Categorical aspects of the Kollár-Shepherd-Barron correspondence

(joint w. Giancarlo Urzúa)

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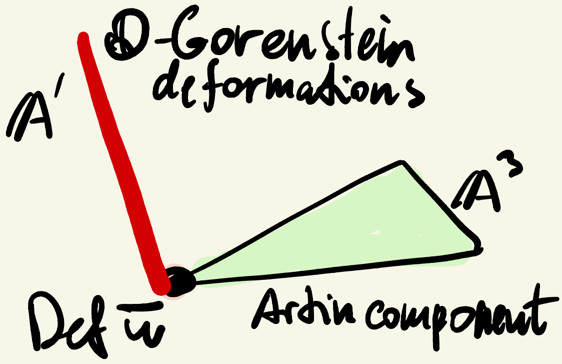
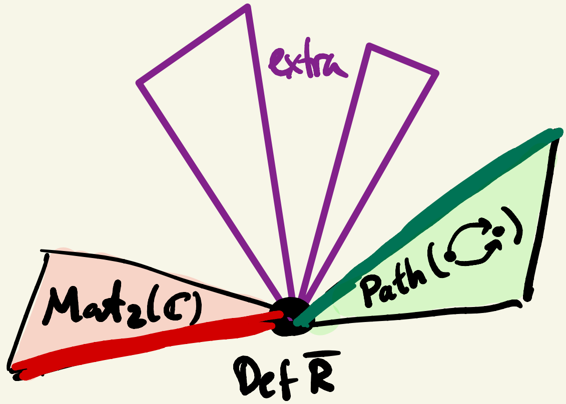
Classical examples of reducible deformation spaces:

(Pinkham) deformations of surface singularities
$\bar{W}$ = cone over RNC in $\mathbb{P}^4$



(Gabriel) deformations of finite-dim. non-comm. algebras
$\bar{R} = \mathbb{C}[x,y,z]/(x,y,z)^2$



Note: Path algebras of quivers (and Morita equiv.) don't deform, so they correspond to open subsets in irreducible components of Def $\bar{R}$

Theorems (T.-Urzua) Def $\bar{W} \xrightarrow{\perp}$ Def $\bar{R}$ (TBD)

for every 2D toroidal singularity (= cqs) $\bar{W}$.

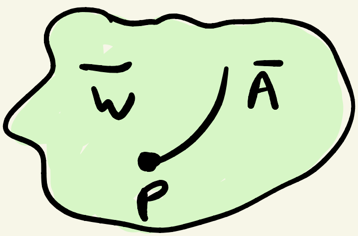
General deformations of $\bar{W}$ correspond to deformations of $\bar{R}$ to path algebras of quivers (or Morita-equivalent)

$\{\text{irred. comp. of Def}\,\bar{W}\} \xrightarrow{\perp} \{\text{irred. comp. Def}\,\bar{R}\}$

$\mathbb{Q}$G deformations of $\bar{W} \xrightarrow{\perp}$ semi-simple deformations of $\bar{R}$ (one direction due to Kawamata)

Techical assumptions: $\frac{1}{\Delta}(1,\Omega) \subset \overline{W}$ projective surface that satisfies various conditions (on class group, unobstructedness of deformations, etc) – list in the paper but there is a simple choice e.g. $\overline{W} = \mathbb{P}(1,\Omega,\Delta)$. Harder choices of $\overline{W}$ are interesting for applications to moduli (e.g. of Dolgachev surfaces in the paper)

Kawamata iterated extensions:



$\exists\ \bar{A}$ étale-locally a toric coordinate at P, in particular it generates a local class group

