GEOMETRY OF CHOW QUOTIENTS OF GRASSMANNIANS

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Abstract

We consider Kapranov's Chow quotient compactification of the moduli space of ordered n-tuples of hyperplanes in P^{r-1} in linear general position. For $r = 2$, this is canonically identified with the Grothendieck-Knudsen compactification of $M_{0,n}$ which has, among others, the following nice properties:

(1) *modular meaning: stable pointed rational curves;*

(2) *canonical description of limits of one-parameter degenerations;*

(3) *natural Mori theoretic meaning: log-canonical compactification.*

We generalize (1) and (2) to all (r, n)*, but we show that (3), which we view as the deepest, fails except possibly in the cases* $(2, n)$*,* $(3, 6)$*,* $(3, 7)$*,* $(3, 8)$*, where we conjecture that it holds.*

Contents

1. Introduction and statement of results

Let $\mathbb{P}(r, n)$ be the space of ordered n-tuples of hyperplanes in $\mathbb{P}^{r-1}$. Let $\mathbb{P}^\circ(r, n)$ be the open subset that is in linear general position. PGL_r acts freely on $\mathbb{P}^\circ(r, n)$ with the quotient $X(r, n)$. The space $X(2, n)$ is usually denoted $M_{0,n}$ and has the Grothendieck-Knudsen compactification $\overline{M}_{0,n}$ with many remarkable properties.

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1.1. Properties (of $M_{0,n} \subset \overline{M}_{0,n}$)

(1) $\overline{M}_{0,n}$ has a natural moduli interpretation; namely, it is the moduli space of stable n-pointed rational curves.

(2) $M_{0,n} \subset \overline{M}_{0,n}$ is canonically associated to the open variety $M_{0,n}$; namely, it is the log canonical compactification (see [KM, Lemma 3.6]). In particular, the pair $(\overline{M}_{0,n}, \partial\overline{M}_{0,n})$ has log canonical singularities.

(3) Given power series $f_1(z), \ldots, f_n(z)$ in one variable which we think of as a one-parameter family in $M_{0,n}$, one can ask: What is the limiting stable n-pointed rational curve in $\overline{M}_{0,n}$ as $z \to 0$? There is a beautiful answer, due to Kapranov [K2], in terms of the Bruhat-Tits tree for PGL_2.

Question 1.2

Is there a compactification $X(r, n) \subset \overline{X}(r, n)$ which satisfies any or all of the properties of Section 1.1?

In this article we consider Kapranov's Chow quotient compactification $\overline{X}(r, n)$, introduced in [K1]. This carries a flat family of pairs of schemes with boundary

$$p : (\mathscr{S}, \mathscr{B}) \to \overline{X}(r, n), \tag{1.2.1}$$

the so-called family of visible contours, generalising the universal family over $\overline{M}_{0,n}$. Lafforgue in [L2] gave a precise description of the fibres (S, B), showing in particular that each pair has toroidal singularities.

1.3. Moduli interpretation as in Section 1.1(1)

$\overline{X}(r, n)$ is a natural moduli space of semi log canonical pairs (the higher-dimensional Mori-theoretic generalisation of stable pointed curves). This is a recent result of Hacking [H]. We observed the same result independently. Together with Hacking, we give a proof in [HKT].

1.4. Log canonical model

The initial motivation for this article was the elementary observation (see Proposition 2.18) that $X(r, n)$ is minimal of log general type and thus (assuming general conjectures of Mori theory) has a log canonical compactification. Unfortunately, this is not, in general, the Chow quotient:

Let $\tilde{X}(r, n) \to \overline{X}(r, n)$ be the normalisation.

THEOREM 1.5

$\tilde{X}(3, n)$ with its boundary fails to be log canonical for $n \geq 9$ (for $n \geq 7$ in characteristic 2). $\tilde{X}(4, n)$ is not log canonical for $n \geq 8$.

We give a proof in §3. In the positive direction we speculate that the two compactifications agree in the remaining cases.

CONJECTURE 1.6
$\tilde{X}(r, n)$ is the log canonical model of $X(r, n)$ in the cases

$$(2, n), \qquad (3, 6), \qquad (3, 7), \qquad (3, 8)$$

and those obtained by the canonical duality $\overline{X}(r, n) = \overline{X}(n - r, n)$ (see [K1, Theorem 2.3]). Moreover, in these cases the pair $(\overline{X}(r, n), B)$ has toroidal singularities.

For $(2, n)$, the conjecture is true by Section 1.1(2). Together with Hacking, we have proved it for $(3, 6)$ using Theorem 2.19, and we expect the remaining two cases to follow in the same way.

Question 1.7
What is the log canonical compactification $X(r, n) \subset \overline{X}_{\mathrm{lc}}(r, n)$?

We believe that this compactification is of compelling interest, as it gives a birational model with reasonable boundary singularities of a compactification of $X(r, n)$ whose boundary components meet in absolutely arbitrary ways.

1.8. Mnëv's universality theorem
The boundary

$$\mathbb{P}(r, n) \backslash \mathbb{P}^{\circ}(r, n)$$

is a union of $\binom{n}{r}$ Weil divisors. The components have only mild singularities; however, they meet in very complicated ways. Let Y be an affine scheme of finite type over $\operatorname{Spec} \mathbb{Z}$. By [L2, Theorem 1.8], there are integers n, m and an open subset

$$U \subset Y \times \mathbb{A}^m$$

such that the projection $U \to Y$ is surjective, and U is isomorphic to a boundary stratum of $\mathbb{P}(3, n)$. (A boundary stratum for a divisorial boundary means the locally closed subset of points that lie in each of a prescribed subset of the irreducible components, but no others.)

We prove in Theorem 3.13 that singularities of the pair $(\overline{X}(3, n), B)$ are also, in general, arbitrary. Now by Proposition 2.18 and the finite generation conjecture of Mori theory, $X(3, n) \subset \overline{X}_{\mathrm{lc}}(3, n)$ gives a canonical compactification in which the boundary has log canonical singularities. We do not know whether $\overline{X}_{\mathrm{lc}}(3, n)$ maps to $\overline{X}(r, n)$; if it does, $\overline{X}_{\mathrm{lc}}$ would give an absolutely canonical way of (partially) resolving a boundary whose strata include all possible singularities.

1.9. Realization via Bruhat-Tits buildings as in Section 1.1(3)
Throughout the article k is a fixed algebraically closed field. Let $R = k[[z]]$, and let K be its quotient field. $V = k^r$ and $V_T = V \otimes_k T$ for a k-algebra T. We write $\mathscr{P}$ for projective spaces of quotients (or, equivalently, hyperplanes), $\mathbb{P}$ for spaces of lines.

Begin with a collection

$$\mathscr{F} := \{f_1, \ldots, f_n\} \subset V_K, \tag{1.9.1}$$

any r of which are linearly independent, and thus give a K-point of $X(r, n)$. We think of $\mathscr{F}$ as the equations of n one-parameter families in $\mathbb{P}^{r-1}$. Following [K2] (where the case $r = 2$ is treated), we ask the following.

Question 1.10
What is the limit as $z \to 0$; that is, in the pullback of the family (1.2.1) along the associated R-point of $\overline{X}(r, n)$, what is the special fibre?

We give a canonical solution, in terms of the Bruhat-Tits building $\mathscr{B}$. Proofs and further related results are given in §§4 – 7. Recall that $\mathscr{B}$ is a set of equivalence classes of R-lattices in V_K. We assume that the reader is familiar with basic notions associated with $\mathscr{B}$, all of which are reviewed in §4.

Definition 1.11
For a finite $Y \subset \mathscr{B}$, let $\mathbb{S}_Y$ be the *join* of the projective bundles $\mathscr{P}(M)$, $[M] \in Y$; that is, fix one, and take the closure of the graph of the product of the birational maps from this projective bundle to all the others. Let $S_Y \subset \mathbb{S}_Y$ be the special fibre of $p : \mathbb{S}_Y \to \operatorname{Spec}(R)$.

By [M, Theorem 2.2], if Y is convex, then p is semistable (i.e., $\mathbb{S}_Y$ is nonsingular) and S_Y has smooth irreducible components and normal crossings. Following [F], we refer to $\mathbb{S}_Y$ as the Deligne scheme. It represents a natural functor (see Theorem 5.2).

Definition 1.12
We define the *membrane* $[\mathscr{F}] \subset \mathscr{B}$ to be classes of lattices M which have a basis given by scalar multiples of some r elements from $\mathscr{F}$.

For a lattice M and a nontrivial element $f \in V_K$, there is a unique $a > 0$ such that $z^a f \in M \backslash zM$. We define $f^M := z^a f \subset M$ and define $f^{\overline{M}}$ its image under $M \to \overline{M} := M/zM$. For a subset $\Theta \subset V_K$, let $\Theta^{\overline{M}} := \{f^{\overline{M}} \mid f \in \Theta\}$.

We call $[M]$ *stable* if $\mathscr{F}^{\overline{M}}$ contains $r + 1$ elements in linear general position. Let $\mathrm{Stab} \subset [\mathscr{F}]$ be the set of stable classes. Stab is finite (see Lemma 5.19). Let $Y \subset \mathscr{B}$ be any finite convex subset containing Stab (e.g., the convex hull [Stab], which is finite;

see Lemma 4.8). Let $\mathbb{B}_i \subset \mathbb{S}_Y$ be the closure of the hyperplane

$$\{f_i = 0\} \subset \mathscr{P}(V_K) \subset \mathbb{S}_Y$$

on the generic fibre of $p : \mathbb{S}_Y \to \mathrm{Spec}(R)$. Let $\mathbb{B} = \sum \mathbb{B}_i$. The pair $(\mathbb{S}_Y, \mathbb{B})$ represents natural functors (see Definition-Lemma 5.21).

THEOREM 1.13
$\mathbb{S}_Y$, $\mathbb{B}_i$ are nonsingular, and the divisor $S_Y + \mathbb{B}$ has normal crossings. The 1-forms $\mathrm{dlog}(f_i/f_j)$ define globally generating sections of the vector bundle $\Omega^1_{\mathbb{S}/R}(\log \mathbb{B})$. The image of the associated map to the Grassmannian

$$\mathbb{S}_Y \to \mathrm{Spec}(R) \times G(r-1, n-1)$$

is $\mathscr{S} \to \mathrm{Spec}(R)$, the pullback of the family (1.2.1) along the R-point of $\overline{X}(r,n)$ defined by $\mathscr{F}$. In particular, the relative log canonical bundle $K_{\mathbb{S}_Y} + \mathbb{B}$ is relatively globally generated and big, and $\mathbb{S}_Y \to \mathscr{S}$ is the relative minimal model and is crepant.

For the definition and basic properties of bundles of relative log differentials, see §9. We note that the crepant semistable model $(\mathbb{S}_Y, \mathbb{B})$ is in many ways preferable to its minimal model $\mathscr{S} \to \mathrm{Spec}(R)$. For example, dropping the last hyperplane induces a natural regular birational map

$$\mathbb{S}_{[\mathrm{Stab}(\mathscr{F})]} \to \mathbb{S}_{[\mathrm{Stab}(\mathscr{F}')]}$$

for $\mathscr{F}' = \mathscr{F} \backslash \{f_n\}$, but for $r \geq 3$ the associated rational map between minimal models is not, in general, regular. There are examples where regularity fails already with $(r,n) = (3,5)$.

The special fibre S_Y can be read off directly from $Y \subset [\mathscr{F}]$. There is one component $\tilde{\mathscr{P}}(\overline{M})$, a certain blowup of the projective space $\mathscr{P}(\overline{M})$, for each $[M] \in Y$, glued in a canonical way given by the simplicial complex structure of $\mathscr{B}$ (see Remark 5.14). The semistable model (S_Y, B) depends on the choice of the finite convex subset $Y \subset [\mathscr{F}]$, but there is a canonical infinite version, with no boundary, which dominates all finite S_Y (see §7).

1.14. Relation to tropical geometry
There are several connections between this work and tropical geometry. For example, $[\mathscr{F}]$ is naturally homeomorphic to the tropicalisation of the r-dimensional subspace in K^n defined by the rows of the matrix with columns f_i (see Theorem 4.11).

2. Various toric quotients of the Grassmannian
In this section we fix notation and provide a background on quotients of Grassmanians, due mostly to Kapranov and Lafforgue. The new results are Theorem 2.3, Lemma 2.4, Theorem 2.17, Proposition 2.18, and Theorem 2.19.

2.1. Chow quotients (see [KSZ], [GKZ])

Let H be an algebraic group acting on a projective variety $\mathbb{P}$ (which in our applications is a projective space, whence the choice of notation). Let $\mathbb{P}^0 \subset \mathbb{P}$ be a (sufficiently) generic open H-invariant subset. There is a natural map to the Chow variety,

$$\mathbb{P}^0/H \to \mathrm{Chow}(\mathbb{P}), \qquad x \mapsto \overline{Hx}.$$

The Chow quotient $\mathbb{P}//H$ is defined to be the closure of the image of this map.

We begin by describing this in the case where H is an algebraic torus acting on a projective space $\mathbb{P}$. Let $P \subset \mathbb{X}_{\mathbb{R}}$ be a convex polytope with vertices in $\mathbb{X}$, the character lattice of H. We denote vertices of P by the same letter. Let V be the k-vector space with a basis $\{z_p \mid p \in P\}$. The torus H acts on V by the formula $h \cdot z_p = p(h) z_p$. Let $\mathbb{P} := \mathbb{P}(V)$. For any $S \subset \mathbb{P}$, let $\mathrm{Supp}(S) \subset P$ be the set of coordinates that do not vanish on S. Let $\mathbb{P}^0 = \{p \in \mathbb{P} \mid \mathrm{Supp}(p) = P\}$.

Take the big torus $\mathscr{H} = \mathbb{G}_m^P$ with its obvious action on V. We can assume, without loss of generality, that $H \subset \mathscr{H}$. This is equivalent to $\langle P \rangle_{\mathbb{Z}} = \mathbb{X}$, where for any $S \subset \mathbb{X}$ we denote by $\langle S \rangle_{\mathbb{Z}}$ the minimal sublattice containing S. $\mathbb{P}//H$ is a projective $\mathscr{H}$-toric variety with open orbit $\mathbb{P}^0/H \subset \mathbb{P}//H$ and canonical $\mathscr{H}$-equivariant Chow polarisation. By a toric variety, we mean a variety with an action of a torus having a dense open orbit. We do not assume that the action is effective or that the variety is normal. The fan of the normalisation $\Psi : \mathbb{P}//_n H \to \mathbb{P}//H$ can be described as follows. A triangulation T of P (with all vertices in the set of vertices of P) is called coherent if there exists a concave piecewise affine function on P whose domains of affinity are precisely maximal simplices of T. It gives rise to a polyhedral cone $C(T) \subset \mathbb{R}^P$ of the maximal dimension which consists of all functions $\psi : P \to \mathbb{R}$ such that $\psi_T : P \to \mathbb{R}$ is concave, where ψ_T is given by affinely interpolating ψ inside each simplex of T. Cones $C(T)$ (and their faces) for various T give a complete fan $\mathscr{F}(P)$. Lower-dimensional faces of $\mathscr{F}(P)$ correspond to (coherent) polyhedral decompositions $\underline{P}$ of P. More precisely, $C(\underline{P})$ is the set of concave functions affine on each polytope of $\underline{P}$.

On occasion, we abuse notation and refer to the collection of maximal-dimensional polytopes of a polyhedral decomposition as a polyhedral decomposition itself. We have the orbit decomposition

$$\mathbb{P}//H = \bigsqcup_{\underline{P}} (\mathbb{P}//H)_{\underline{P}}$$

(and a similar one for $\mathbb{P}//_n H$) indexed by polyhedral decompositions. A cycle $X \in (\mathbb{P}//H)_{\underline{P}}$ is the union of toric orbits with multiplicities

$$X = \sum_{P' \in \underline{P}} m_{P'} X_{P'}, \qquad \mathrm{Supp}(X_{P'}) = P', \qquad m_{P'} = [\mathbb{X} : \langle P' \rangle_{\mathbb{Z}}]. \tag{2.1.1}$$

If $m_{P'} = 1$ for any $P' \in \underline{P}$, then we say that X is a *broken toric variety*. If

$$\langle P'' \rangle_{\mathbb{Z}} = \mathbb{X} \cap \langle P'' \rangle_{\mathbb{Q}}$$

for any face P'' of a polytope $P' \in \underline{P}$, then we call $\underline{P}$ unimodular.

Definition 2.2
A decomposition $\underline{P}$ is called *central* if $\underline{P} = \{C, S_1, \ldots, S_r\}$, where $S_i \cap S_j \subset \partial P$. We call C the central polytope. Let $U_{\underline{P}} \subset \mathbb{P}//_n H$ be an affine open toric subset with fan $C(\underline{P})$. It contains $(\mathbb{P}//_n H)_{\underline{P}}$ as the only closed orbit.

