PA$ is a point, $\mathcal{P} \subset PA$ is a plane and $p \in \mathcal{P}$. Then the flag $\{p, \mathcal{P}\}$ is called *a fan* if any line l such that $p \in l \subset \mathcal{P}$ is the projectivization of a 2-dimensional subalgebra. The point p is called *the vertex* of the fan.

Dimension count shows that if the algebra A is generic and $\{p, \mathcal{P}\}$ is a fan, then there are no 2-dimensional commutative subalgebras L satisfying the condition

$$p \in PL \subset \mathcal{P}.$$

It follows that if $p = P\langle v \rangle$, then the operator $[v, \cdot]$ has the Jordan form

$$\operatorname{diag}[0, \lambda, \lambda, -2\lambda], \quad \lambda \neq 0.$$

Conversely, it is clear that any vector having the induced operator with such a Jordan form corresponds to a vertex of some fan. To any fan $\{p, \mathcal{P}\}$ assign the line $F = [p, \mathcal{P}] \subset \mathcal{P}$. Clearly, this line is well-defined (for any fan of a generic algebra). Let us call it *the green line* of the fan. Obviously, the green line of any fan of a generic algebra lies on the associated cubic surface. Arguing as in the proof of Theorem 3.4, one can see that the green lines of fans do not intersect with the projectivizations of the commutative subalgebras. By the same argument, distinct fans cannot have the same green line. By the well-known description of 27 lines on a cubic surface, there exist exactly 10 lines $I_1, \dots, I_{10}$ that do not intersect PL_1 and PL_2. Namely, consider any three lines of five lines that are determined in Theorem 3.5. Then the cubic contains three lines that intersect all of them: the projectivizations of the commutative subalgebras and one of I_i's.

Theorem 3.6. *A generic 4-dimensional anticommutative algebra with zero trace contains exactly 10 fans; their green lines coincide with the lines on the associated cubic surface that do not intersect the projectivizations of the commutative subalgebras.*

Proof. For a generic algebra A, there exists a line $l \subset PA$ satisfying the following conditions:
(1) $l \subset K$;
(2) l does not intersect the projectivizations of the commutative subalgebras;
(3) l intersects the projectivizations of some 3-dimensional subalgebras C_1 and C_2 at some points of lines defined by Theorem 3.5. (Recall that points of these lines are characterized as points such that the images of their induced operators lie in the 3-dimensional subalgebras, see the previous subsection.)
(4) l does not lie in the projectivization of any 3-dimensional subalgebra (Lemma 2.1 and dimension count show that the projectivization of any 3-dimensional subalgebra of a generic algebra intersects the associated cubic at a line defined by Theorem 3.5 and a smooth conic).

We shall fix in A a convenient basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and show that the conditions (1)–(4) imply that l is the green line of some fan. Let $l \cap PC_1 = \langle \varepsilon_1 \rangle$ and $l \cap PC_2 = \langle \varepsilon_2 \rangle$. Since l does not intersect the projectivizations of the commutative subalgebras, C_1 and C_2 are exactly the images of the operators $[\varepsilon_1, \cdot]$ and $[\varepsilon_2, \cdot]$. Clearly, $P(C_1 \cap C_2)$ is a line that does not intersect l, otherwise l would lie in the projectivization of some 3-dimensional subalgebra. We take ε_3 and ε_4 in $C_1 \cap C_2$; moreover, we define ε_3 as $[\varepsilon_1, \varepsilon_2]$.

The matrix of the operator $[x\varepsilon_1 + y\varepsilon_2, \cdot]$ in the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ has the form

$$\begin{pmatrix} 0 & 0 & ax & dx \\ 0 & 0 & gy & ly \\ -y & x & bx + hy & fx + my \\ 0 & 0 & cx + ky & -bx - hy \end{pmatrix}, \tag{3.1}$$

where a, b, c, d, f, g, h are some structure constants. Clearly, $l \subset K$ iff

$$\begin{cases} c(l-d) + b(g-a) = 0, \\ k(l-d) + h(g-a) = 0. \end{cases} \tag{3.2}$$

It can be proved by a direct calculation that l is the green line of some fan iff the system of equations in variables z and w

$$\begin{cases} az + dw = 1, \\ gz + wl = 1, \\ cz = bw, \\ kz = hw \end{cases} \tag{3.3}$$

is consistent.