We define a sequence of sheaves inductively

$$D_0 = \mathcal{O}(-\bar{A})$$

$$0 \to D_0 \otimes \mathrm{Ext}^1(D_0, D_0)^\vee \to D_1 \to D_0 \to 0$$

$$0 \to D_0 \otimes \mathrm{Ext}^1(D_1, D_0)^\vee \to D_2 \to D_1 \to 0$$

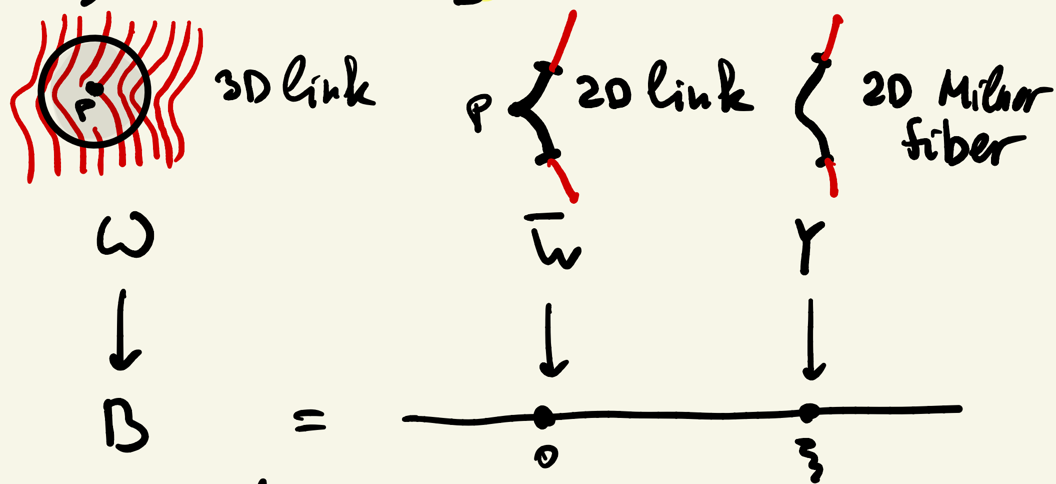
$$\vdots$$

Apriori this can go on forever but in our case the sequence stabilizes with

$D_\infty = \overline{F}$ Kawamata vector bundle on $\overline{W}$

$\overline{R} = \mathrm{End}(\overline{F})$ Δ-dim Kalck–Karmazyn algebra

Topology of smoothing over a smooth curve B



Categorification

Th (Karmazyn-Kuznetsov-Shinder)

SOD $D^b(\bar{W}) = \langle D^b(\bar{R}\text{-mod}), \mathcal{B}^{\bar{W}} \rangle$, $\mathcal{B}^{\bar{W}} \subset D^{perf}(\bar{W})$

Cor (Kalck-Karmazyn) $D_{sing}(\bar{W}) = D_{sing}(\bar{R})$

Th 1 (T.-Urzúa) The SOD deforms:

$$D^b(W) = \langle D^b(\mathcal{R}\text{-mod}), \mathcal{B}^W \rangle$$

B-linear SOD

with specializations

over 0: KKS S.O.D.

$D^b(\bar{W}) = \langle D^b(\bar{R}\text{-mod}), \mathcal{B}^{\bar{W}} \rangle$

categorified 2D link

over ξ: SOD of smoothing

$D^b(Y) = \langle D^b(\hat{R}\text{-mod}), \mathcal{B}^Y \rangle$

categorified Milnor fiber

categorified 3D link

what are algebras $\mathcal{R}$ and $\hat{\mathcal{R}}$?

v.b. $\bar{F}$ on $\bar{W}$ deforms uniquely to a v.b. $\mathcal{F}$ on $\mathcal{W}$

v.b. $\hat{F} = \mathcal{F}|_Y$ (we call $\bar{F}, \mathcal{F}, \hat{F}$ Kawamata bundles)

$$\mathcal{R} = \pi_* \mathcal{E}nd(\mathcal{F}) \text{ v.b. on } B \text{ of rank } \Delta$$

$$\mathcal{R}_0 = \bar{\mathcal{R}} = \mathrm{End}_{\bar{W}}(\bar{F}) \qquad \mathcal{R}_{\xi} = \hat{\mathcal{R}} = \mathrm{End}_Y(\hat{F})$$

Note: SOD are expected to deform

Functor $\hat{\mathcal{R}}\text{-Mod} \to \mathrm{QCoh}(Y)$

$$\mathrm{Coker}(\hat{\mathcal{R}}^I \xrightarrow[A]{} \hat{\mathcal{R}}^J) \longmapsto \mathrm{Coker}(\hat{F}^I \xrightarrow[A]{} \hat{F}^J)$$

induces $D^-(\hat{\mathcal{R}}\text{-mod}) \xrightarrow{\otimes \hat{F}} D^-(\mathrm{coh}\, Y)$

Tricky: prove boundedness $D^b(\hat{\mathcal{R}}\text{-mod}) \xrightarrow{\otimes \hat{F}} D^b(Y)$