THEOREM 2.3
If $\underline{P}$ is unimodular and central, then $\Psi(U_{\underline{P}})$ is quasi-smooth (i.e., $\Psi|_{U_{\underline{P}}}$ is bijective, and $U_{\underline{P}}$ is smooth).

Proof
To show that $U_{\underline{P}}$ is smooth, we have to show that $C(\underline{P}) = (\mathbb{Z}_{\geq 0})^r$ up to global affine functions. Let $f \in C(\underline{P})$. Then f is a concave locally affine function. Let f' be an affine function equal to f on C. So $f - f'$ is a concave locally affine function that vanishes on C. Let f_i, $1 \leq i \leq r$, be a primitive (i.e., not divisible by an integer) concave locally affine function that vanishes on $P \backslash S_i$. Then $f - f'$ is a linear combination of f_i's with nonnegative coefficients.

To show that $\Psi|_{U_{\underline{P}}}$ is bijective, it suffices to prove that $\Psi|_{(\mathbb{P}//_n H)_{\underline{P}}}$ is bijective. Indeed, other strata in $U_{\underline{P}}$ correspond to decompositions coarser than $\underline{P}$ which are automatically unimodular and central, so we can use the same argument.

For a moment, let $\underline{P}$ be any unimodular decomposition. The following construction is a variation of the Ishida complex (see [O]). Let $\underline{P}^i$ be the set of i-codimensional faces of polytopes in $\underline{P}$ which do not belong to the boundary ∂P. Fix some orientation of each $Q \in \underline{P}^i$. Let A be an abelian group. Consider the complex $C^{\bullet}_{\mathrm{Aff}}(\underline{P}, A)$ with $C^i_{\mathrm{Aff}}(\underline{P}, A) = \bigoplus_{Q \in \underline{P}^i} \mathrm{Aff}(Q, A)$, where $\mathrm{Aff}(Q, A)$ is the group of affine maps $Q \to A$. The differential $d^i : C^i_{\mathrm{Aff}}(\underline{P}, A) \to C^{i+1}_{\mathrm{Aff}}(\underline{P}, A)$ is a direct sum of differentials $d^{Q,R}$ for $Q \in \underline{P}^i$, $R \in \underline{P}^{i+1}$. If R is not a face of Q, then $d^{Q,R} = 0$. Otherwise, $d^{Q,R}$ is the restriction map $\mathrm{Aff}(Q, A) \to \mathrm{Aff}(R, A)$ taken with a negative sign if the fixed orientation of R is opposite to the orientation induced from Q. Let $H^{\bullet}_{\mathrm{Aff}}(\underline{P}, A)$ be the cohomology of $C^{\bullet}_{\mathrm{Aff}}(\underline{P}, A)$. It is clear that $H^0_{\mathrm{Aff}}(\underline{P}, A)$ is the set of piecewise affine functions $P \to A$.

LEMMA 2.4
If $H^1_{\mathrm{Aff}}(\underline{P}, \mathbb{Z}) = 0$, then $\Psi|_{(\mathbb{P}//_n H)_{\underline{P}}}$ is bijective.

Proof
We identify $\mathscr{H}$ with maps $P \to \mathbb{G}_m$. Elements of $\mathscr{H}$ of order N are maps $P \to \mu_N$, and any map $a: P \to \mathbb{Z}$ gives a one-parameter subgroup (1-PS) $z \mapsto \{p \mapsto z^{a(p)}\}$.

Let $X \in (\mathbb{P}//H)_{\underline{P}}$ be as in (2.1.1). Let $x \in \Psi^{-1}(X)$. We claim that $\mathscr{H}x \to \mathscr{H}X$ is bijective. Since $\mathscr{H}_x \subset \mathscr{H}_X$, it suffices to prove that the stabilizer $\mathscr{H}_X$ is connected.

Let $h \in \mathscr{H}_X$. Then $h \in \mathscr{H}_{X_{P'}}$ for any $P' \in \underline{P}$. But if e is a generic point of $X_{P'}$, then

$$\mathscr{H}_{X_{P'}} = \{h \in \mathscr{H} \mid h \cdot e \in X_{P'}\} = \{h \in \mathscr{H} \mid \exists h_{P'} \in H,\ h \cdot e = h_{P'} \cdot e\}.$$

It follows that $h(p) = h_{P'}(p)$, and hence h is affine on each P'. We see that $\mathscr{H}_X = H^0_{\mathrm{Aff}}(\underline{P}, \mathbb{G}_m)$.

It is enough to show that any element $h \in \mathscr{H}_X$ of a finite order N embeds in a 1-PS $\gamma \subset \mathscr{H}_X$. So let $h \in H^0_{\mathrm{Aff}}(\underline{P}, \mu_N)$. We have the exact sequence

$$0 \to C^\bullet_{\mathrm{Aff}}(\underline{P}, \mathbb{Z}) \xrightarrow{\cdot N} C^\bullet_{\mathrm{Aff}}(\underline{P}, \mathbb{Z}) \to C^\bullet_{\mathrm{Aff}}(\underline{P}, \mu_N) \to 0.$$

Since $H^1_{\mathrm{Aff}}(\underline{P}, \mathbb{Z}) = 0$, there exists an element of $H^0_{\mathrm{Aff}}(\underline{P}, \mathbb{Z})$ which maps to h. The corresponding 1-PS contains h and belongs to the stabilizer $\mathscr{H}_X$. □

Back in our situation, it is clear that $\underline{P}^1 = \{F_1, \dots, F_p\}$ is the set of codimension-one faces of C which are not on the boundary of P. We want to use Lemma 2.4. Let $c \in C^1_{\mathrm{Aff}}(\underline{P}, \mathbb{Z})$, so $c = (f_1, \dots, f_p)$, where f_i is affine on F_i. For each i, we have $F_i = C \cap S_j$ for some j. We can choose $g_j \in \mathrm{Aff}(S_j, \mathbb{Z})$, which restricts on f_i (taking into account the orientation). Then c is equal to the differential of the cochain $\tilde{c}$, where $\tilde{c}(C) = 0$ and $\tilde{c}(S_i) = g_i$. □

2.5. Matroid polytopes (see [GGMS], [L2])
Let $H = \mathbb{G}_m^n$ be the torus acting on $\mathbb{P} = \mathbb{P}\big(\bigwedge^r k^n\big)$. The weights $e_{i_1} + \dots + e_{i_r} \in \mathbb{R}^n$ are the vertices of the hypersymplex

$$\Delta(r, n) := \Big\{(x_1, \dots, x_n) \in \mathbb{R}^n \;\Big|\; \sum x_i = r,\ 1 \geq x_i \geq 0\Big\}.$$

It has $2n$ faces $\{x_i = 0\}$ and $\{x_i = 1\}$. The Plücker embedding $G(r, n) \subset \mathbb{P}$ induces a closed embedding $G(r, n)//H \subset \mathbb{P}//H$. For a subset $I \subset N$, we write $x_I = \sum_{i \in I} x_i$ and consider x_I as a function on $\Delta(r, n)$; in particular, $x_I = r - x_{I^c}$.

A matroid C of rank r on the set $N = \{1, \dots, n\}$ gives rise to a *matroid polytope* $P \subset \Delta(r, n)$, a convex hull of vertices $e_{i_1} + \dots + e_{i_r}$ for any base $\{i_1, \dots, i_r\} \subset N$. It is known that P is defined by inequalities $x_I \leq \operatorname{rank} I$. In almost all our applications, $P = \Delta(r, n)$, and so we adopt the following notational convention throughout the article. If we drop the polytope P from notation, it is assumed to be $\Delta(r, n)$ for a pair

(r, n) clear from the context. Let

$$\mathbb{P}^P = \{x \in \mathbb{P} \mid \operatorname{Supp}(x) \subset P\}, \qquad \mathbb{P}^{P,0} = \{x \in \mathbb{P}^P \mid \operatorname{Supp}(x) = P\}.$$

In [L2, Theorem 2.1] Lafforgue defines a subfan of $\mathscr{F}(P)$ whose cones are in one-to-one correspondence with matroid decompositions $\underline{P}$ of P (i.e., tilings of P by matroid polytopes). This is a fan because a polyhedral decomposition coarser than a matroid decomposition is a matroid decomposition. (Moreover, if a convex polytope $Q \subset \Delta(r, n)$ admits a tiling by matroid polytopes, then Q itself is a matroid polytope; see [L1].) The associated toric variety is denoted $\mathscr{A}^P$. Just by definition, $\mathscr{A}^P$ is the toric open subset in the normalisation of the Chow quotient:

$$\mathscr{A}^P \subset \mathbb{P}^P//_n H. \tag{2.5.1}$$

Orbits in $\mathscr{A}^P$ correspond to matroid decompositions. Notice that the action of $\mathbb{G}_m^P$ on $\mathscr{A}^P$ is not effective: the kernel $(\mathbb{G}_m^P)_\emptyset \subset \mathbb{G}_m^P$ is the subtorus of affine maps $P \to \mathbb{G}_m$. Let $\mathscr{A}_\emptyset^P := \mathbb{G}_m^P/(\mathbb{G}_m^P)_\emptyset$.

For any face Q of P, Lafforgue defines a toric *face map* $\mathscr{A}^P \to \mathscr{A}^Q$. The corresponding map of fans is given by the restriction of piecewise affine functions from P to Q. In particular, the image of the orbit $\mathscr{A}_{\underline{P}}^P$ is $\mathscr{A}_{\underline{P'}}^Q$, where $\underline{P'}$ is the matroid decomposition of Q obtained by intersecting polytopes in $\underline{P}$ with Q.

Lafforgue introduces a second normal toric variety $\tilde{\mathscr{A}}^P$ for the torus $\tilde{\mathscr{A}}_\emptyset^P := \mathbb{G}_m^P/\mathbb{G}_m$ and a map of toric varieties

$$\tilde{\mathscr{A}}^P \to \mathscr{A}^P \tag{2.5.2}$$

extending the natural quotient map $\tilde{\mathscr{A}}_\emptyset^P \twoheadrightarrow \mathscr{A}_\emptyset^P$. The torus orbits of $\tilde{\mathscr{A}}^P$ are in one-to-one correspondence with $(\underline{P}, P')$ for $\underline{P}$, a matroid decomposition, and $P' \in \underline{P}$, one of the matroid polytopes. By [L2, Proposition IV.3], (2.5.2) is projective and flat, with geometrically reduced fibres, and there exists an equivariant closed embedding

$$\tilde{\mathscr{A}}^P \subset \mathscr{A}^P \times \mathbb{P}^P. \tag{2.5.3}$$

The fibre of (2.5.2) over a closed point of $\mathscr{A}_{\underline{P}}^P = (\mathbb{P}^P//_n H)_{\underline{P}}$ is a broken toric variety (2.1.1) in $\mathbb{P}^P$. All multiplicities are equal to 1 because of the following fundamental observation (see [GGMS]):

$$\textit{any matroid decomposition is unimodular.} \tag{2.5.4}$$

In fact, (2.5.2) is the pullback of the universal Chow family over the Chow quotient $\mathbb{P}^P//H$ along the map $\mathscr{A}^P \subset \mathbb{P}^P//_n H \to \mathbb{P}^P//H$.

For each maximal face P' of P, the pair $(\emptyset, P')$, where $\emptyset$ denotes the trivial decomposition (just P and its faces), corresponds to an irreducible boundary divisor of $\tilde{\mathscr{A}}^P$. Denote the union of these boundary divisors as $\tilde{\mathscr{B}}^P \subset \tilde{\mathscr{A}}^P$. In the case where $P = \Delta(r, n)$, there are $2n$ such boundary divisors, corresponding to the maximal faces $\{x_i = 0\}, \{x_i = 1\}$ of $\Delta(r, n)$. We indicate by $\tilde{\mathscr{B}}_i$ the divisor corresponding to $\{x_i = 1\}$. Boundary divisors of $\tilde{\mathscr{A}}^P$ induce boundary divisors B on fibres of (2.5.2) for each maximal face of P. For $P = \Delta(r, n)$, we write B_i for the divisor corresponding to $\tilde{\mathscr{B}}_i$.

2.6. Realizable matroids

Let $C = \{L_i\}_{i \in N}$ be a configuration of n hyperplanes in $\mathbb{P}^{r-1}$. Then independent subsets of the corresponding *realizable matroid* (denoted by the same letter C) are subsets of linearly independent hyperplanes. C has rank r if there is at least one independent r-tuple. By the *multiplicity* of a point $p \in \mathbb{P}^{r-1}$ with respect to C we mean the number of hyperplanes in C which contain p (i.e., the usual geometric $\operatorname{mult}_p C$ if we view C as a divisor in $\mathbb{P}^{r-1}$). Recall that a hyperplane arrangement is called *connected* if its group of automorphisms is trivial. It is well known that C is a connected arrangement if and only if P_C has a maximal dimension.

Let $X_C(r, n)$ be the corresponding moduli space (i.e., N-tuples of hyperplanes with incidence as specified by C modulo PGL_r). We recall the Gelfand-MacPherson correspondence. Consider an $(r \times n)$-matrix M_C with columns given by equations of hyperplanes of C (defined up to a scalar multiple). The row space of M_C gives a point $x \in G(r, n) \subset \mathbb{P}\big(\bigwedge^r k^n\big)$. The convex hull of $\operatorname{Supp}(x) \subset \Delta(r, n)$ is P_C. The locally closed subscheme

$$G^{P,0}(r, n) = G(r, n) \cap \mathbb{P}^{P,0}$$

is called a thin Schubert cell. Of course, $G^{\Delta(r,n),0}(r, n) = G^0(r, n)$. Thus $X_C(r, n)$ is identified with the quotient of (the reduction of) $G^{P_C,0}(r, n)/H$.

Next, we consider Lafforgue's main object, $\overline{\Omega}^P$, which we consider only in the case $P = \Delta(r, n)$. We use a different construction from his, as it is a quicker way of describing the scheme structure: $\overline{\Omega}$ is the subscheme of $\mathscr{A}$ over which the fibres of (2.5.2) are contained in $G(r, n)$.

PROPOSITION 2.7

The Lafforgue space $\overline{\Omega} \subset \mathscr{A}$ is $\varphi^{-1}\big(\operatorname{Hilb}(G(r, n))\big)$, where $\varphi : \mathscr{A} \to \operatorname{Hilb}\big(\mathbb{P}\big(\bigwedge^r (k^n)\big)\big)$ is the map induced by (2.5.3).

Proof

As Lafforgue pointed out to us, this follows from [L2, Theorems 4.4, 4.22]. □

2.8. Structure map

We have the composition

$$\overline{\Omega} \subset \mathscr{A} \to \mathscr{A}/\mathscr{A}_\emptyset$$

(where the last map is the stack quotient), which Lafforgue calls the *structure map*. In particular, this endows $\overline{\Omega}$ with a stratification by locally closed subschemes, $\overline{\Omega}_{\underline{P}}$ (the restriction of the corresponding toric stratum of $\mathscr{A}$), parameterized by matroid decompositions $\underline{P}$ of $\Delta(r, n)$. The Lafforgue stratum $\overline{\Omega}_{\underline{P}}$ is empty if the matroid decomposition $\underline{P}$ contains nonrealizable matroids. The stratum for the trivial decomposition, $\emptyset$ (meaning the only polytope is $\Delta(r, n)$), is an open subset

$$\overline{\Omega}_{\emptyset} = X(r, n) \subset \overline{\Omega},$$

which Lafforgue calls the *main* stratum. Lafforgue proves that $\overline{\Omega}$ is projective and thus gives a "compactification" of $X(r, n)$—quotes are used because his space is, in general, reducible, as we observe in Proposition 3.10.

2.9

We denote the closure of $\overline{\Omega}_{\emptyset}$ in $\overline{\Omega}$ by $\overline{X}_L(r, n)$. There are immersions

$$X(r, n) \subset \overline{X}_L(r, n) \subset \mathscr{A} \subset \mathbb{P}//_n H$$

(the first and last open, the middle one closed) and

$$X(r, n) \subset \overline{X}(r, n) \subset \mathbb{P}//H$$

(open followed by closed). It follows that there exists a finite birational map

$$\overline{X}_L(r, n) \to \overline{X}(r, n). \tag{2.9.1}$$

In particular, $\overline{X}(r, n)$ and $\overline{X}_L(r, n)$ have the same normalisation $\tilde{X}(r, n)$.

2.10. Toric family

We denote the pullback of $\tilde{\mathscr{A}} \to \mathscr{A}$ to $\overline{\Omega}$ by $\mathscr{T} \to \overline{\Omega}$ ($\mathscr{T}$ denotes toric). By definition, $\mathscr{T} \subset \overline{\Omega} \times G(r, n)$.