Write the third order minors of (3.1) that may be not equal to zero:

$$
\begin{gathered}
xy^2(al-dg),\\
x^2y(al-dg),\\
xy(g(bx+hy)+l(cx+ky)),\\
y^2(g(bx+hy)+l(cx+ky)),\\
xy(a(bx+hy)+d(cx+ky)),\\
x^2(a(bx+hy)+d(cx+ky)).
\end{gathered}
\tag{3.4}
$$

Suppose that $al-dg=0$. Then there exist x and y not equal to zero simultaneously such that all minors are equal to 0. Indeed, the system of equations

$$
\begin{cases} g(bx+hy)+l(cx+ky)=0,\\ a(bx+hy)+d(cx+ky)=0 \end{cases}
$$

is equivalent to the matrix equation

$$
\begin{pmatrix} a & d\\ g & l\end{pmatrix}\begin{pmatrix} b & h\\ c & k\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}=0,
$$

which has a nonzero solution. But this property implies that l intersects the commutative subalgebras. Hence $al-dg\neq 0$. Since ε_1 and ε_2 do not lie in the commutative subalgebras, we get a system of inequalities that is necessary for l not to intersect the commutative subalgebras:

$$
\begin{cases} al-dg\neq 0,\\ ab+dc\neq 0,\\ gh+kl\neq 0.\end{cases}
\tag{3.5}
$$

Go back to the system (3.2). It is a system of linear equations in the variables $l-d$ and $g-a$. By (3.5), $al-dg\neq 0$; therefore this system has a nontrivial solution. Hence

$$
ch-bk=0.
\tag{3.6}
$$

Now we can solve (3.3). By (3.5), k and h are not equal to zero simultaneously. Suppose that $k\neq 0$ (the second case is similar). Put $z=hw/k$. Then the two last equations (3.3) are the identities by (3.6). Substituting $z=hw/k$ in the first two equations, we get the system

$$
\begin{cases} (ah+dk)w=1,\\ (gh+lk)w=1.\end{cases}
$$

This system has a solution because $gh+kl\neq 0$ by (3.5) and $gh+kl=ah+dk$ by the second equation in (3.2). This completes the proof of Theorem 3.6. $\square$

Notice that we interpreted only 17 lines on the associated cubic surface in terms of the corresponding algebra, but these lines uniquely determine the other lines and the cubic surface itself.

4. "Affinization" of commutative algebras

Consider now GL_n-modules $\mathcal{C}$ and $\mathcal{C}_0$. Theorem 1.2 asserts that generic algebras in these modules have exactly $2^n - 1$ one-dimensional subalgebras. Easy dimension count shows that there are no *quadratic nilpotents* (i.e. vectors $v \neq 0$, such that $(v, v) = 0$) in these algebras. Hence the set of 1-dimensional subalgebras is identified with the set of nonzero idempotents. Thus a generic algebra contains exactly 2^n idempotents, including zero. In this section we shall give an elementary proof of this fact together with a construction ("affinization") eliminating the difference between 0 and other idempotents.

Suppose A is a commutative algebra defined by a multiplication $(\cdot\,,\cdot)$ on a vector space $V \simeq \mathbb{C}^n$. It is uniquely determined by the quadratic map

$$V \to V, \quad v \mapsto Q(v) \equiv (v, v).$$

Let us construct a canonical linear isomorphism between V^* and some linear space of affine quadratic functions on V. This isomorphism takes each linear form f to the function

$$v \mapsto f\big(v - Q(v)\big).$$

We can embed V in $P\mathbb{C}^{n+1}$ with homogeneous coordinates $x_1, \ldots, x_n, T$ as the complement of the hyperplane $T = 0$. Affine quadratic functions on V can be canonically identified with elements of $\mathrm{S}^2(\mathbb{C}^{n+1})^*$. So our construction defines a regular map

$$\mathcal{C} \to \mathrm{Gr}\left(n, \mathrm{S}^2(\mathbb{C}^{n+1})^*\right), \tag{4.1}$$

i.e., any algebra defines a n-dimensional linear system of quadrics in $P\mathbb{C}^{n+1} = \mathbb{P}^n$. It is clear that the idempotents of the algebra coincide with the base points of this system lying in the finite part of the projective space; lines spanned by nonzero quadratic nilpotents correspond to the base points at infinity. Let us show that this construction "almost identifies" commutative algebras and n-dimensional linear systems of quadrics in $\mathbb{P}^n$.

Theorem 4.1. *Images of $\mathcal{C}$ or $\mathcal{C}_0$ under the map* (4.1) *are sections of* PGL_{n+1}*-variety* $\mathrm{Gr}(n, \mathrm{S}^2(\mathbb{C}^{n+1})^*)$.