Once known, $\otimes \hat{F}$ fully faithful, adjoints exist

$\Rightarrow D^b(Y) = \langle D^b(\hat{\mathcal{R}}\text{-mod}), \mathcal{B}^Y \rangle$

Note: SOD on smooth proj. varieties are rare

Next: the structure of $\hat{\mathcal{R}}$ and $D^b(\hat{\mathcal{R}}\text{-mod})$

Kollár-Shepherd-Barron correspondence

(version of Behnke-Christoffersen)

$\mathrm{Def}(\overline{W})_{red} = \bigcup_{\substack{\text{M-resolution} \\ W^+ \to \overline{W}}} \mathrm{Def}^{W^+}(\overline{W})$ ← smooth components

deformations that can be lifted to a $\mathbb{Q}$-Gorenstein deformation of W^+ (after a finite base change)

$\left[\begin{array}{c} K_{W^+/\overline{W}} \text{ nef} \\ W^+ \text{ has Wahl sing.} \end{array} \right]$

Example: If W^+ = minimal resolution of $\overline{W}$

$\Rightarrow \mathrm{Def}^{W^+}(\overline{W})$ = Artin component

Th. 2 (T.-Urzúa)

Y sufficiently general on $\mathrm{Def}^{W^+}(\overline{W})$

$\Rightarrow \hat{R}$ is Morita-equivalent to a path algebra of a quiver (which depends on W^+)

Cor $\hat{R}$ does not deform $\Rightarrow \psi : \mathrm{Def}\,\overline{W} \hookrightarrow \mathrm{Def}\,\overline{R}$ maps each irreducible component $\mathrm{Def}^{W^+}\overline{W}$ to a unique irreducible component of $\mathrm{Def}\,\overline{R}$

Cor There are many quivers that can be embedded into derived categories of surfaces

[Answers several questions raised by Belmans-Raedschelders]

Note: $D^b(Y)$ may contain exceptional collections but their endomorphism algebras are typically path algebras of quivers **with relations**

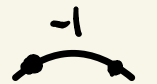
Note: **Orlov** proved that every quiver is realized on a variety of sufficiently large dimension.

Obstructions for surfaces, e.g. possible **Mukai** lattices

Example from K-SB $\frac{1}{19}(1,7)$ has 3 M-resolutions

minimal resolution (self-intersections)

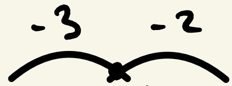
$$\frac{19}{7} = 3 - \cfrac{1}{4 - \cfrac{1}{2}}$$



$\binom{2}{1}$ $\binom{3}{1}$

$\binom{n}{a} = \frac{1}{n^2}(1, na-1)$

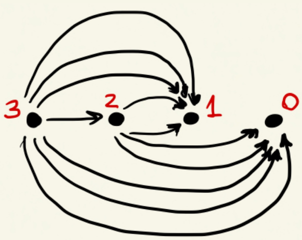
(self-intersections on the resolution)



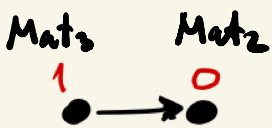
$\binom{2}{1}$

Wahl singularity

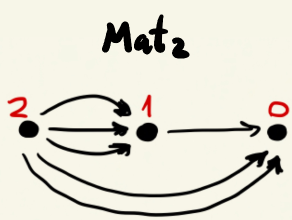
Corresponding quivers:



$\dim \overline{R} = \dim \widehat{R} = 19$

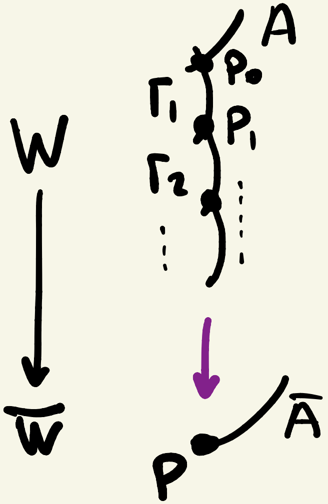


$9 + 4 + 6 = 19$



$1 + 4 + 1 + 2 + 6 + 2 + 3 = 19$

Crucial observation $\exists$ braid group action on all Wahl resolutions $W \to \overline{W}$



Definition of a Wahl resolution:

(1) partial toroidal resolution

(2) $P_i = \begin{pmatrix} n_i \\ a_i \end{pmatrix} = \frac{1}{n_i^2}(1, n_i a_i - 1)$

Wahl singularity

($n_i = 1 \Rightarrow$ smooth point)

$$\begin{array}{l} \mathrm{Def}^{\mathbb{Q}G} W \simeq \mathbb{A}^N \text{ (unobstructed)} \\ \quad \downarrow \text{ (blow-down deformations)} \\ \mathrm{Def}\, \overline{W} \end{array}$$

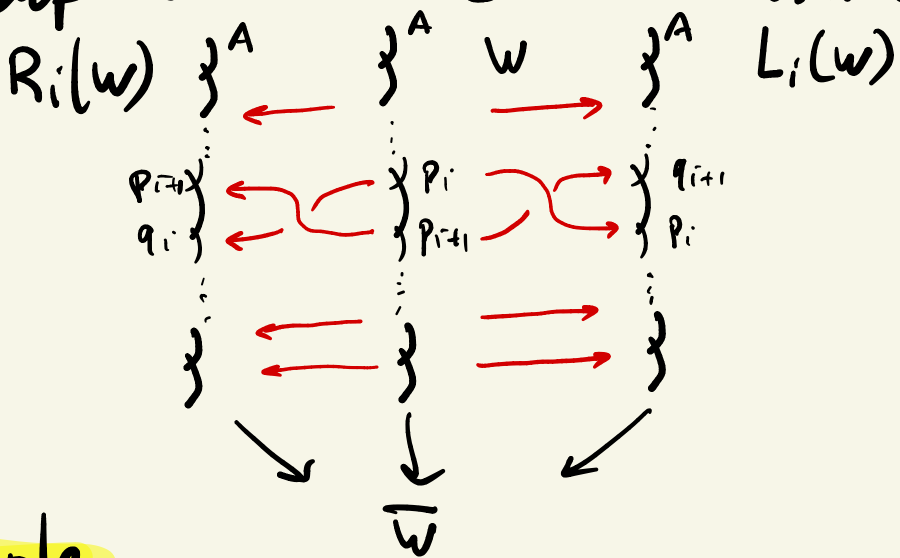
(3) For a general curve $B \to \mathrm{Def}^{\mathbb{Q}G} W$, the corresponding 3-fold contraction $\mathcal{W} \to \overline{\mathcal{W}}$ is small ($\Leftrightarrow$ general fibers are isomorphic to Y)

Note: True for M-resolutions $W^+ \to \overline{W}$

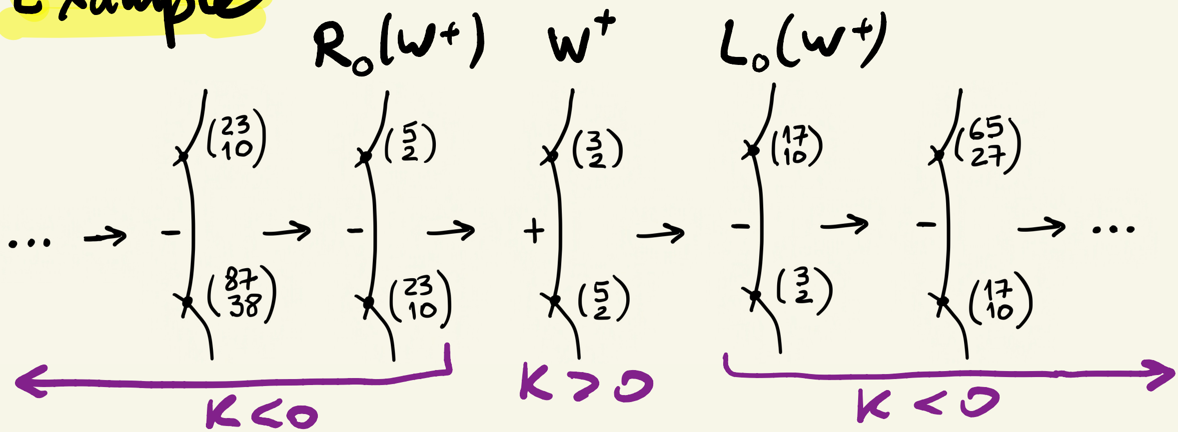
Note: Given a Wahl resolution $W \to \overline{W}$, run MMP for its general $\mathbb{Q}G$ deformation. Terminates with an M-resolution $W^+ \to \overline{W}$