Kapranov [K1, Theorem 1.5.2] shows that the natural Chow family

$$\mathscr{T} \to \overline{X}(r, n), \quad \mathscr{T} \subset \overline{X}(r, n) \times G(r, n),$$

is flat. The family $\mathscr{T} \to \overline{X}_L(r, n)$ is the pullback of $\mathscr{T} \to \overline{X}(r, n)$ along (2.9.1).

Let $\mathscr{B}, \mathscr{B}_i \subset \mathscr{T}$ be the restrictions of the boundary divisors $\tilde{\mathscr{B}}, \tilde{\mathscr{B}}_i \subset \tilde{\mathscr{A}}$.

2.11. Family of visible contours

Let $G_e(r-1, n-1) \subset G(r, n)$ be the subspace of r-planes containing the fixed vector $e = (1, \ldots, 1)$. Kapranov defines the *family of visible contours*

$$\mathscr{S} = \mathscr{T} \cap \big(\overline{X}(r, n) \times G_e(r-1, n-1)\big) \subset \overline{X}(r, n) \times G(r, n).$$

Kapranov shows that the family $\mathscr{S}$ is flat and that the associated map

$$\overline{X}(r,n) \to \mathrm{Hilb}\big(G_e(r-1,n-1)\big) \tag{2.11.1}$$

is a closed embedding. There is a similar family over $\overline{\Omega}$. (Lafforgue calls it $\tilde{\mathbb{P}}(\mathscr{E})$.)

Definition 2.12
Let $\mathscr{S} \subset \mathscr{T}$ be the scheme-theoretic intersection

$$\mathscr{S} := \mathscr{T} \cap [\overline{\Omega} \times G_e(r-1,n-1)] \subset [\overline{\Omega} \times G(r,n)].$$

H acts on $\tilde{\mathscr{A}}$, trivially on $\mathscr{A}$, and $\tilde{\mathscr{A}} \to \mathscr{A}$ is H-equivariant. Thus H acts on $\mathscr{T}$ (and trivially on $\overline{\Omega}$), so that $\mathscr{T} \to \overline{\Omega}$ is equivariant.

Let $\mathscr{B}, \mathscr{B}_i \subset \mathscr{S}$ indicate the restriction of $\mathscr{B}, \mathscr{B}_i \subset \mathscr{T}$. We note that $\mathscr{B} \subset \mathscr{S}$ is the union of $\mathscr{B}_i$, as the n components of $\mathscr{B} \subset \mathscr{T}$ corresponding to the faces $x_i = 0$ of $\Delta(r,n)$ are easily seen to be disjoint from $G_e(r-1,n-1)$.

The fibres of $(\mathscr{S}, \mathscr{B}) \to \overline{\Omega}$ have singularities like (or better than) those of $(\mathscr{T}, \mathscr{B})$, as follows from the following transversality result.

PROPOSITION 2.13 (see [L2, page xv])
The natural map $\mathscr{S} \to \mathscr{T}/H$ to the quotient stack (or, equivalently, $\mathscr{S} \times H \to \mathscr{T}$) is smooth.

Proof
For the reader's convenience, we recall Lafforgue's elegant construction. Let

$$\mathscr{E} \subset G(r,n) \times \mathbb{A}^n$$

be the universal rank r subbundle, and let $\overset{\circ}{\mathscr{E}} \subset \mathscr{E}$ be the inverse image under the second projection of the open subset $H \subset \mathbb{A}^n$ (i.e., the subset with all coordinates nonzero). H obviously acts freely on $\overset{\circ}{\mathscr{E}}$, and the quotient is canonically identified with $G_e(r-1,n-1)$. This gives a smooth map

$$G_e(r-1,n-1) = \overset{\circ}{\mathscr{E}}/H \to G(r,n)/H.$$

Now for any H-equivariant $\mathscr{T} \to G(r,n)$, the construction pulls back. □

Note, in particular, that this shows the following.

COROLLARY 2.14
$\mathscr{S} \subset \mathscr{T}$ is regularly embedded, with normal bundle the pullback of the universal quotient bundle of $G_e(r-1,n-1)$.

2.15

A precise description of the fibres of $\mathscr{S}$ is given in [L2, Chapter 5]. Let $S \subset T$ be a closed fibre of $\mathscr{S} \subset \mathscr{T}$ over a point of $\overline{\Omega}_{\underline{P}}$. We have by Proposition 2.13 a smooth *structure map* $S \to T/H$, and so S inherits a stratification from the orbit stratification of T/H, parameterized by $P \in \underline{P}$. In particular, the facets (maximal-dimensional polytopes) of $\underline{P}$ correspond to irreducible components, and the stratum S_P (which are the points of S which lie only on the irreducible component corresponding to P) is the complement in $\mathbb{P}^{r-1}$ to a connected arrangement of n hyperplanes with associated matroid polytope P (see §2.6). The irreducible component itself is the log canonical compactification of S_P, as follows, for example, from Theorem 2.17. For $r = 3$, this compactification is smooth, and it is described by Lemma 8.8.

2.16. Log structures and toric stacks

For basic properties of log structures and toric stacks, we refer to [O11, §5]. Any log structure we use in this article is toric; that is, the space comes with an evident map to a toric variety, and we endow the space with the pullback of the toric log structure on the toric variety. In fact, we do not make any use of the log structure itself, only the bundles of log (and relative log) differentials, all of which are computed by the following basic operation. (Our notation is chosen with an eye to its immediate application.)

Let $q : \tilde{\mathscr{A}} \to \mathscr{A}$ be a map of toric varieties so that the map of underlying tori is a surjective homomorphism with kernel H. We have the smooth map

$$\tilde{\mathscr{A}} \to \tilde{\mathscr{A}}/H$$

(where the target is the stack quotient) and, in particular, its relative cotangent bundle, which is canonically identified with a trivial bundle with fibre the dual of the Lie algebra to H. We denote the bundle

$$\Omega^1_q(\log) = \Omega^1_{\tilde{\mathscr{A}}/(\tilde{\mathscr{A}}/H)} \tag{2.16.1}$$

as q is log smooth, and this is its bundle of relative log differentials, as follows from [O11, Proposition 5.14] and [O12, §3.7].

For a map $\overline{\Omega} \to \mathscr{A}$, consider the pullback

$$\mathscr{T} := \tilde{\mathscr{A}} \times_{\mathscr{A}} \overline{\Omega} \to \overline{\Omega}.$$

Then (2.16.1) pulls back to the relative cotangent bundle for

$$\mathscr{T} \to \mathscr{T}/H.$$

$\mathscr{T} \to \overline{\Omega}$ is again log smooth, with this (trivial) bundle of relative log differentials.

Now suppose that $\mathscr{S} \subset \mathscr{T}$ is a closed subscheme, so that the map $\mathscr{S} \times H \to \mathscr{T}$ (or, equivalently, $\mathscr{S} \to \mathscr{T}/H$) is smooth. Then the relative cotangent bundle for

$$\mathscr{S} \to \mathscr{T}/H$$

is a quotient of the pullback of $\Omega^1_{\mathscr{T}/(\mathscr{T}/H)}$ ($p : \mathscr{S} \to \overline{\Omega}$ is log smooth) with bundle of relative log differentials

$$\Omega^1_p(\log) = \Omega^1_{\mathscr{S}/(\mathscr{T}/H)}.$$

THEOREM 2.17

The visible contour family $p : \mathscr{S} \to \overline{\Omega}$ is log smooth. Its bundle of log differentials

$$\Omega^1_p(\log) = \Omega^1_{\mathscr{S}/(\mathscr{T}/H)}$$

is a quotient of the pullback of $\Omega^1_{\tilde{\mathscr{A}}/(\tilde{\mathscr{A}}/H)}$, which is the trivial bundle $\tilde{\mathscr{A}} \times V_n$. Fibres (S, B) are semi log canonical, and the restriction of the Plücker polarisation to $S \subset G_e(r-1, n-1)$ is $\mathscr{O}(K_S + B)$.

Proof

Let $(S, B) \subset (T, B)$ be closed fibres of $(\mathscr{S}, \mathscr{B}) \subset (\mathscr{T}, \mathscr{B})$. (T, B) is semi log canonical, and $\mathscr{O}(K_T + B)$ is canonically trivial (e.g., by [A, Lemma 3.1]). (S, B) is now semi log canonical by (2.13), and by adjunction $\mathscr{O}(K_S + B)$ is the determinant of its normal bundle, which is the Plücker polarisation by (2.14). The other claims are immediate from Proposition 2.13 and the general discussion in §2.16. □

The initial motivation for this article was the following elementary observation.

PROPOSITION 2.18

$X(r, n)$ is minimal of log general type. Its first log canonical map is a regular immersion.

Proof

Fixing the first $r + 1$ hyperplanes identifies $X(r, n)$ with an open subset of $U^{n-(r+1)}$, where $U \subset \mathbb{P}^{r-1}$ is the complement to B, the union of $r + 1$ fixed hyperplanes in linear general position. Since $K_{\mathbb{P}^{r-1}} + B = \mathscr{O}(1)$, the first log canonical map on U is just the open immersion $U \subset \mathbb{P}^{r-1}$. □

We have the following criterion to guarantee that $X(r, n) \subset \overline{X}_L(r, n)$ is a log minimal model. Let $T_p(\log)$ be the dual bundle to $\Omega^1_p(\log)$ on $\mathscr{S}$ (i.e., the relative tangent bundle to $\mathscr{S} \to \mathscr{T}/H$). The sheaf $\Omega^1_{\overline{X}_L(r,n)/k}(\log B)$ is defined in Definition 9.1.

THEOREM 2.19
If $R^2 p_*(T_p(\log))$ vanishes at a point of $\overline{X}_L(r,n) \subset \overline{\Omega}$, then $\overline{\Omega} \to \mathscr{A}/\mathscr{A}_\emptyset$ is smooth, $\overline{X}_L(r,n) = \overline{\Omega}$, $\overline{\Omega}$ is normal, and the pair $(\overline{X}_L(r,n), B)$ has toroidal singularities, near the point.

If $R^2 p_*(T_p(\log))$ vanishes identically along $\overline{X}_L(r,n)$, then $\Omega^1_{\overline{X}_L(r,n)/k}(\log B)$ is locally free, globally generated, and its determinant $\mathcal{O}(K_{\overline{X}_L} + B)$ is globally generated and big. In particular, $X(r,n) \subset \overline{X}_L(r,n)$ is a log minimal model.

Proof
By [L2, Theorems 4.25(ii), 5.15], vanishing of R^2 implies that the structure map is smooth. Now suppose that the structure map is smooth along $\overline{X}_L(r,n)$. The bundle of log differentials for the toric log structure on a normal toric variety is precisely the bundle in Definition 9.1, which implies the analogous statement for $(\overline{X}_L(r,n), B)$. The bundle of differentials is the cotangent bundle of the structure sheaf, and thus a quotient of the cotangent bundle to $\tilde{\mathscr{A}} \to \tilde{\mathscr{A}}/\tilde{\mathscr{A}}_\emptyset$ which by §2.16 is canonically trivial, whence the global generation. Now $K_{\overline{X}_L(r,n)} + B$ is big by Proposition 2.18. □

3. Singularities of $(\overline{X}(r,n), B)$

In this section we prove Theorem 1.5. The very simple idea is as follows. The notion of log canonical pair $(\overline{X}, \sum B_i)$ generalises normal crossing. In particular, if all the irreducible components B_i are $\mathbb{Q}$-Cartier, then log canonical implies at least that the intersection of the B_i has the expected codimension (see Proposition 3.18). We prove Theorem 1.5 by observing that well-known configurations give points of $\overline{X}(r,n)$ lying on too many boundary divisors. The main work is to show that these points are actually in the closure of the generic stratum and that the boundary divisors are Cartier near these points.

3.1. Divisor B_I

Let $I \subset N$. It is easy to see that $\{x_I \le k\} \subset \Delta(r,n)$ is a realizable matroid polytope for any $0 < k < r$. The corresponding hyperplane arrangement $\{L_1, \dots, L_n\}$ is as follows: the only condition we impose is

$$\operatorname{codim} \bigcap_{i \in I} L_i = k.$$

This polytope has full dimension if and only if $|I| > k$.

It follows that if $|I| > k$ and $|I^c| > r-k$, then there is a matroid decomposition of $\Delta(r,n)$ with two polytopes $\{x_I \ge k\}$ and $\{x_I \le k\}$. The corresponding stratum of $\mathscr{A}$ is obviously maximal among boundary strata. We denote its closure (and corresponding subschemes of $\overline{\Omega}$, $\overline{X}_L(r,n)$, etc.) by B_I.

For example, let $r = 3$. In the configuration with polytope $\{x_{I^c} \leq 1\}$, lines of I^c are identified, and lines of I are generic. In the configuration $\{x_I \leq 2\}$, the lines of I have a common point of incidence, and lines of I^c are generic.

3.2. Central configurations and matroids

PROPOSITION 3.3

Let $\mathscr{I}$ be an index set, and for each $\alpha \in \mathscr{I}$, let $I_\alpha \subset N$ be a subset such that $|I_\alpha| \geq r$ and

$$|I_\alpha \cap I_\beta| \leq r - 2 \quad \textit{for } \alpha \neq \beta. \tag{3.2.1}$$

Let us call $S \subset N$ independent if $|S| < r$ or $|S| = r$ and $S \not\subset I_\alpha$ for any $\alpha \in \mathscr{I}$. This gives a structure of a matroid on N.

Proof

We only have to check that for any $S \subset N$, all maximal independent subsets in S have the same number of elements. It suffices to prove that if S contains an independent set T, $|T| = r$, then any independent subset $R \subset S$ can be embedded in an independent subset with r elements. We can assume that $|R| = r - 1$. If $R \not\subset I_\alpha$ for any $\alpha \in \mathscr{I}$, then we can just add any element to R. If $R \subset I_\alpha$ for some α, then this α is unique by (3.2.1), and we add to R an element of T which is not contained in I_α. □

Definition 3.4

We call matroids of this form *central*. A polytope P_C of a central matroid C is given by inequalities $x_{I_\alpha} \leq r - 1$ for all $\alpha \in \mathscr{I}$. Let $P_\alpha \subset \Delta(r, n)$ be the matroid polytope $x_{I_\alpha} \geq r - 1$. Let $\underline{\mathscr{I}} = \{P_C\} \cup \{P_\alpha\}_{\alpha \in \mathscr{I}}$.

LEMMA 3.5

$\underline{\mathscr{I}}$ is a central decomposition of $\Delta(r, n)$ (see Definition 2.2 for the definition) with central polytope P_C. For each subset $\mathscr{I}' \subset \mathscr{I}$, $\underline{\mathscr{I}'}$ is a matroid decomposition, coarser than $\underline{\mathscr{I}}$, and all matroid decompositions coarser than $\underline{\mathscr{I}}$ occur in this way.

Proof

To show that $\underline{\mathscr{I}}$ is a central decomposition, it suffices to check that $P_\alpha \cap P_\beta$ is on the boundary for any $\alpha \neq \beta$. (This implies, in particular, that any interior point of any wall $\{x_{I_\alpha} = r - 1\}$ belongs to exactly two polytopes, P_C and P_α.) Assume that $x \in P_\alpha \cap P_\beta$ is an interior point of $\Delta(r, n)$. Then $x_{I_\alpha \cap I_\beta} < r - 2$ by (3.2.1). (Otherwise, $x_i = 1$ for any $i \in I_\alpha \cap I_\beta$, and therefore x is on the boundary.) Therefore $x_{I_\alpha \setminus I_\beta} = x_{I_\alpha} - x_{I_\alpha \cap I_\beta} > 1$ and $x_{I_\alpha \cap I_\beta} = x_{I_\alpha \setminus I_\beta} + x_{I_\beta} > r$, a contradiction.

Any matroid decomposition coarser than $\underline{\mathcal{I}}$ is obviously central and can be obtained by combining P_C with several P_α's. This has the same effect as taking these α's out of $\mathcal{I}$. □

PROPOSITION 3.6
Let $U_{\underline{\mathcal{I}}}$ be the affine open toric subset of $\mathcal{A}$, as in Definition 2.2. Then $U_{\underline{\mathcal{I}}}$ is smooth and bijective to $\Psi(U_{\underline{\mathcal{I}}}) \subset \mathbb{P}(\Lambda^r k^n)//H$. Let $U = U_{\underline{\mathcal{I}}} \cap \overline{\Omega}$. $U \subset \overline{X}_L(r,n)$ maps finitely and homeomorphically onto its image in $\overline{X}(r,n)$.

Proof
The proof follows from Theorem 2.3. □

Definition 3.7
We say that a hyperplane arrangement C is *central* if a pair $(\mathbb{P}^{r-1}, C)$ has normal crossings on the complement to a zero-dimensional set. If $r = 3$, it simply means that there are no double lines. Let $\mathcal{I} \subset \mathbb{P}^{r-1}$ be the set of points of multiplicity at least r. Then a matroid of C is a central matroid that corresponds to subsets $I_\alpha \subset N$ of hyperplanes containing $\alpha \in \mathcal{I}$.