Proof. Consider an n-dimensional linear system of quadrics. Suppose that there exists a point such that n elements of a basis of this system intersect transversally at this point (this case is generic). Some projective transformation takes this point to $(0 : ... : 0 : 1)$. Then the elements of the basis have the form

$$(4.2)\quad Tf_1(x_1,...,x_n) - Q_1(x_1,...,x_n),...,Tf_n(x_1,...,x_n) - Q_n(x_1,...,x_n),$$

where f_j are linear functions and Q_j are quadratic functions. The equations of the tangent spaces of the quadrics (4.2) at the point $(0 : ... : 0 : 1)$ have the form $f_1 = 0, \dots, f_n = 0$. Since these quadrics intersect transversally, we see that $f_1, \dots, f_n$ are linearly independent. Now we can change the basis of our linear system in such a way that $f_i \equiv x_i$. We get exactly the affine quadrics that span a linear system corresponding to the commutative algebra A with the multiplication

$$Q\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} Q_i(x_1, \dots, x_n) e_i.$$

Consider a projective transformation of $\mathbb{P}^n$ of the form

$$(4.3)\qquad x_i \mapsto x_i, \quad (1 \le i \le n), \quad T \mapsto T + \sum_{i=1}^{n} \alpha_i x_i.$$

This transformation takes our linear system to the system that corresponds to the algebra $A + \tilde{A}$, where $\tilde{A}$ is the algebra with the multiplication table

$$(e_i, e_j) = \frac{\alpha_i}{2} e_j + \frac{\alpha_j}{2} e_i, \quad 1 \le i, j \le n.$$

Denote by F the linear form $\mathrm{Tr}(v, \cdot)$ on V. Clearly, $F(e_i) = \frac{n+1}{2}\alpha_i$. Therefore, some projective transformation (4.3) takes our system to a system that lies in the image of $\mathcal{C}_0$. □

Corollary 4.1. *A generic algebra in $\mathcal{C}$ and $\mathcal{C}_0$ contains exactly 2^n idempotents, including zero.* □

5. Generic 3-dimensional commutative algebra

Let A be a commutative algebra with zero trace and underlying vector space $V = \mathbb{C}^3 \subset \mathbb{P}^3$. Then the construction of the previous section gives a 3-dimensional linear system of quadrics in $\mathbb{P}^3$, which is canonically identified with V^*. Consider all singular quadrics in this system. The projectivization of this set is a quartic in PV^*. We call it *the associated quartic*. This construction gives a nonzero quadratic SL_3-equivariant map

$$\mathcal{C}_0 \to S^4 V.$$

This map is dominant (see [KM]). We shall use "affinization" to prove the following theorem.

Suppose A is a generic commutative 3-dimensional algebra in V, Q is the corresponding quartic in PV^*. The 28 bitangents of Q can be identified with 28 points in PV.

Theorem 5.1. ([KM]) *These points are*

$$\mathbb{C}v_i, \quad \mathbb{C}(v_i - v_j, v_i - v_j),$$

where v_i are 7 nonzero idempotents and $i \neq j$.

Proof. We shall prove the generalization of this theorem to the case of a generic linear system of quadrics in $\mathbb{P}^3$; The theorem is an easy corollary of this result. Our quartic and its bitangents lie in the projectivization of the linear system. Consider 2 distinct points $v_1, v_2 \in \mathbb{P}^3$ that belong to all quadrics in the linear system. The number of such pairs is equal to 28. Now consider all quadrics containing a line that connects v_1 and v_2. These quadrics form a line in the projectivization of the linear system. Let us prove that this line is a bitangent to the quartic of singular quadrics. Fix a basis in $\mathbb{C}^4$ with first two vectors lying in v_1 and v_2. Then the matrices of quadrics of our line have the form

$$G = \begin{pmatrix} 0 & A^T \\ A & B \end{pmatrix},$$

and therefore,

$$\det G = (\det A)^2.$$

Hence this line intersects the quartic of singular quadrics at two double points, i.e., it is a bitangent.

Now suppose that the linear system corresponds to a generic commutative algebra. Then its quadrics intersect at the idempotents of the algebra. Denote by f the covector corresponding to some quadric that contains the line connecting two idempotents. Consider two cases.

Let the idempotents be 0 and v_i. Then we have $f(\alpha v_i - Q(\alpha v_i)) = \alpha(1-\alpha)f(v_i) = 0$ for any α, therefore $f(v_i) = 0$.

Let the idempotents be v_i v_j. Then we have $f((\alpha v_i + (1-\alpha)v_j) - Q(\alpha v_i + (1-\alpha)v_j)) = \alpha(1-\alpha)f(v_i - 2(v_i, v_j) + v_j) = 0$ for any α, therefore $f((v_i - v_j, v_i - v_j)) = 0$. This completes the proof of Theorem 5.1. □

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