Lemma (based on earlier work Hacking–T–Urzúa on Mori's conjecture on extremal nbhds IA, IIA)

$\exists$ transitive braid group action on $\left\{\begin{array}{l}\text{Wahl resolutions} \\ W \text{ with the same } W^+\end{array}\right\}$



Example

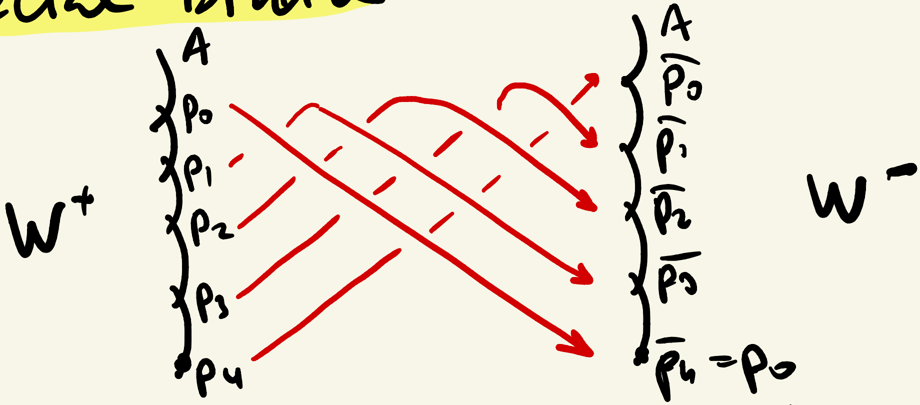


$R_0(W^+) = [3\,5\,2](1)[3\,2\,2\,2\,6\,2\,5\,2]$ $[2\,5](1)[3\,5\,2] = W^+$

$\overline{W} = [2\,4\,2\,5\,2] = \frac{1}{94}(1,55)$

(numbers are − self-intersections in the resolution)

Special brand



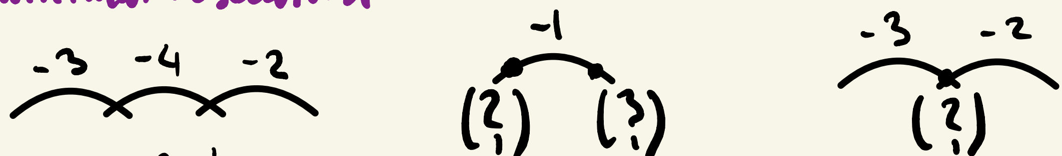
M-resolution $K \geq 0$

N-resolution $K \leq 0$

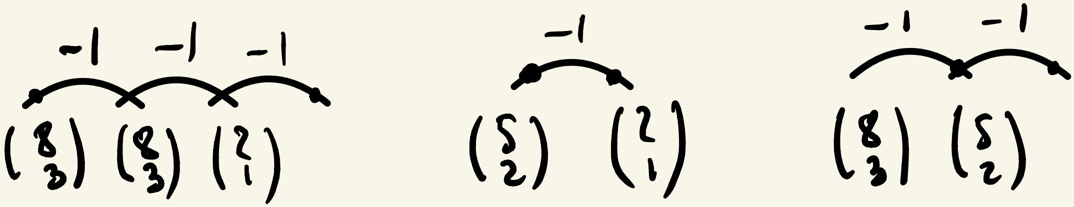
Rk Typically, there are infinitely many K-negative resolution but the N-resolution is the "first" one

Example from K-SB $\frac{1}{19}(1,7)$ has 3 M-resolutions

minimal resolution



N-reselutions

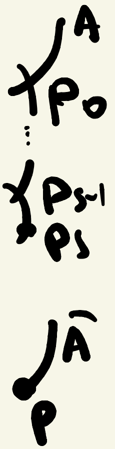


Th3 (T.-Urzua)

Categorification of KSB correspondence

Wahl resolution and its smoothing

$$D^b(W) = \langle D^b(R_s\text{-mod}), \dots, D^b(R_0\text{-mod}), \mathcal{B}^{\overline{W}} \rangle$$



$$\begin{tikzcd} W \arrow[r, rightsquigarrow] \arrow[d] & Y \arrow[d, equal] \\ \overline{W} \arrow[r, rightsquigarrow] & Y \end{tikzcd}$$