Definition 3.8
Fix a hyperplane $L \subset \mathbb{P}^{r-1}$. For each subset $J \subset N$, $|J| \geq r$, let Q_J be the moduli space of J-tuples of hyperplanes, L_j, $j \in J$, in $\mathbb{P}^{r-1}$ such that the entire collection of hyperplanes, together with L, is in linear general position modulo automorphism of $\mathbb{P}^{r-1}$ preserving L.

Note that Q_J is a smooth variety of dimension $(r-1)(|J|-r)$. Intersecting with the fixed hyperplane L gives a natural smooth surjection

$$Q_J \to X(r-1, |J|).$$

LEMMA 3.9
Let C, $\mathcal{I}$, $\underline{\mathcal{I}}$ be as in Definition 3.7. For each $\alpha \in \mathcal{I}$, we have a natural map $X_C(r,n) \to X(r-1, |I_\alpha|)$ taking the hyperplanes through a. Let

$$M = \prod_{\alpha \in \mathcal{I}} X(r-1, |I_\alpha|) \qquad \text{and} \qquad Q = \prod_{\alpha \in \mathcal{I}} Q_{I_\alpha}.$$

There is a natural identification

$$\overline{\Omega}_{\underline{\mathcal{I}}} = X_C(r,n) \times_M Q.$$

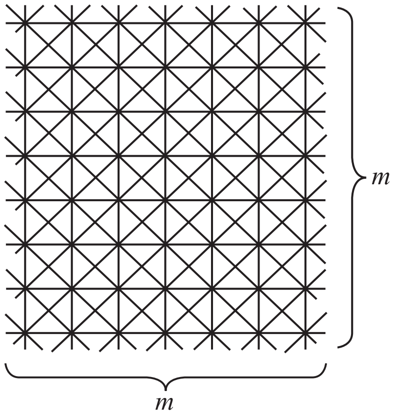

Figure 1

In particular,

$$\dim(\overline{\Omega}_{\underline{\mathscr{I}}}) = \dim\bigl(X_C(r,n)\bigr) + \sum_{\alpha \in \mathscr{I}} (|I_a| - r).$$

Proof

The proof is immediate from [L2, §3.6]. $\square$

Next, we demonstrate that Lafforgue's space $\overline{\Omega}$ is reducible.

PROPOSITION 3.10

Let C be the following configuration of $6m-2$ lines in $\mathbb{R}^2$ (see Figure 1).

Let $\mathscr{I}$ be its multiple points, as in Definition 3.7. Then $\dim \overline{\Omega}_{\underline{\mathscr{I}}} \geq m^2$. The Lafforgue space $\overline{\Omega}^{\Delta(3,n)}$ is not irreducible for large n.

Proof

The configuration C has at least m^2 points of multiplicity 4, so the inequality is immediate from Lemma 3.9. The final remark follows as the main component $\overline{X}_L(3, 6m-2)$ of $\overline{\Omega}^{\Delta(3,6m-2)}$ has dimension $12m-12$. $\square$

However, for a large class of central configurations, the stratum $\overline{\Omega}_{\underline{\mathscr{I}}}$ belongs to the closure of the main stratum.

3.11. Lax configurations

We say that a central hyperplane arrangement C is *lax* if there is a total ordering on N such that for each $i \in N$, points on L_i of multiplicity *strictly greater* than r with respect to $N_{\leq i}$ are linearly independent. For example, the configuration in

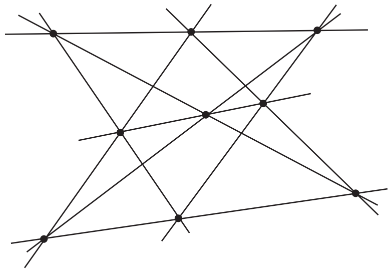

Figure 2

Proposition 3.10 is not lax for $m \geq 4$. On the other hand, any configuration of lines in $\mathbb{P}^2$ without points of multiplicity 4 or more is lax.

THEOREM 3.12
The notation is as in Definition 3.7. Assume that C is lax.

(1) *The stratum $\overline{\Omega}_{\underline{I}}$ is contained (set theoretically) in $\overline{X}_L(r,n) \subset \overline{\Omega}$.*

(2) *Let $U = U_{\underline{\mathscr{I}}} \cap \overline{\Omega}$, where $U_{\underline{\mathscr{I}}} \subset \mathscr{A}$ is the smooth toric affine open set of Proposition 3.6. Let $\tilde{U} \to U$ be the normalisation. Then U is an irreducible open factorial subset of $\overline{X}_L(r,n) \subset \overline{\Omega}$, smooth in codimension one. Moreover, the boundary strata B_{I_α} are Cartier, generically smooth, and irreducible on U, their union is the boundary, and their scheme-theoretic intersection is the stratum $\overline{\Omega}_{\underline{\mathscr{I}}}$.*

(3) *Let $\tilde{U} \to U$ be the normalisation, and let $\tilde{B} \subset \tilde{U}$ be the reduction of the inverse image of B. If $K_{\tilde{U}} + \tilde{B}$ is log canonical at a point in the inverse image of $p \in \overline{\Omega}_{\underline{\mathscr{I}}}$, then the stratum has pure codimension $|\mathscr{I}|$ in U near p; that is,*

$$\sum_{\alpha \in \mathscr{I}} (|I_\alpha| - r + 1) + \dim X_C(r,n) = n(r-1) - r^2 + 1.$$

We postpone the proof until the end of this section.

Proof of Theorem 1.5
Consider the Brianchon-Pascal configuration (see [HC], [D]) of nine lines with $|\mathscr{I}| = 9$ and $|I_\alpha| = 3$ for all α (see Figure 2).

It is easy to compute that $\dim X_C(3,9) = 2$. Now apply Theorem 3.12: the left-hand side in Theorem 3.12(3) is equal to 11, but the right-hand side is 10. If $n \geq 10$, add generic lines.

There is an even better configuration of nine lines with $|\mathscr{I}| = 12$ and $|I_\alpha| = 3$ for all α. Namely, fix a smooth plane cubic. Every line containing two distinct inflection points contains exactly three. This gives a configuration of 12 lines. Furthermore, each inflection point lies on exactly three lines, and these are all the intersection points of the configuration. This is the famous Hesse Wendepunkt-configuration (see [HC],

[D]). Let C be the dual configuration. Now apply Theorem 3.12: the left-hand side in Theorem 3.12(3) is equal to 12, but the right-hand side is 10.

For the characteristic 2, use the Fano configuration (see [GGMS, Theorem 4.5]), and argue as above: the left-hand side in Theorem 3.12(3) is equal to 7, but the right-hand side is 6.

In the (4, 8) case, take the configuration of eight planes in $\mathbb{P}^3$ given by the faces of the octahedron. There are 12 points of multiplicity 4 (i.e., lying on four of the planes), while $\tilde{X}(4,8)$ is 9-dimensional. If $n \geq 9$, add generic planes. □

THEOREM 3.13
The boundary strata of $(\overline{X}(3,n), B)$ for lax configurations have arbitrary singularities; that is, their reductions give reductions of all possible affine varieties defined over $\mathbb{Z}$ (up to products with $\mathbb{A}^1$).

Proof
By Theorem 3.12, it suffices to prove that $\overline{\Omega}_{\underline{\mathscr{I}}}^{\Delta(3,n)}$ for lax configurations C, $\underline{\mathscr{I}}$ satisfies Mnëv's theorem (see [L2, Theorem 1.14]); that is, given affine variety Y over $\mathbb{Z}$, there are integers n, m, an open set $U \subset Y \times \mathbb{A}^m$ with $U \to Y$ surjective, and a lax configuration C with n lines such that U is isomorphic to the reduction of the Lafforgue stratum $\overline{\Omega}_{\underline{\mathscr{I}}}^{\Delta(3,n)}$. One can follow directly the proof of Mnëv's theorem: Lafforgue constructs an explicit configuration that encodes the defining equations for Y, and it is easy to check that this configuration is lax. The ordering of lines (in Lafforgue's notation) should be as follows: lines $[0, 1_\alpha, P_\alpha, \infty_\alpha]$ and the infinite line should go first (at the end of the process there will be many points of multiplicity greater than 3 along them); then take all auxiliary lines in the order of their appearance in Lafforgue's construction. □

Now we proceed with the proof of Theorem 3.12.

3.14. Face maps and cross-ratios
The collection of $X(r,n)$ has a hypersimplicial structure: there are obvious maps $B_i : X(r,n) \to X(r,n-1)$ (dropping the ith hyperplane) and $A_i : X(r,n) \to X(r-1, n-1)$ (intersecting with the ith hyperplane). These maps extend to maps of Chow quotients (see [K1, Theorem 1.6]) and to maps of Lafforgue's varieties $\overline{X}_L \subset \overline{\Omega} \subset \mathscr{A}$ (see [L2, Theorem 2.4]). For $\mathscr{A}$, these maps are just restrictions of face maps in §2.5 corresponding to faces $\{x_i = 0\} \simeq \Delta(r, n-1)$ and $\{x_i = 1\} \simeq \Delta(r-1, n-1)$.

In particular, let $V, W \subset N$ be subsets such that $|V| = 4$, $|W| = r-2$, $V \cap W \neq \emptyset$. Then dropping all hyperplanes not in $V \cup W$ and intersecting with all hyperplanes in W give cross-ratio maps

$$CR_{V,W} : X(r,n) \to X(2,4) = M_{0,4} = \mathbb{P}^1 \backslash \{0, 1, \infty\} \qquad (3.14.1)$$

and

$$CR_{V,W} : \overline{\Omega} \to \overline{X}(2,4) = \overline{M}_{0,4} = \mathbb{P}^1.$$

It follows that $CR_{V,W}(\overline{\Omega}_{\underline{P}}) \subset \mathbb{P}^1 \backslash \{0,1,\infty\}$ if and only if $\underline{P}$ does not break

$$\Delta_{V,W}(2,4) = \bigcap_{i \notin V \cup W} \{x_i = 0\} \cap \bigcap_{i \in W} \{x_i = 1\}.$$

$\Delta(2,4)$ is an octahedron, and values $\{0,1,\infty\}$ correspond to three decompositions of $\Delta(2,4)$ into two pyramids.

To write (3.14.1) as a cross-ratio, let $V = \{i_1 i_2 i_3 i_4\}$ with $i_1 < i_2 < i_3 < i_4$. Let $L_1, \dots, L_n$ be a collection of hyperplanes in $X(r,n)$. Consider an $(r \times n)$-matrix M with columns given by equations of these hyperplanes. Then

$$CR_{V,W}(L_1, \dots, L_n) = \frac{\mathrm{Det}_{i_1 i_2 W}\, \mathrm{Det}_{i_3 i_4 W}}{\mathrm{Det}_{i_1 i_3 W}\, \mathrm{Det}_{i_2 i_4 W}},$$

where each Det_T is an $(r \times r)$-minor of M with columns given by T.

Let $C = \{L_1, \dots, L_n\}$ be any configuration as in §3, and let $x_0 \in G(r,n)$ be a point that corresponds to C under the Gelfand-MacPherson transform. Let $(\mathscr{X}, x_0) \subset G(r,n)$ be a pointed curve such that $\mathscr{X} \cap G^0(r,n) \neq \emptyset$. Let $F : G(r,n)^0 \to X(r,n)$ be the canonical H-torsor. Then

$$p_0 = \lim_{x \to x_0} F(x) \in X_L(r,n)$$

belongs to $\overline{\Omega}_{\underline{P}}$, where $\underline{P}$ is a matroid decomposition of $\Delta(r,n)$ containing P_C. Indeed, it is clear that x_0 is contained in the fibre of the universal family (2.5.2) over p_0, so $P_C = \mathrm{Supp}(x_0)$ is in $\underline{P}$.

PROPOSITION 3.15
Let C be central, as in Definition 3.7. If

$$\lim_{x \to x_0} CR_{V,W}(x) \notin \{0,1,\infty\}$$

for any $W \subset I_\alpha$, $|V \cap I_\alpha^c| = 1$, $\alpha \in \mathscr{I}$, then $\underline{P} = \underline{\mathscr{I}}$.

Proof
Any decomposition containing P_C is a refinement of $\underline{\mathscr{I}}$. So it remains to prove the following combinatorial statement: any realizable matroid decomposition $\underline{P}$ refining $\underline{\mathscr{I}}$ is equal to $\underline{\mathscr{I}}$, provided that $\underline{P} \cap \Delta(2,4) = \Delta(2,4)$ for any face $\Delta(2,4) \subset \Delta(r,n)$ which belongs to the boundary of some P_α and such that exactly one face of this octahedron $\Delta(2,4)$ belongs to the wall $x_{I_\alpha} = r - 1$. (This is a condition equivalent to $W \subset I_\alpha$, $|V \cap I_\alpha^c| = 1$.)

Restrictions of $\underline{P}$ and $\underline{\mathscr{I}}$ to the faces of $\Delta(r,n)$ have the same form. Also, if $r=2$, then the claim follows, for example, from the explicit description of matroid decompositions of $\Delta(2,n)$ (see [K1, Theorem 1.3]), so we can argue by induction, and it remains to prove the following: any realizable matroid decomposition $\underline{P}$ refining $\underline{\mathscr{I}}$ is equal to $\underline{\mathscr{I}}$, provided that $\underline{P}|_F=\underline{\mathscr{I}}|_F$ for any face $F=\{x_i=1\}$ of $\Delta(r,n)$, $r>2$.

Assume, on the contrary, that a certain $P_\alpha\in\underline{\mathscr{I}}$ is broken into pieces. Choose a polytope $Q\subset\underline{P}\cap P_\alpha$ such that the boundary of Q contains the face $F=\{x_l=1\}\cap P_\alpha$, $l\notin I_\alpha$. A polytope Q is realizable. In the corresponding configuration D, the hyperplane L_l is multiple (of multiplicity $|I_\alpha^c|$), and intersections of hyperplanes L_j, $j\in I_\alpha$, with L_l are in general position (because $F\subset Q$). It follows that D is central (except that L_l is multiple). If $Q\neq P_\alpha$, then there is at least one degeneracy; hyperplanes L_j, $j\in J\subset I_\alpha$, $|J|=r$, pass through a point $\beta\notin L_l$. Since not all hyperplanes L_i, $i\in I_\alpha$ pass through β, there exist indices $k,k'\in J$ and $i\in I_\alpha$ such that a line $\bigcap_{i\in J\setminus\{k,k'\}}L_i$ intersects L_k and $L_{k'}$ at β, L_l at L_i at two other distinct points. It follows that $\Delta_{\{k,k',l,i\},J\setminus\{k,k'\}}(2,4)$ is broken by $\underline{P}$. $\square$

Proof of Theorem 3.12(1)

Let M_C be as in §2.6 for a fixed lax hyperplane arrangement C. Let $Z\subset\overline{\Omega}_{\underline{\mathscr{I}}}$ be the fibre over the point of $X_C(r,n)$ given by C, in the product decomposition Lemma 3.9.

We consider lines $x_M:\mathbb{A}^1\to\mathscr{M}(r,n)$, $x_M(z)=M_C+zM$ for $M\in\mathscr{M}(r,n)$, and the induced regular map (which we abusively denote by the same symbol) $x_M:\mathbb{A}^1\to\overline{X}_L(r,n)$. We consider the limit of x_M as $z\to 0$.

We assume that N has the lax order of §3.11, so for any l, points on L_l of multiplicity greater than r with respect to $L_1,\ldots,L_{l-1}$ are linearly independent. Let p_l be the number of these points. A moment's thought, and Lemma 3.9 yields the equality

$$\sum_{\alpha\in\mathscr{I}}(|I_a|-r)=\sum_i p_i=\dim(Z).$$

We now construct the columns of M. Suppose that the first $l-1$ columns of M are already constructed, and consider column l. Let $e_i\in L_l$, $i=1,\ldots,p_l$, be points of multiplicity greater than r. We include these e_i's in the basis $e_1,\ldots,e_r$ and write the lth column in the dual basis. Let $V_i,W_i\subset N$, $i=1,\ldots,p_n$, be any choice of subsets as in §3.14 such that $V_i=\{i_1,i_2,i_3,l\}$, $|V_i\cap I_{e_i}^c|=1$, $W_i\subset I_{e_i}$.

CLAIM 3.16

For $i=1,\ldots,p_l$,

$$\lim_{z\to 0}CR_{V_i,W_i}\bigl(x_M(z)\bigr)$$

does not depend on M_{jl} for $j \neq i$ and depends nontrivially on M_{il} (i.e., we can make this limit any general value by varying M_{il}).

We can assume, without loss of generality, that $i_1 \notin I_{e_i}$. (Otherwise, take an appropriate automorphism of a cross-ratio function.) Note that the claim implies the result. First, it is clear that any single choice of subsets W, V as in Proposition 3.15 can be chosen as W_i, V_i for some i and l. So all these cross-ratios are generic (i.e., take on values other than $\{0, 1, \infty\}$) for general M. Now by Proposition 3.15, the limit point is in Z. Now by the claim, we can vary $\dim(Z)$ of the cross-ratios completely independently by varying M. Since Z is smooth and connected by Lemma 3.9, it thus follows that $Z \subset \overline{X}_L(r, n)$ (set theoretically), and so since C was arbitrary, this completes the proof.

Let $W := W_i$. Then

$$\lim_{z \to 0} CR_{V_i, W_i}\big(x_M(z)\big) = \lim_{z \to 0} \frac{\mathrm{Det}_{i_1 i_2 W}\, \mathrm{Det}_{i_3 l W}}{\mathrm{Det}_{i_1 i_3 W}\, \mathrm{Det}_{i_2 l W}}.$$

Notice that $\lim_{z \to 0} \mathrm{Det}_{i_1 i_2 W}$ and $\mathrm{Det}_{i_1 i_3 W}$ are not zero; by assumption, L_{i_1} does not pass through e_i, but projections of any $r - 1$ hyperplanes in I_{e_i} from e_i are linearly independent.

So we have to demonstrate that

$$\lim_{z \to 0} \frac{\mathrm{Det}_{i_3 l W}}{\mathrm{Det}_{i_2 l W}}$$

does not depend on M_{jl} for $j \neq i$ and depends not trivially on M_{il}. Indeed, the constant terms of $\mathrm{Det}_{i_3 l W}$ and $\mathrm{Det}_{i_2 l W}$ vanish; let us find coefficients at z. The ith rows of the corresponding submatrices of M_C are trivial, so we can expand both determinants along this row and get

$$\lim_{z \to 0} \frac{\mathrm{Det}_{i_3 l W}}{\mathrm{Det}_{i_2 l W}} = \frac{M_{i i_3} R_{i i_3} + M_{il} R_{il} + \cdots}{M_{i i_2} R_{i i_2} + M_{il} R_{il} + \cdots},$$

where R_{ij} are cofactors of the corresponding submatrices of M_C. These cofactors are not trivial because projections of any $r - 1$ hyperplanes in I_{e_i} from e_i are linearly independent. So we see that the limit indeed does not depend on M_{jl} for $j \neq i$ and is a Möbius function in M_{il}. This function can be made nontrivial by adding an open condition $M_{i i_3} R_{i i_3} \neq M_{i i_2} R_{i i_2}$. □

PROPOSITION 3.17
Let C be a lax configuration with multiple points $\mathscr{I}$. If $|\mathscr{I}| \geq 2$, then $\operatorname{codim}_{\overline{X}_L(r,n)} \overline{\Omega}_{\underline{\mathscr{I}}} \geq 2$.

Proof
We proceed by induction on $\sum |I_\alpha|$ using Lemma 3.9 and the following observation: a configuration near a connected configuration is connected. We compare quantities $\dim X_C(r,n) + \sum_{\alpha\in\mathscr{I}}(|I_a| - r)$ for various configurations. Since all of them will be connected, we can replace $X_C(r,n)$ by its PGL_r-torsor $\mathbb{P}_C(r,n)$, the space of all configurations with prescribed multiplicities.

Assume first that there are some points of multiplicity greater than r. Take the last hyperplane L in the lax order which contains such a point. Move L a little bit to take it off this point, but keep all other points of multiplicity greater than r on L. (This is possible because they are linearly independent.) If there are no such points, keep some point of multiplicity r (if there are any of them on L). Then the dimension of the configuration space will increase by at least one, the sum $\sum |I_\alpha| - r$ will decrease by at most one, and $|\mathscr{I}|$ is still at least 2. At the end, there will be at least two points A, B of multiplicity r and no points of higher multiplicity. Now take a hyperplane through A and move it, keeping B if it belongs to this hyperplane. This will increase the dimension of the configuration space by at least one, but the result will still not be generic and thus having codimension at least one. □

Proof of Theorem 3.12(2)
From Lemma 3.9, the generic stratum of B_{I_a} is smooth and connected and codimension one in U. By Proposition 3.17, all other boundary strata of U are lower dimensional. The boundary of U is (by definition) the scheme-theoretic inverse image of the boundary of $U_{\underline{\mathscr{I}}}$, and so it is Cartier and, in particular, pure codimension one, by Theorem 2.3. It follows that the B_{I_a} are irreducible and Cartier, and their union is the full boundary. They are generically smooth by Lemma 3.9. The proof of Proposition 2.18 shows that their complement, the main stratum $X(r,n)$, is isomorphic to an open subset of affine space and thus has a trivial divisor class group. Thus U is factorial. Now it is smooth generically along the Cartier divisors B_{I_a} by Lemma 3.9. In the open set $U_{\underline{\mathscr{I}}} \subset \mathscr{A}$, the stratum $\mathscr{A}_{\underline{\mathscr{I}}}$ is the scheme-theoretic intersection of the boundary divisors that contain it. (This is true in any toric variety.) Thus $\overline{\Omega}_{\underline{\mathscr{I}}}$ is scheme-theoretically the intersection of the boundary divisors of U that contain it. □

The proof of Theorem 3.12(3) now follows from Lemma 3.9 and Proposition 3.18.

PROPOSITION 3.18
Let X be a normal variety. Let B_i be irreducible Q-Cartier Weil divisors. If $K_X + \sum B_i$ is log canonical, then the intersection

$$B_1 \cap B_2 \cap \cdots \cap B_n$$

is (either empty or) pure codimension n.

Proof
We can intersect with a general hyperplane to reduce to the case when n is the dimension of X and then apply [Ko, Theorem 18.22]. □

4. The membrane

We begin with some background on buildings; for more details, see [Br]. We follow the notation of §1.9. We follow [S] for elementary definitions and properties of simplicial complexes.

Definition 4.1
The *total Grassmannian* $\mathrm{Gr}(V)$ is a simplicial complex of dimension $r-1$. Its vertices are nontrivial subspaces of V. (In particular, V itself is a vertex.) A collection of distinct subspaces forms a simplex if and only if they are pairwise incident (i.e., one is contained in the other), from which it easily follows that $m-1$ simplices correspond to flags of nontrivial subspaces

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_m. \tag{4.1.1}$$

The compactified *Bruhat-Tits building* $\overline{\mathscr{B}}$ is a set of equivalence classes of nontrivial free R-submodules of V_K, where M_1, M_2 are equivalent if and only if $M_1 = cM_2$ for some $c \in K^*$. The Bruhat-Tits building $\mathscr{B} \subset \overline{\mathscr{B}}$ is the subset of equivalence classes of lattices (i.e., free submodules of rank r). As a set, $\mathscr{B} = \mathrm{PGL}_r(K)/\mathrm{PGL}_r(R^*)$.

$\mathscr{B}$ is a simplicial complex of dimension $r-1$. $[M_1], \ldots, [M_m] \in \mathscr{B}$ span an $(m-1)$-simplex σ if and only if we can choose representatives so

$$zM_m = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m. \tag{4.1.2}$$

LEMMA 4.2 (see [Br])
For any lattice $[\Lambda] \in \mathscr{B}$, there exists a canonical isomorphism

$$\mathrm{Star}_\Lambda \mathscr{B} \simeq \mathrm{Gr}(\overline{\Lambda}),$$

where $\overline{\Lambda} = \Lambda/z\Lambda$. More generally, for any simplex σ (see (4.1.2)) in $\mathscr{B}$,

$$\mathrm{Star}_\sigma \mathscr{B} \simeq \mathrm{Gr}(M_m/M_{m-1}) * \cdots * \mathrm{Gr}(M_1/M_0),$$

where $*$ denotes the join of simplicial complexes.

Proof
Indeed, lattices between $z\Lambda$ and Λ are obviously in incidence-preserving bijection with subspaces of $\overline{\Lambda}$. Similarly, lattices that fit in the flag σ correspond bijectively to subspaces in one of M_i/M_{i-1}. It is worth mentioning that for any simplex σ of $\mathrm{Gr}(V)$ as in (4.1.1), $\mathrm{Star}_\sigma \mathrm{Gr}(V) = \mathrm{Gr}(U_m/U_{m-1}) * \cdots * \mathrm{Gr}(U_1/U_0)$. □

Definition 4.3
Let $[\Lambda] \in \overline{\mathscr{B}}$. We denote by

$$R_\Lambda : \overline{\mathscr{B}} \to \operatorname{Star}_\Lambda \mathscr{B}$$

the map that sends a submodule M to $M^\Lambda + z\Lambda$. More generally, for any simplex (4.1.2) in $\mathscr{B}$ there is a map $R_\sigma : \overline{\mathscr{B}} \to \operatorname{Star}_\sigma \mathscr{B}$ defined as follows: $R_\sigma(M) = M^{\Lambda_k} + \Lambda_i$, where i is maximal such that $M^{\Lambda_k} \not\subset \Lambda_i$. Finally, we denote by $\operatorname{Res}_\Lambda$ and $\operatorname{Res}_\sigma$ compositions of R_Λ and R_σ with isomorphisms of Lemma 4.2.

LEMMA 4.4
R_Λ and R_σ are retraction maps of simplicial complexes.

Proof
It is clear that the R_σ-map preserves incidence and is the identity on $\operatorname{Star}_\sigma \mathscr{B}$. □

4.5. Convex structure
A subset of $\overline{\mathscr{B}}$ is called *convex* if it is closed under finite R-module sums. For a collection of free R-submodules $\{M_\alpha\}$, their *convex hull* in $\overline{\mathscr{B}}$, denoted $[M_\alpha]$, is the subcomplex with vertices (with representatives) of form $\sum c_\alpha M_\alpha$, $c_\alpha \in K$. This is obviously the smallest convex subset that contains all the $[M_\alpha]$. Similarly, a subset of $\operatorname{Gr}(V)$ is called convex if it is closed under Minkowski sums. We write $[U_\alpha]$ for the convex hull of subspaces $\{U_\alpha\} \subset \operatorname{Gr}(V)$.

It is clear that stars of simplices in $\mathscr{B}$ and $\operatorname{Gr}(V)$ are convex and that isomorphisms (see Lemma 4.2) preserve the convex structure.

Example 4.6
For $T = \{g_1, \dots, g_r\}$ a basis of V_K, the convex hull $[T]$ is called an *apartment*. It is the set of equivalence classes $[M]$ such that M has an R-basis $c_1 g_1, \dots, c_r g_r$ for some $c_i \in K$.

Retractions commute with convex hulls.

PROPOSITION 4.7
Let $\{M_\alpha\} \subset \overline{\mathscr{B}}$ be a subset, and let $\sigma \subset \mathscr{B}$ be a simplex. Then

$$[R_\sigma M_\alpha] = R_\sigma [M_\alpha].$$

If $\sigma \subset [M_\alpha]$, then both sides are also equal to $\operatorname{Star}_\sigma [M_\alpha]$.

We leave the proof of this proposition as an exercise for the reader.

LEMMA 4.8 (see [F])
The convex hull of a finite subset of $\mathscr{B}$ is finite.

LEMMA 4.9
The membrane $[\mathscr{F}]$ is the convex hull of $\{Rf_1, \dots, Rf_n\}$ and the union of apartments $[T]$ for subsets $T \subset \mathscr{F}$, $|T| = r$.

Proof
The proof is immediate from the definitions and Nakayama's lemma. □

4.10. Membranes as tropical subspaces
We begin by recalling the construction of the tropical variety (also called a non-Archimedean amoeba or a Bieri-Groves set; see [SS] for details). Let $H = \mathbb{G}_m^n$ be an algebraic torus. Let $\overline{K}$ be the field of generalised Puiseux series $\sum_{\alpha\in I\subset\mathbb{R}} c_\alpha z^\alpha$, where I is a locally finite subset of $\mathbb{R}$, bounded below (and is allowed to vary with the series). There is an evaluation map

$$\mathrm{ord}: \ H(\overline{K}) \to H(\overline{K})/H(\overline{R}) = \mathbb{R}^n,$$

where $\overline{R} \subset \overline{K}$ is the subring of series for which $I \subset \mathbb{R}_{\geq 0}$. For any subvariety $Z \subset H$, $\mathrm{ord}(Z)$ is called the *tropicalisation* of Z. It is a polyhedral complex of dimension $\dim Z$. If Z is invariant under dilations, then $\mathrm{ord}(Z)$ is invariant under diagonal translations, and we consider

$$\mathrm{Ord}(Z) = \mathrm{ord}(Z) \qquad \mod \mathbb{R}(1, \dots, 1).$$

Let $\mathscr{F} := \{Rf_1, \dots, Rf_n\}$ be as in Lemma 4.9. Consider the map

$$\Phi: \ V_K^\vee \to K^n, \qquad F \mapsto \big(F(f_1), \dots, F(f_n)\big).$$

Let $Z = \Phi(V_K^\vee) \cap H$. Then Z is of course the intersection with H of the r-plane spanned by the rows of the $(r \times n)$-matrix with columns given by f_i's. Its tropicalisation $\mathrm{Ord}(Z) \subset \mathbb{R}^{n-1}$ is called *a tropical projective subspace*.

For any simplicial complex C, we denote by $|C|$ the corresponding topological space (obtained by gluing physical simplices). Recall that $|\mathscr{B}|$ can be identified with the space of equivalence classes of additive norms on V_K, where an additive norm N is a map $V_K(K) \to \mathbb{R} \cup \{\infty\}$ such that

$$N(cv) = \mathrm{ord}(c) + N(v) \quad \text{for any } c \in K,\ v \in V_K(K),$$
$$N(u+v) \geq \min\big(N(u), N(v)\big) \quad \text{for any } u, v \in V_K(K),$$

and

$$N(u) = \infty \quad \text{iff } u = 0.$$

Two additive norms are equivalent if they differ by a constant. For a norm N, let $\tilde{\Psi}(N) = (N(f_1), \dots, N(f_n)) \in \mathbb{R}^n$. Now consider

$$\Psi : |\mathscr{B}| \to \mathbb{R}^{n-1}, \qquad \Psi([N]) = \tilde{\Psi}(N) \mod \mathbb{R}(1, \dots, 1).$$

The map is continuous because the topology on $|\mathscr{B}|$ is exactly the topology of pointwise convergence of norms. The following theorem is our version of the tropical Gelfand-MacPherson transform.

THEOREM 4.11
Ψ induces a homeomorphism $|[\mathscr{F}]| \simeq \mathrm{Ord}(Z)$.

Proof
Let Ω be Drinfeld's symmetric domain, the complement to the union of all K-rational hyperplanes in $V_K^\vee(\overline{K})$. There is a surjection (see [Dr])

$$D : \Omega \to |\mathscr{B}|, \qquad F \mapsto [v \to \operatorname{ord} F(v)] \quad \text{for } v \in V_K(K);$$

here we interpret $|\mathscr{B}|$ as the set of equivalence classes of norms. The following diagram is obviously commutative:

$$\begin{tikzcd} \Omega \arrow[r, "D"] \arrow[d, "\Phi"] & {|\mathscr{B}|} \arrow[d, "\Psi"] \\ H(\overline{K}) \arrow[r, "\mathrm{Ord}"] & \mathbb{R}^{n-1} \end{tikzcd}$$

It follows that $\mathrm{Im}(\Psi) \subset \mathrm{Ord}(Z)$.

For any lattice $\Lambda \in \mathscr{B}$, the corresponding norm N_Λ is

$$N_\Lambda(v) = \{-a \mid z^a v \in \Lambda \backslash z\Lambda\} \in \mathbb{Z}.$$

In particular, $\Psi(\mathscr{B}) \subset \mathbb{Z}^{n-1}$. Also, it follows easily from definitions that Ψ is affine on simplices of $|\mathscr{B}|$ and uniformly continuous. Since $\mathrm{Ord}(Z)$ is a polyhedral complex, it remains to check that for any $\mathbb{Q}$-point of $\mathrm{Ord}(Z)$, there exists a unique $\mathbb{Q}$-point of $|[\mathscr{F}]|$ which maps onto it. (A $\mathbb{Q}$-point of $|\mathscr{B}|$ means a point of some simplex with rational barycentric coordinates.) Now we can pass from K to Puiseux series $k[[z^{1/m}]]$ with sufficiently large m (this does not change either $\mathrm{Ord}(Z)$ or $|[\mathscr{F}]|$; see also Proposition 6.4 for another version of this barycentric trick), and it remains to check the latter statement for $\mathbb{Z}$-points. Replacing f_i's by $z^{a_i} f_i$'s, we can assume that this point is $O = (0, \dots, 0)$. Now we claim that if $O \in \operatorname{ord}(Z)$, then $\tilde{\Psi}(\Lambda) = O$ for $[\Lambda] \in [\mathscr{F}]$ if and only if $\Lambda = Rf_1 + \cdots + Rf_n$.