$\mathcal{B}$-linear deformation

$$D^b(Y) = \langle \underbrace{E_s^W, \dots, E_0^W}_{\|}, \mathcal{B}^Y \rangle$$

$$\cup$$

$$D^b(Y) = \langle D^b(\hat{R}\text{-mod}), \mathcal{B}^Y \rangle$$

$\mathcal{B}$-linear deformation

$$D^b(\overline{W}) = \langle D^b(\overline{R}\text{-mod}), \mathcal{B}^{\overline{W}} \rangle$$

Here E_i^W is the Hacking exceptional vector bundle associated with $\mathbb{Q}$-Gorenstein smoothing of $P_i \in W$. It has rank n_i if $P_i = \binom{n_i}{a_i}$.

Braid group action on $D^b(Y)$ by mutations of exceptional collection $\{E_s^W, \dots, E_0^W\}$ correspond to mutations of Wahl resolutions (up to shift)

$\forall \sigma \in \text{BraidGroup}, \{E_s^w, \ldots, E_0^w\} \xrightarrow{\sigma} \{E_s^{\sigma(w)}, \ldots, E_0^{\sigma(w)}\}$

SOD typically deforms but does not specialize. However, in our case mutated SODs specialize if we simultaneously change ("mutate") the special fiber.

A key corollary is that every mutation of this exceptional collection of vector bundles is an exceptional collection of vector bundles (and not exceptional sheaves or complexes)

Corollaries:

$\langle E_s^+, \ldots, E_0^- \rangle$ associated with M-resolution W^+ has Ext^1 only in forward direction

$\langle E_s^-, \ldots, E_0^- \rangle$ associated with N-resolution W^- is strong (has Hom only)

Deformed Kawamata bundle $\hat{F} = \bigoplus (E_i^-)^{\mathrm{rk}\, E_{s-i}^+}$

Deformed Kalck-Karmazyn algebra $\hat{R} = \mathrm{End}(\hat{F})$ is hereditary ($\simeq$ path algebra of a quiver)

Can we write explicit equations for a deformation $\bar{R} \rightsquigarrow \hat{R}$ of finite-dim algebras?

$\Delta/\Omega = b_1 - \frac{1}{b_2 - \cdots} =: [b_1 \dots b_s]$ $\quad \Delta/(\Delta-\Omega) = [a_1 \dots a_\ell]$

Assume $\frac{1}{\Delta}(1,\Omega)$ not Gorenstein $\iff \ell \geq 2$

Riemenschneider quasideterminants:

$$\begin{bmatrix} z_0 & & z_1 & & \cdots & z_{\ell-1} & & z_\ell \\ & z_1^{a_1-2} & & z_2^{a_2-2} & \cdots & z_{\ell-1}^{a_{\ell-1}-2} & z_\ell^{a_\ell-2} & \\ z_1 & & z_2 & & \cdots & z_\ell & & z_{\ell+1} \end{bmatrix}$$

$$\hat{\mathcal{O}}(\bar{w}) = \frac{\mathbb{C}[[z_0 \dots z_{\ell+1}]]}{\langle \circ\circ - \circ\circ \dots \circ\circ \rangle}$$

Kalck-Karmazyn algebra

$$\bar{R} = \frac{\mathbb{C}\langle z_0 \dots z_{\ell+1} \rangle}{\langle \circ\circ, \circ \dots \circ, z_0, z_{\ell+1} \rangle}$$

order: "toric coordinates"

Example $\frac{1}{4}(1,1)$ $\quad \frac{4}{1} = [4]$ $\quad \frac{4}{3} = [222]$

$$\hat{\mathcal{O}}(\bar{w}) = \frac{\mathbb{C}[[z_0 \dots z_4]]}{2\times 2 \text{ minors} \begin{bmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}}$$

$$\overline{R} = \frac{\mathbb{C}\langle z_1 z_2 z_3\rangle}{\langle z_i z_j\rangle}$$ fat point on $\mathbb{A}^3$