Suppose that $\tilde{\Psi}(Rf_1 + \cdots + Rf_n) \neq O$; that is, suppose that $f_j \in z(Rf_1 + \cdots + Rf_n)$ for some j. By Nakayama's lemma, we can assume, without loss of generality, that $Rf_1 + \cdots + Rf_n = Rf_1 + \cdots + Rf_r$. Therefore $f_j \in z(Rf_1 + \cdots + Rf_r)$. But then for any $F \in V_K^\vee(\overline{K})$, if $\operatorname{ord} F(f_i) = 0$ for $i \leq r$, then $\operatorname{ord} F(f_j) > 0$. But this contradicts $O \in \operatorname{ord}(Z)$.

Now take any lattice $\Lambda = Rz^{a_1} f_1 + \cdots + Rz^{a_n} f_n$. We can assume, without loss of generality, that $\Lambda = Rz^{a_1} f_1 + \cdots + Rz^{a_r} f_r$. Let $\tilde{\Psi}(\Lambda) = O$. Then $f_i \in \Lambda$ for any i; therefore $\Lambda = Rf_1 + \cdots + Rf_n$. □

5. Deligne schemes

Now we turn to the proof of Theorem 1.13. We follow the notation of the introduction and §4. Here we prove that the pair $(\mathbb{S}_Y, S_Y + \mathbb{B})$ of Theorem 1.13 has normal crossings (see Theorem 5.23). Global generation is considered in §6.

5.1. Deligne functor (see [F])

Let $Y \subset \mathscr{B}$ be a finite set. The *Deligne functor* $\mathbb{S}_Y$ is the functor from R-schemes to sets, a T-valued point q which consists of a collection of equivalence classes of line bundle quotients

$$q_M : M_T \twoheadrightarrow L(M_T)$$

for each lattice $[M] \in Y$, where $M_T := T \times_R M$, where two quotients are equivalent if they have the same kernel, satisfying the following compatibility requirements.

- For each inclusion $i : N \hookrightarrow M$, there is a commutative diagram

$$\begin{array}{ccc} N_T & \xrightarrow{q_N} & L(N_T) \\ {\scriptstyle i_T}\big\downarrow & & \big\downarrow \\ M_T & \xrightarrow{q_M} & L(M_T) \end{array}$$

- Multiplication by $c \in K^*$ gives an isomorphism

$$\ker q_M \xrightarrow{\cdot c} \ker q_{cM}.$$

It is clear from this definition that $\mathbb{S}_Y$ is represented by a closed subscheme

$$\mathbb{S}_Y \subset \prod_{[M] \in Y} \mathscr{P}(M),$$

$(\mathbb{S}_Y)_K = \mathscr{P}(V_K)$, and $\mathbb{S}_Y$ contains the Mustafin join (see Definition 1.11).

THEOREM 5.2 (see [F])
Assume that Y is finite and convex. Then $\mathbb{S}_Y$ is smooth and irreducible. (In particular, it is isomorphic to the Mustafin join.) Its special fibre $S_Y = (\mathbb{S}_Y)_k$ has normal crossings.

We begin by explaining Faltings's proof of Theorem 5.2, recalling and expanding on the three paragraphs of [F, page 167]. This is the substance of §5.3 – Remark 5.18. For this, $Y \subset \mathscr{B}$ is an arbitrary finite convex subset. Beginning with Lemma 5.19, our treatment diverges from [F]. We specialize to convex subsets $Y \subset [\mathscr{F}]$ as in Theorem 1.13 and consider singularities of the natural boundary.

5.3. Maximal lattices (see [F])
Consider a k-point of $\mathbb{S}_Y$; that is, consider a compatible family of one-dimensional k-vector space quotients

$$q_M : \overline{M} \twoheadrightarrow L(\overline{M}), \quad [M] \in Y.$$

This gives a partial order on Y: $[N] \leq_q [M]$ if and only if the composition

$$\overline{N} = \overline{N^M} \to \overline{M} \twoheadrightarrow L(\overline{M})$$

induced by inclusion $N^M \subset M$ is surjective and thus, by compatibility, canonically identified with q_N. In this case, we also say that q_M does not vanish on N.

A lattice $[M] \in Y$ is called *maximal* for q if it is maximal with respect to the order $\leq_q$. In other words, q_N vanishes on M for any $[N] \in Y$, $[N] \neq [M]$.

LEMMA 5.4 (see [F])
Maximal lattices form a nonempty simplex σ as in (4.1.2). The corresponding k-point of $\mathbb{S}_Y$ is equivalent to a collection of hyperplanes

$$H_i \subset M_i/M_{i-1}, \quad m \geq i \geq 1,$$

which do not contain $\mathrm{Res}_\sigma[M]$ *for any* $[M] \in Y$.

Definition 5.5
Let $[M] \in Y$. We let $\tilde{\mathscr{P}}(\overline{M}) \subset \mathbb{S}_Y$ be the subfunctor of compatible quotients such that for each $[N] \in Y$, $N = N^M$, the quotient $q_N : N_T \twoheadrightarrow L(N_T)$ vanishes on $(N \cap zM)_T \subset N_T$.

It is clear that $\tilde{\mathscr{P}}(\overline{M})$ is represented by a closed subscheme of $\mathbb{S}_Y$.

LEMMA 5.6
The k-points of $\tilde{\mathscr{P}}(\overline{M}) \subset \mathbb{S}_Y$ are precisely the set of k-points of $\mathbb{S}_Y$ for which M is a maximal lattice.

Proof

Consider a k-point q of $\tilde{\mathscr{P}}(\overline{M})$. Suppose that $M \subsetneq N$, $[N] \in Y$. Then $N^M = z^k N$ for some $k > 0$. By the definition of $\tilde{\mathscr{P}}(\overline{M})$, $q_{z^k N}$ vanishes on $z^k N \cap zM$. So by compatibility of quotients with scaling, q_N vanishes on $N \cap z^{1-k} M$, which contains M. Thus M is a maximal lattice.

Conversely, suppose that M is maximal for a k-point q. Take $[N] \in Y$ such that $N = N^M$. By maximality, $q_{z^{-1}N+M}$ vanishes on M. Thus by compatibility, q_{N+zM} vanishes on zM; thus, again by compatibility, q_N vanishes on $N \cap zM$. So the point lies in $\tilde{\mathscr{P}}(\overline{M})$. □

Definition 5.7

Let V be a finite-dimensional k-vector space, and let $\mathscr{W} \subset \mathrm{Gr}(V)$ be a finite convex collection of subspaces that includes V. Let $\mathrm{BL}(\mathscr{P}(V), \mathscr{W})$ be the functor from k-schemes to sets which assigns to each T the collection of line bundle quotients $W_T \twoheadrightarrow L(W_T)$, $W \in \mathscr{W}$, W_T the pullback, compatible with the inclusion maps between the W; that is, the composition

$$A_T \to B_T \xrightarrow{q_B} L(B_T)$$

factors through $q_A : A_T \twoheadrightarrow L(A_T)$ for $A \subset B$, $A, B \in \mathscr{W}$.

PROPOSITION 5.8

There is a canonical identification

$$\tilde{\mathscr{P}}(\overline{M}) = \mathrm{BL}\left(\mathscr{P}(\overline{M}), \mathrm{Res}_M(Y)\right).$$

Proof

The proof is immediate from the definitions. □

PROPOSITION 5.9

$\mathrm{BL}(\mathscr{P}(V), \mathscr{W})$ *is represented by the closure of the graph of the product of canonical rational maps* $\mathscr{P}(V) \dashrightarrow \mathscr{P}(W)$*,* $W \in \mathscr{W}$*. Furthermore,* $\mathrm{BL}(\mathscr{P}(V), \mathscr{W})$ *is smooth.*

Proof

We induct on the number of subspaces in $\mathscr{W}$. When $\mathscr{W} = \{V\}$, the result is obvious. In any case, it is clear that the functor is represented by a certain closed subscheme

$$X \subset \prod_{W \in \mathscr{W}} \mathscr{P}(W).$$

Let $\mathscr{P}^0(V) \subset \mathscr{P}(V)$ be an open subset of quotients that do not vanish on any $W \subset \mathscr{W}$. Then $\mathscr{P}^0(V)$ is an open subset of X; its closure X' in X is the closure of the graph in the statement.

Take a fixed closed point q^0. We show that $X = X'$ near q^0. Let $W \in \mathscr{W}$ be a maximal subspace such that q_V^0 vanishes on W. If there are none, then $q^0 \in \mathscr{P}^0(V)$, and so $X = X'$ near q^0.

Let $D \subset X$ be the subscheme of compatible quotients so that q_V vanishes on W, and let $D^0 \subset D$ be a sufficiently small neighbourhood of q^0. Let $\mathscr{W}_W \subset \mathscr{W}$ be those subspaces contained in W; clearly, $\mathscr{W}_W$ is convex. Take any $E \in \mathscr{W}$ and $q \in D^0$. If $E \notin \mathscr{W}_W$, then q_V^0 (and hence q_V) does not vanish on E, from which it follows that $E \to L(V)$ is surjective and thus identified with q_E. It follows easily that D^0 is represented by an open subset of $\mathrm{BL}(\mathscr{P}(W), \mathscr{W}_W) \times \mathscr{P}(V/W)$. In particular, by induction, D^0 is connected and smooth of dimension $\dim V - 2$.

We claim that $D^0 \subset X'$. As D^0 is integral, it is enough to check this on some open subset of D^0. We consider the open subset where q_W does not vanish on any $E \in \mathscr{W}_W$, and q_V does not vanish on any $E \notin \mathscr{W}_W$. This is naturally identified with an open subset for $\mathscr{W} = \{V, W\}$, and so we reduce to this case. But in this case it is easy to see that $X = X'$ is the blowup of $\mathscr{P}(V)$ along $\mathscr{P}(V/W)$, and so obviously, $D^0 \subset X'$.

By the claim, X has dimension at least $\dim V - 1$ along D^0. $D \subset X$ is locally principal, defined by the vanishing of a map between the universal quotient line bundles for W and V. In order to prove that X is smooth and equal to X' along D^0, it suffices to use the following well-known fact. Let (A, m) be a local Noetherian ring of Krull dimension at least d. Assume that A/f is regular of dimension at most $d - 1$ for $f \in m$. Then A is regular of dimension d. □

Definition 5.10
Define the *depth* of $W \in \mathscr{W}$ to be the largest $d \geq 0$ so that there is a proper flag

$$W = W_0 \subset W_1 \subset \cdots \subset W_d = V$$

with $W_i \in \mathscr{W}$. Let $\mathscr{W}_{\leq m} \subset \mathscr{W}$ be the subset of subspaces of depth at most m. Let $\mathscr{W}_m \subset \mathscr{W}_{\leq m}$ be the subset of subspaces of depth exactly m.

Notice that $\mathrm{BL}(\mathscr{P}(V), \mathscr{W}_0) = \mathscr{P}(V)$ and $\mathrm{BL}(\mathscr{P}(V), \mathscr{W}_{\leq N}) = \mathrm{BL}(\mathscr{P}(V), \mathscr{W})$ for $N \gg 0$. Thus the next proposition shows that the canonical map $\mathrm{BL}(\mathscr{P}(V), \mathscr{W}) \to \mathscr{P}(V)$ is an iterated blowup along smooth centers.

PROPOSITION 5.11
A forgetful functor

$$p : \mathrm{BL}\left(\mathscr{P}(V), \mathscr{W}_{\leq m+1}\right) \to \mathrm{BL}\left(\mathscr{P}(V), \mathscr{W}_{\leq m}\right) \tag{3.14.1}$$

is represented by the blowup along the union of the strict transforms of $\mathscr{P}(V/W) \subset \mathscr{P}(V)$ for $W \in \mathscr{W}_{m+1}$ (which are pairwise disjoint).

Proof
Let $W \in \mathscr{W}_{m+1}$. We claim that the strict transform of $\mathscr{P}(V/W)$ represents the subfunctor X_W of $\mathrm{BL}(\mathscr{P}(V), \mathscr{W}_{\leq m})$ of compatible quotients such that q_E vanishes on $E \cap W$ for all $E \in \mathscr{W}_{\leq m}$. This subfunctor is naturally identified with $\mathrm{BL}(\mathscr{P}(V/W), \mathscr{W}^W_{\leq m})$, where $\mathscr{W}^W_{\leq m}$ is the (obviously convex) collection of subspaces $(E+W)/W \subset V/W$ for $E \in \mathscr{W}_{\leq m}$. By Proposition 5.9, it is smooth and connected and thus the strict transform. For disjointness: if $W', W'' \in \mathscr{W}_{m+1}$, then $\tilde{W} := W' + W'' \in \mathscr{W}_{\leq m}$, and it is not possible for $q_{\tilde{W}}$ to vanish both on W' and on W''; thus the strict transforms are disjoint.

The map (3.14.1) is obviously an isomorphism outside the union of subfunctors X_W. Take $W \in \mathscr{W}_{m+1}$. The inverse image

$$p^{-1}(X_W) \subset \mathrm{BL}\left(\mathscr{P}(V), \mathscr{W}_{\leq m+1}\right)$$

is naturally identified with $\mathscr{P}(W) \times \mathrm{BL}(\mathscr{P}(V/W), \mathscr{W}^W_{\leq m})$. In particular, by Proposition 5.9 it is a smooth connected Cartier divisor. It follows that the exceptional locus of p is the disjoint union of these divisors. Thus p factors through the proscribed blowup, and the induced map to the blowup will have no exceptional divisors and is thus an isomorphism (as domain and image are smooth). □

Definition 5.12
For a subset $\sigma \subset Y$, define the intersection

$$\tilde{\mathscr{P}}(\overline{\sigma}) := \bigcap_{M \in \sigma} \tilde{\mathscr{P}}(\overline{M}) \subset \mathbb{S}_Y.$$

PROPOSITION 5.13
$\tilde{\mathscr{P}}(\overline{\sigma})$ is nonempty if and only if σ is a simplex (4.1.2). Consider the convex subset $\mathrm{Res}_\sigma(Y)$, a collection of convex subsets $\mathscr{W}_i \subset \mathrm{Gr}(M_i/M_{i-1})$. There is a canonical identification

$$\tilde{\mathscr{P}}(\overline{\sigma}) = \prod_{m \geq i \geq 1} \mathrm{BL}\left(\mathscr{P}(M_i/M_{i-1}), \mathscr{W}_i\right) =: \mathrm{BL}\left(\mathscr{P}(\overline{\sigma}), \mathrm{Res}_\sigma Y\right).$$

Proof
By Lemma 5.6, the k-points of the intersection are exactly those for which all $M \in \sigma$ are maximal. Thus if it is nonempty, σ is a simplex by Lemma 5.4. The expression for the intersection is immediate from the definition of Res (see Definition 4.3) and the functorial definitions of $\tilde{\mathscr{P}}(\overline{M})$ and BL. □

Remark 5.14
Observe by Proposition 5.8 – Proposition 5.13 that the special fibre S_Y has normal crossings. Moreover, it can be canonically defined purely in terms of the subcomplex

$Y \subset \mathscr{B}$. Indeed, by Proposition 5.8 its irreducible components and their intersections are encoded by the $\mathrm{BL}(\mathscr{P}(\overline{\sigma}), \mathrm{Res}_\sigma(Y))$ for simplices $\sigma \subset Y$, and by Proposition 4.7 we have canonical identifications

$$\mathrm{Res}_\sigma Y = \mathrm{Star}_\sigma Y \subset \mathrm{Star}_\sigma \mathscr{B} = \mathrm{Star}_{\overline{\sigma}} \mathrm{Gr}(\overline{\Lambda}_m).$$

Definition 5.15
Let $\sigma \subset Y$ be a simplex. Let $U(\sigma) \subset \mathbb{S}_Y$ be the open subset whose complement is the closed subset of the special fibre given by the union of $\tilde{\mathscr{P}}(\overline{N})$, $[N] \in Y \backslash \sigma$.

LEMMA 5.16
$U(\sigma)$ is the union of the generic fibre together with the open subset of the special fibre consisting of all k-points whose simplex of maximal lattices (see §5.3) is contained in σ. It represents the following subfunctor. Let σ be the simplex (4.1.2). For $[M] \in Y$, choose minimal i so that $M^{M_m} \subset M_i$. A T-point of $\mathbb{S}_Y$ is a point of $U(\sigma)$ if and only if the composition

$$M_T \to (M_i)_T \twoheadrightarrow L\big((M_i)_T\big)$$

is surjective for all $[M] \in Y$.

Proof
The proof is immediate from Lemma 5.6 and the definitions. □

Note by Lemma 5.4 that the $U(\sigma)$ for $\sigma \subset Y$ give an open cover of $\mathbb{S}_Y$. Faltings proves that $U(\sigma)$ is nonsingular, and semistable over $\mathrm{Spec}(R)$, by writing down explicit local equations (see [F, page 167]). This can also be seen from the following.

PROPOSITION 5.17
Let $\sigma \subset Y$ be the simplex (4.1.2). Let $U \subset \mathscr{P}(M_m)$ be the open subset of quotients $M_m \to L$ such that $N^{M_m} \to L$ is surjective for all $[N] \in Y \backslash \sigma$. Let $q : \mathrm{BL}(\mathscr{P}(M_m), \overline{\sigma}) \to \mathscr{P}(M_m)$ be the iterated blowup of $\mathscr{P}(M_m)$ along the flag of subspaces of its special fibre

$$\mathscr{P}(M_m/M_{m-1}) \subset \mathscr{P}(M_m/M_{m-2}) \subset \cdots \subset \mathscr{P}(M_m/M_1) \subset \mathscr{P}(M_m/M_0) = \mathscr{P}(\overline{M_m});$$

that is, blow up first the subspace

$$\mathscr{P}(M_m/M_{m-1}) \subset \mathscr{P}(\overline{M_m}) \subset \mathscr{P}(M_m),$$

then the strict transform of $\mathscr{P}(M_m/M_{m-2})$, and so on. There is a natural isomorphism

$$U(\sigma) \to q^{-1}(U).$$

Remark 5.18
When $\sigma = [M]$, Proposition 5.17 is immediate from Lemma 5.16. As this is the only case of Proposition 5.17 which we need, we omit the proof, which in any case is analogous to (and simpler than) those of Propositions 5.9 and 5.11. Proposition 5.17 can also be deduced from the claim on [F, page 168] that for any $[N] \in Y$, the natural map $\mathbb{S}_Y \to \mathscr{P}(N)$ is a composition of blowups with smooth centers (which Faltings describes).

Now fix $\mathscr{F}$ as in (1.9.1).

LEMMA 5.19
Let $Z \subset N$ be a subset with $|Z| = r+1$. There is a unique stable lattice $[\Lambda_Z] \in [\mathscr{F}]$ such that the limits f_Z^{Λ} are generic (i.e., any r of them is an R-basis). In particular, there are finitely many stable lattices, and Stab *is finite. If we reorder so that $Z = \{0, 1, \ldots, r\}$ and express*

$$f_0 = z^{a_1} p_1 f_1 + \cdots + z^{a_r} p_r f_r$$

with $a_i \in \mathbb{Z}$, $p_i \in R^*$, then $\Lambda_Z = Rz^{a_1} f_1 + \cdots + Rz^{a_r} f_r$.

Proof
It is clear that for Λ_Z as given, the limits $\mathscr{F}^{\Lambda_Z}$ are in general position, so Λ_Z is stable. For uniqueness, assume that the limits $\mathscr{F}_Z^{\Lambda}$ are in general position. Then f_i^{Λ}, $r \geq i \geq 1$, are an R-basis of Λ. Define $b_i \in \mathbb{Z}$ by $z^{b_i} f_i = f_i^{\Lambda}$. Scaling Λ, we may assume that $b_i \geq a_i$ with equality for some $r \geq i \geq 1$. Thus $f_0 = \underline{f_0^{\Lambda}}$. Then $b_i = a_i$ for all i; otherwise, $\overline{f_0^{\Lambda}}$ is in the span of some proper subset of the f_i^{Λ}, $r \geq i \geq 1$. So $\Lambda = \Lambda_Z$. $\square$

5.20. Notation
For a subset $I \subset N$, let $V_I \subset V_K$ be the vector subspace spanned by f_i, $i \in I$, and let $V^I := V/V_I$. For each lattice $M \subset V$, let M^I be its image in V^I; that is, let $M^I := M/M \cap V_I$.

Let $Y \subset [\mathscr{F}]$ be a finite convex collection containing Stab. One checks immediately that the collection of equivalence classes

$$Y^I := \{[M^I]\}_{[M] \in Y}$$

is convex.

DEFINITION-LEMMA 5.21
Let $\mathbb{B}_i \subset \mathbb{S}_Y$ be the subfunctor of compatible quotients such that q_M vanishes on f_i^M for all $[M] \in Y$. Then $\mathbb{B}_i$ is the nonsingular Deligne scheme for $Y^{\{i\}}$ and is the closure of the hyperplane on the generic fibre

$$\{f_i = 0\} \subset \mathscr{P}(V_K) \subset \mathbb{S}_Y.$$

Proof
Clearly, $M \cap V_I = f_i^M R$, so $\mathbb{B}_i = \mathbb{S}_{Y^{\{i\}}}$. The rest follows from Theorem 5.2. □

PROPOSITION 5.22
Let $[M] \in [\mathscr{F}]$ be a maximal lattice for a k-point of

$$\bigcap_{i \in I} \mathbb{B}_i \subset \mathbb{S}_Y.$$

Then the limits f_i^M, $i \in I$, are independent over R; that is, they generate an R-direct summand of M of rank $|I|$.

Proof
We consider the corresponding simplex of maximal lattices

$$zM = M_0 \subset M_1 \subset \cdots \subset M_m = M.$$

For each $m \geq s \geq 1$, let $I_s \subset I$ be those i such that $f_i^M \in M_s \backslash M_{s-1}$. Clearly, it is enough to show that the images of f_i^M, $i \in I_s$, in M_s/M_{s-1} are linearly independent. By scaling (which allows us to move any of the M_i to the M_1 position), it is enough to consider $s = 1$ and show that the images of f_i, $i \in I_1$, are linearly independent in M/zM. Or, suppose not. Choose a minimal set whose images are linearly dependent, which after reordering we may assume are $f_0, f_1, \ldots, f_p$. With further reordering, we may assume that

$$f_1^M, f_2^M, \ldots, f_r^M$$

are an R-basis of M or, equivalently, that their images give a basis of M/zM. Renaming the f_i, we can assume that $f_i^M = f_i$. Now consider the unique expression

$$f_0 = \sum_{i=1}^{r} z^{a_i} p_i f_i \tag{5.22.1}$$

with $a_i \in \mathbb{Z}$, $p_i \in R^*$. By construction, $a_i \geq 0$. Since the images of $f_0, \ldots, f_p$ in M/zM are a minimal linearly dependent set, it follows that $a_i \geq 1$ for $i > p$, and

$a_i = 0$ for $p \geq i \geq 1$. Now let

$$\Lambda := Rf_1 + \cdots + Rf_p + Rz^{a_{p+1}} f_{p+1} + \cdots + Rz^{a_r} f_r.$$

Note that $f_i^{M_1} = f_i$ for $p \geq i \geq 0$, so by assumption q_{M_1} vanishes on these f_i. Notice that $z^{a_t} f_t \in zM_k = M_0 \subset M_1$ for $t \geq p+1$. Thus by Theorem 5.2, q_{M_1} vanishes on these $z^{a_t} f_t$. Thus q_{M_1} vanishes on Λ. But by (5.22.1) and Lemma 5.19, Λ is stable. In particular, $[\Lambda] \in Y$. Clearly, $\Lambda \subset M_1$, but $\Lambda \not\subset M_0$. So the vanishing of $q_{M_1}|_\Lambda$ violates Lemma 5.4. □

THEOREM 5.23

Let $Y \subset [\mathscr{F}]$ be a finite convex set containing all the stable lattices. For any subset $I \subset N$, the scheme-theoretic intersection

$$\bigcap_{i \in I} \mathbb{B}_i \subset \mathbb{S}_Y$$

is nonsingular and (empty or) of codimension $|I|$ and represents the Deligne functor $\mathbb{S}_{Y^I}$. The divisor $S_Y + \mathbb{B} \subset \mathbb{S}_Y$ has normal crossings.

Proof

By Theorem 5.2, it is enough to show that the intersection $\mathscr{I}$ represents the Deligne functor. It is obvious from the definitions and Definition-Lemma 5.21 that $\mathscr{I}$ represents the subfunctor of $\mathbb{S}_Y$ of compatible quotients q_M which vanish on f_i^M, $i \in M$, while $\mathbb{S}_{Y^I}$ is the subfunctor where q_M vanishes on $M \cap V_I$. Since $f_i^M \in M \cap V_I$, clearly

$$\mathbb{S}_{Y^I} \subset \mathscr{I}$$

is a closed subscheme. Since by Theorem 5.2, $\mathbb{S}_{Y^I}$ is flat over R, it is enough to show that they have the same special fibres. And so it is enough to show that the subfunctors agree on $T = \mathrm{Spec}(B)$ for B a local ($k = R/zR$)-algebra, with residue field k. By maximal for such a T-point we mean maximal for the closed point. We take a family of compatible quotients vanishing on all f_i^M and show that they actually vanish on $V_I \cap M$. By Lemma 5.4, it is enough to consider maximal M. But by Proposition 5.22, f_i^M, $i \in I$, are independent over R, so in fact they give an R-basis of $V_I \cap M$. □

6. Global generation

Here we complete the proof of Theorem 1.13.

6.1. Bubbling

We choose an increasing sequence of finite convex subsets $Y_i \subset \mathscr{B}$ whose union is the membrane $[\mathscr{F}]$; the existence, for example, follows from Lemma 4.8. We assume

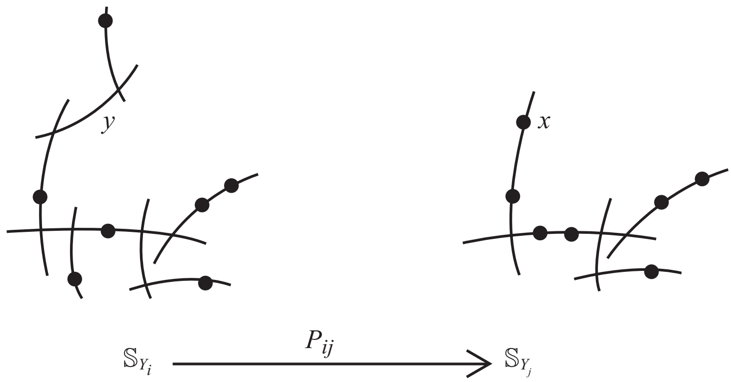

Figure 3

that Stab $\subset Y_i$. We have the natural forgetful maps

$$p_{i,j} : \mathbb{S}_{Y_i} \to \mathbb{S}_{Y_j}, \quad i \geq j.$$

By Theorem 5.23, we have the vector bundle $\Omega^1_p(\log \mathbb{B})$ on $\mathbb{S}_{Y_i}$ (see §9).

PROPOSITION 6.2
Given a closed point $x \in \mathbb{S}_{Y_j}$ for all i sufficiently large, there is a closed point $y \in \mathbb{S}_{Y_i} \backslash \mathbb{B}$, so that $p_{i,j}(y) = x$ (see Figure 3).

Proof
Say that x lies on $\tilde{\mathscr{P}}_{Y_j}(\overline{M})$, $[M] \in Y_j$. Choose i sufficiently big so that

$$[M + z^{-1} f^M R] \in Y_i$$

for all $f \in \mathscr{F}$. Clearly, $p_{i,j}(\tilde{\mathscr{P}}_{Y_i}(\overline{M})) = \tilde{\mathscr{P}}_{Y_j}(\overline{M})$. It remains to show that $\tilde{\mathscr{P}}_{Y_j}(\overline{M})$ is disjoint from $\mathbb{B} \subset \mathbb{S}_{Y_j}$. Suppose that $q \in \tilde{\mathscr{P}}_{Y_j}(\overline{M}) \cap \mathbb{B}_k$. Let $N = M + z^{-1} f_k^M R$. Clearly, $z^{-1} f_k^M = f_k^N$, so by definition of $\mathbb{B}_k$, q_N vanishes on $z^{-1} f_k^M$. But q_N vanishes on M by definition of a maximal lattice. So $q_N = 0$, a contradiction. □

6.3. Barycentric subdivision trick
Next, we introduce a convenient operation. Let $R' = k[[z^{1/m}]]$, and let $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ be the associated finite map. Let M_v be a collection of lattices in V_K, and let Y be their convex hull. Let $M'_v := M_v \otimes_R R'$, and let Y' be their convex hull.

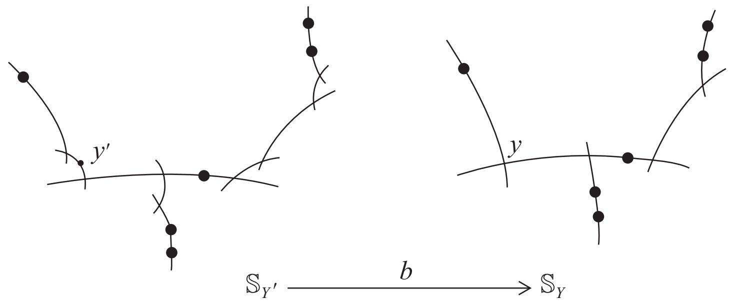

Figure 4

PROPOSITION 6.4
There is a commutative diagram

$$\begin{tikzcd}
\mathbb{S}_{Y'} \arrow[r, "b"] \arrow[d, "p'"'] & \mathbb{S}_Y \arrow[d, "p"'] \\
\mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

with all arrows proper. If $m \geq r$, then given a k-point $y \in \mathbb{S}_Y$ there is a k-point $y' \in \mathbb{S}_{Y'}$ in its inverse image which lies on a unique irreducible component of the special fibre $S_{Y'}$ (see Figure 4).

Proof
For any R-object X, we denote by X' the base change to R'. It is clear that $\mathbb{S}'_Y$ represents the functor $\mathbb{S}_{\tilde{Y}}$ defined as in Theorem 5.2 but for the nonconvex collection $\tilde{Y} = \{M' \mid M \in Y\}$. Since $\tilde{Y} \subset Y'$, there is a forgetful map $\mathbb{S}_{Y'} \to \mathbb{S}'_Y$ sending compatible collections of quotients to compatible collections. This implies the commutative diagram above.

Now let $m \geq r$, let $\omega = z^{1/m}$, and let $y \in \mathbb{S}_Y$ be a k-point with the simplex of maximal lattices

$$\sigma = \{M_0 = zM_k \subset M_1 \subset \cdots \subset M_k\}.$$

By Lemma 5.4, y is determined by a collection of hyperplanes $H_i \subset M_i/M_{i-1}$ which do not contain any $\mathrm{Res}_\sigma\, M$ for $M \in Y$. Let N be the R'-lattice

$$N = M'_1 + \omega M'_2 + \omega^2 M'_3 + \cdots + \omega^{k-1} M'_k.$$

Observe that $\omega^i M_i' \subset \omega N$, $k \geq i \geq 0$. ($M_0' = \omega^{m-k}\omega^k M_k'$, and for $k > 0$ the inclusion is clear.) Thus we have a map

$$\bigoplus_{i=1}^{k} M_i'/M_{i-1}' \xrightarrow{\oplus\cdot\omega^{i-1}} N/\omega N. \tag{6.4.1}$$

This map is clearly surjective; thus it is an isomorphism, as its domain and range are r-dimensional k-vector spaces. By the injectivity of the map, we have

$$\omega^{i-1}M_i' \cap \omega N = \omega^{i-1}M_{i-1}. \tag{6.4.2}$$

Now let $y' \in \mathscr{P}(N/zN)$ be given by any hyperplane $H' \subset N/zN$ which restricts to H_i on M_i/M_{i-1} under (6.4.1). It is enough to show $y' \in U([N])$; then, clearly, y' is sent to y and N is the only maximal lattice of y'. (Or equivalently, by Lemma 5.6, y' lies on a unique irreducible component of the special fibre $S_{Y'}$.) By Lemma 5.4, it is enough to show that H' does not contain $\mathrm{Res}_{[N]}[\Lambda]$ for $[\Lambda] \in Y'$, and by Proposition 4.7, it is enough to check this for $[\Lambda] = [M']$, $[M] \in Y$.

We can assume that $M = M^{M_k}$, $M \subset M_i$, $M \not\subset M_{i-1}$. Then $\mathrm{Res}_\sigma[M] = (M + M_i)/M_{i-1}$, and it follows from (6.4.2) that $(M')^N = \omega^{i-1}M'$; thus we have

$$(\cdot\omega^{i-1})(\mathrm{Res}_\sigma[M]) = \mathrm{Res}_{[N]}[M'],$$

by (6.4.1). Thus H' does not contain $\mathrm{Res}_{[N]}[M']$ since $H_i \not\supset \mathrm{Res}_\sigma[M]$. □

THEOREM 6.5

Under the natural maps $p_{i,j} : \mathbb{S}_{Y_i} \to \mathbb{S}_{Y_j}$ for $j \leq i$, there is an induced isomorphism of vector bundles

$$p_{i,j}^*\big(\Omega_p^1(\log \mathbb{B})\big) \to \Omega_p^1(\log \mathbb{B}).$$

Furthermore, the global sections

$$\mathrm{dlog}\left(\frac{f}{g}\right), \quad f, g \in \mathscr{F},$$

generate $\Omega_p^1(\log \mathbb{B})$ globally. In particular, $\omega_p(\mathbb{B})$ is globally generated.