$\dim \overline{R} = 4$

Example $\frac{1}{8}(1,5)$ $\quad \frac{8}{5} = [2\,3\,2]$ $\quad \frac{8}{3} = [3\,3]$

$$\begin{bmatrix} 0 & \cdot & z_1 & \cdot & z_2 \\ \cdot & z_1 & \cdot & z_2 & \cdot \\ z_1 & \cdot & z_2 & \cdot & 0 \end{bmatrix} \qquad \overline{R} = \frac{\mathbb{C}\langle z_1, z_2\rangle}{\left\langle \begin{matrix} z_1^3, z_2^2 z_1^2, \\ z_1 z_2, z_2^3 \end{matrix} \right\rangle}$$

$\dim \overline{R} = 8$

Let's deform along the Artin component

Preparation: equivalent matrix is

$$\begin{bmatrix} z_0 & z_1 & & \cdots & & z_{\ell-1} & z_\ell^{a_\ell-1} \\ & & z_2^{a_2-2} & \cdots & z_{\ell-1}^{a_{\ell-1}-2} & & \\ z_1^{a_1-1} & z_2 & & \cdots & & z_\ell & z_{\ell+1} \end{bmatrix},$$

e.g. $\begin{bmatrix} z_0 & z_1 & z_2^2 \\ z_1^2 & z_2 & z_3 \end{bmatrix}$ for $\frac{1}{8}(1,5)$

Now perturb entries into

$$\begin{bmatrix} z_0 & z_1 & \cdots & & & z_{l-1}^{(a_{l-1}-2)}\, z_{l-1} & z_l^{(a_l-1)} \\ \vdots & \vdots & z_2^{(a_2-2)} & \cdots & z_{l-1}^{(a_{l-1}-2)} & \vdots & \vdots \\ z_1^{(a_1-1)} & z_2-t_2 & & \cdots & z_{l-1}-t_{l-1} & z_l & z_{l+1} \end{bmatrix}$$

$$z_i^{(r_i)} = z_i^{r_i} - \sum_{j=1}^{r_i} s_j^i z_i^{r_i-j}$$

deformation parameters

Commutative quasideterminants give versal deformation over the Artin component

$$\begin{tikzcd} & \mathrm{Def}\, W^{min} \simeq \mathbb{A}^N_{t_i, s_j^{(i)}} \arrow[d] \\ W \arrow[ur] & \mathrm{Def}\, \overline{W} \end{tikzcd}$$

(unobstructed)

blow down deformations

the image is the Artin component

Example

$$\begin{tikzcd} & \mathrm{Def}\, W^{min} = \mathbb{A}^3_{t_1,t_2,t_3} \arrow[d] \\ W \arrow[ur] & \mathrm{Def}\, \frac{1}{4}(1,1) \end{tikzcd}$$

$$\hat{\mathcal{O}}(W) = \mathbb{C}[[t_i, z_j]] \Big/ \text{2×2 minors of } \begin{bmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 - t_1 & z_2 - t_2 & z_3 - t_3 & z_4 \end{bmatrix}$$

Non-commutative quasideterminants deform the Kalck Karmazyn algebra:

$$R = \mathbb{C}[[t_i]]\langle z_1, z_2, z_3\rangle \Big/ z_i z_j = t_j z_i$$

All isomorphic! (unless $t_1 = t_2 = t_3$)

$t_1 = 1,\ t_2 = t_3 = 0$

$z_i z_1 = z_i \quad , z_i z_j = 0 \quad j = 2, 3$

$$\hat{R} = \text{Path Algebra}\left(\begin{tikzcd} 1-z_1 \bullet \arrow[r, bend left, "z_2"] \arrow[r, bend right, "z_3"'] & z_1 \end{tikzcd} \right)$$

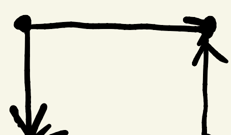
$\frac{1}{8}(1,5)$ example

$$\begin{bmatrix} 0 & z_1 & z_2^{(2)} \\ z_1^{(2)} & z_2 & 0 \end{bmatrix}$$

$$R = \mathbb{C}[[s_1^1, s_1^2, s_2^1, s_2^2]]\langle z_1, z_2\rangle \Big/ \langle z_1 z_1^{(2)}, z_2^{(2)} z_1^{(2)}, z_1 z_2, z_2^{(2)} z_2\rangle$$

4-parameter deformation of $\bar{R}$

General deformations all isomorphic

$\hat{R} = \text{Path Algebra}\Big($ affine Dynkin quiver  $\Big)$