Proof

Since the sections are pulled back from $\mathbb{S}_{Y_j}$, the last remark implies the first. Furthermore, to prove that the given sections generate at some point $y \in \mathbb{S}_{Y_j}$, it is enough to prove that they generate at some inverse image point on $\mathbb{S}_{Y_i}$. Thus by Proposition 6.2, it is enough to prove that they generate at a point $y \in \mathbb{S}_i \setminus \mathbb{B}$.

By Proposition 6.4, we have the proper (generically finite) map

$$\mathbb{S}_{Y_i'} \to \mathbb{S}_{Y_i}.$$

Clearly, the $\mathrm{dlog}(f/g)$ pull back to the analogous forms on the domain, so by Proposition 6.4, we may assume that y has a unique maximal lattice M. Now by Remark 5.18, the natural map

$$\mathbb{S}_{Y_i} \to \mathscr{P}(M)$$

is an isomorphism of $y \in U([M])$ onto an open subset of $\mathscr{P}(M)$ which misses all of the hyperplanes

$$f_i^M = 0.$$

Note that $\mathscr{F}^M$ contains an R-basis of M. Reordering, say, f_k^M, $r \geq k \geq 1$, gives such a basis. Then it is enough to show that the (regular) forms

$$\mathrm{dlog}\left(\frac{f_k^M}{f_1^M}\right), \quad r \geq k \geq 2,$$

give trivialisation of the ordinary cotangent bundle over the open set in question, which is obvious. □

6.6. Minimal model

By Theorem 6.5, the line bundle $\omega_p(\mathbb{B})$ is globally generated. We consider the p-relative minimal model, $\pi : \mathbb{S}_Y \to \overline{\mathbb{S}}$; that is,

$$\overline{\mathbb{S}} := \mathrm{Proj} \bigoplus_m p_*\big(\omega_p(\mathbb{B})^{\otimes m}\big). \tag{6.6.1}$$

Note by Theorem 6.5 that $\overline{\mathbb{S}}$ is independent of Y. Let $\pi_*(\mathbb{B}_i) =: \mathbb{B}_i \subset \overline{\mathbb{S}}$.

THEOREM 6.7

Let $\mathrm{Spec}(R) \to \overline{X}(r,n)$ *be the unique extension of the map which sends the generic fibre to*

$$[\mathbb{P}^{r-1}, L_1 + \cdots + L_n] \in \mathbb{P}^0(r,n)/\mathrm{PGL}_r = X(r,n) \subset \overline{X}(r,n).$$

The pullback of the universal visible contour family $(\mathscr{S}, \mathscr{B})$ *(see Definition 2.12) is* $(\overline{\mathbb{S}}, \mathbb{B})$.

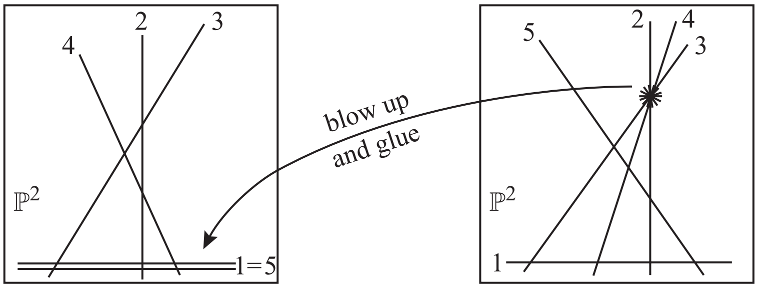

Figure 5

Proof
By Theorem 6.5, we have a natural surjection

$$V_n \otimes \mathcal{O}_{\mathbb{S}_Y} \twoheadrightarrow \Omega^1_p(\log \mathbb{B})$$

inducing a regular map $\mathbb{S}_Y \to G(r-1, n-1)$, which on the general fibre is Kapranov's visible contour embedding of $\mathbb{P}^{r-1}$ given by the bundle of log forms with poles on the n general hyperplanes. This induces a regular map $\mathbb{S}_Y \to \mathscr{S}$, where $\mathscr{S} \to \operatorname{Spec}(R)$ is the pullback of the visible contour family (see Definition 2.12); $\omega_p(\mathbb{B})$ is pulled back from a relatively very ample line bundle (the Plücker polarisation) on $\mathscr{S}$, by Theorem 2.17. Thus $\mathbb{S}_Y \to \mathscr{S}$ factors through a finite map

$$\overline{\mathbb{S}} \to \mathscr{S}.$$

The map is birational, an isomorphism on the generic fibre. By Proposition 2.13, $\mathscr{S}$ is normal. Thus it is an isomorphism. □

6.8. Illustration
Let us look at the first nontrivial example for $r = 3$,

$$\mathscr{F} = \{f_1, f_2, f_3, \quad f_4 = f_1 + f_2 + f_3, \quad f_5 = z^{-1} f_1 + f_2 + f_3\}$$

for f_1, f_2, f_3, the standard basis of $k^3 \subset K^3$. In this case there are two stable lattices,

$$M_1 = Rf_1 + Rf_2 + Rf_3 \qquad \text{and} \qquad M_2 = Rz^{-1} f_1 + Rf_2 + Rf_3.$$

Figure 5 illustrates the limit surface (S, B). Note that $zM_2 \subset M_1 \subset M_2$, so the stable lattices in this case form a 1-simplex, σ; in particular, they are already convex. So we can take $Y = \sigma = \mathrm{Stab}$. The two images in Figure 5 are the configurations of limit lines. The components of S are $\tilde{\mathscr{P}}(\overline{M}_1) = \mathscr{P}(\overline{M}_1)$ and $\tilde{\mathscr{P}}(\overline{M}_2)$, the blowup of $\mathscr{P}(\overline{M}_2)$ at the intersection point $f_2^{\overline{M}_3} = f_3^{\overline{M}_3} = 0$. The two components are glued along $\tilde{\mathscr{P}}(\overline{\sigma}) = \mathscr{P}(M_2/M_1)$, which embeds in $\tilde{\mathscr{P}}(\overline{M}_1)$ as the line $f_1^{\overline{M}_1} = 0$, and in $\tilde{\mathscr{P}}(\overline{M}_2)$ as the exceptional curve.

7. Bubble space

In §6 we chose finite convex $Y \subset [\mathscr{F}]$ containing Stab. Though there is a canonical choice, namely, the convex hull of Stab, it is more aesthetic to take the infinite set $[\mathscr{F}]$. Let Y_i be an increasing sequence of finite convex subsets, containing Stab, with union $[\mathscr{F}]$. Call Y_i *full* along the simplex $\sigma \subset Y_i$ if $\mathrm{Star}_{[M]} Y_i = \mathrm{Star}_{[M]}[\mathscr{F}]$ for all $[M] \in \sigma$. It is clear that if Y_i is full along σ, then so is Y_j for $j \geq i$. Let $U_i \subset Y_i$ be the union of all $U_{Y_i}(\sigma)$ such that Y_i is full along σ. The next lemma shows that we may view U_i as an increasing sequence of open sets. We define $\mathbb{S} = \bigcup_i U_i$.

LEMMA 7.1

If Y_i is full along σ, then $p_{ji}^{-1}(U_{Y_i}(\sigma)) \subset U_{Y_j}(\sigma)$. Moreover, $p_{ji}^{-1}(U_i) \subset U_j$, and the map $p_{ji}^{-1}(U_i) \to U_i$ is an isomorphism.

Proof

Assume that Y_i is full along σ. Take $x \in p_{ji}^{-1}(U_{Y_j}(\sigma))$, and take a maximal lattice $[N] \in Y_j$ for x. Maximal lattices form a simplex, so $[N]$ is adjacent to a lattice in σ, and therefore $[N] \in Y_i$ because Y_i is full along σ. Now it is clear that $[N]$ is maximal for $p_{ji}(x)$, so $[N] \in \sigma$. Thus $x \in U_{Y_i}(\sigma) \subset U_i$.

To show that $p_{ji}^{-1}(U_i) \to U_i$ is an isomorphism, it suffices to check that $p_{ji}^{-1}(U_{Y_i}(\sigma)) \to U_{Y_i}(\sigma)$ is an isomorphism for any $\sigma \subset Y_i$. This map is proper and birational, and its domain and range are nonsingular. So to show that it is an isomorphism, it is enough to check that there are no exceptional divisors and so to check that each irreducible component of the special fibre of the domain maps onto an irreducible component of the special fibre for the image. These components are the (appropriate open subsets of the) $\mathscr{P}_{Y_j}(\overline{M})$, $[M] \in \sigma$, and it is obvious that $\mathscr{P}_{Y_j}(\overline{M}) \twoheadrightarrow \mathscr{P}_{Y_i}(\overline{M})$. □

THEOREM 7.2

$\mathbb{S}$ is nonsingular and locally of finite type. Its special fibre, S_∞, has smooth projective irreducible components and normal crossings. Let $\mathbb{B}_i \subset \mathbb{S}$ be the hyperplane $f_i = 0$ of the generic fibre. $\mathbb{B}_i \subset \mathbb{S}$ is closed and disjoint from S_∞. $\mathbb{B} = \sum \mathbb{B}_i$ has normal crossings. In particular, there is a natural vector bundle $\Omega^1_{\mathbb{S}/R}(\log \mathbb{B})$ whose determinant is $\omega_{\mathbb{S}/R}(\mathbb{B})$. The bundle is globally generated.

There are natural surjective maps $p_i : \mathbb{S} \to \mathbb{S}_{Y_i}$ for all i, and there are isomorphisms

$$p_i^{-1}\big(\Omega^1_p(\log \mathbb{B})\big) = \Omega^1_{\mathbb{S}/R}(\log \mathbb{B}),$$

$$H^0\big(\mathbb{S}_{Y_i}, \Omega^1_{\mathbb{S}_i/R}(\log \mathbb{B})\big) \to H^0\big(\mathbb{S}, \Omega^1_{\mathbb{S}/R}(\log \mathbb{B})\big).$$

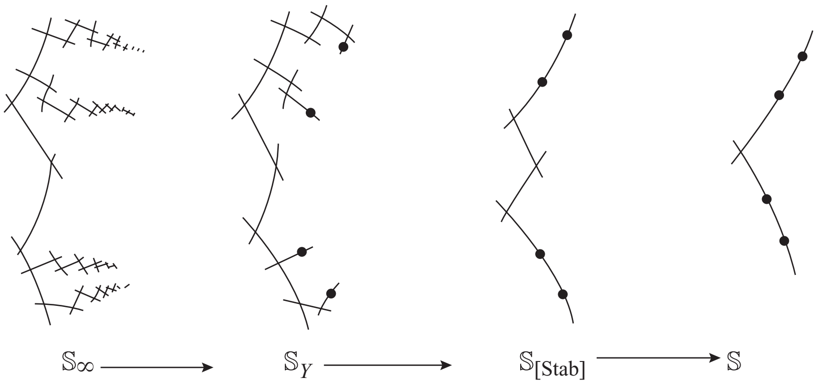

Figure 6

The differential forms $\mathrm{dlog}(f/g)$*,* $f, g \in \mathscr{F}$*, define a natural inclusion*

$$V_n \subset H^0(S_\infty, \Omega^1_{S_\infty}).$$

The sections generate the bundle. The associate map $S_\infty \to G(r-1, n-1)$ *factors through* S_{Y_i} *for all* i*, and its image is the special fibre of the pullback of the family* $\mathscr{S} \to \overline{X}(r, n)$ *for the map* $\mathrm{Spec}(R) \to \overline{X}//H$ *given by* $\mathscr{F}$*.*

Figure 6 illustrates the case where $r = 2$.

Proof
Arguing as in Proposition 6.2, one can show that the special fibre of $U_i \to \mathrm{Spec}(R)$ is disjoint from $\mathbb{B} \subset \mathbb{S}_{Y_i}$. It follows that S_∞ is disjoint from $\mathbb{B}$.

We check that $\mathbb{S} \to \mathbb{S}_{Y_i}$ is surjective for all i. The rest then follows easily from Theorems 6.5 and 6.7. Take $[M] \in Y_i$. It is clear from the definitions that $\tilde{\mathscr{P}}(\overline{M}) \subset \mathbb{S}_{Y_j}$ surjects onto $\tilde{\mathscr{P}}(\overline{M}) \subset \mathbb{S}_{Y_i}$ for $j \geq i$. Moreover, there are only finitely many simplices of $[\mathscr{F}]$ which contain $[M]$ by Lemma 4.4, and for j large, Y_j contains them all, from which it follows that $\tilde{\mathscr{P}}(\overline{M}) \subset U_j$. Thus the image of $\mathbb{S} \to Y_i$ contains $\tilde{\mathscr{P}}(\overline{M})$. The union of the $\tilde{\mathscr{P}}(\overline{M})$ is the full special fibre, so $\mathbb{S} \to Y_i$ is surjective. □

The Deligne functor for $[\mathscr{F}]$ is not represented by a scheme, but $\mathbb{S}$ represents a natural subfunctor. In particular, $\mathbb{S}$ is independent of the choice of a sequence Y_i.

THEOREM 7.3
$\mathbb{S}$ *represents the subfunctor of the Deligne functor for* $[\mathscr{F}]$ *a* T*-valued point of which is a collection of compatible quotients such that each* k'*-point of* T *admits a maximal lattice.*

Proof
Take a k-point of U_i. By Lemma 7.1, any lattice $[M] \in Y_i$ maximal in Y_i for k is maximal in $[\mathscr{F}]$. It follows that U_i is a subfunctor of the functor in the statement, and thus $\mathbb{S}$ is a subfunctor. For the other direction, it is enough to consider T the spectrum of a local ring, with residue field k'. Consider a T-point of the subfunctor in the statement. Note that in the proof of Lemma 5.4, the only place finiteness of Y is used is to establish the existence of a maximal lattice which here we assume. So the k'-point has a simplex of maximal lattices, σ, satisfying Lemma 5.4. For i large, Y_i is full along σ, and now it is clear that the quotients define a T-point of U_i and thus of $\mathbb{S}$. □

8. Limit variety

The matroid decomposition corresponding to the fibre of the visible contour family can be readily obtained from the power series, as we now describe. From the matroid decomposition, one can describe the fibre using [L2, Theorem 5.3]. We assume that the reader is familiar with the variation of geometric invariant theory (VGIT) quotients (see, e.g., [DH]). We note, in particular, that $\Delta(r, n)$ parameterizes PGL_r-ample line bundles on $\mathbb{P}(r, n)$ with nonempty semistable locus.

We call a lattice M connected if the configuration of limiting hyperplanes $\mathscr{F}^{\overline{M}}$ is connected. This obviously implies $[M] \in [\mathscr{F}]$. Of course, stable implies connected. For $r \leq 3$, they are the same (see Lemma 8.6), but they are in general different.

Definition 8.1
Call a polarisation $L \in \Delta(r, n)$ on $\mathbb{P}(r, n)$ *generic* if there are no strictly semistable points. For $[M] \in \mathscr{B}$, let C_M be the arrangement of limiting hyperplanes

$$\{f^{\overline{M}} = 0\} \subset \mathscr{P}(\overline{M}), \quad f \in \mathscr{F},$$

and let $P_M \subset \Delta(r, n)$ be the matroid polytope of C_M (see §3).

LEMMA 8.2
If $L \in \Delta(r, n)$ is a generic polarisation, then there is a unique $[M] \in [\mathscr{F}]$ such that the configuration C_M is L-stable. Moreover, M is connected.

Proof
Let Q be the GIT quotient of $\mathbb{P}(r, n)$ given by L. Q is a fine moduli space for L-stable configurations and carries a universal family, a smooth étale locally trivial $\mathbb{P}^{r-1}$-bundle. $\mathscr{F}$ gives a K-point of Q which extends uniquely to an R-point. The pullback of the universal family is trivial over R (as R is Henselian), and so is $\mathscr{P}(M)$ for some lattice $M \subset V_K$. It is clear that the limit configuration (given by the image in Q of the closed point of R) is equal to C_M. As C_M has no automorphisms, it follows that

this configuration contains r hyperplanes in general position, and so $[M] \in [\mathscr{F}]$. The proof shows that if C_N, C_M are both L-stable, then the rational map $\mathscr{P}(N) \to \mathscr{P}(M)$ is a regular isomorphism (both C_N and C_M are pullbacks of the universal family over Q), which implies that $[N] = [M]$. C_M has a trivial automorphism group, so the final remark holds by definition. □

By the next result, there are only finitely many connected equivalence classes.

THEOREM 8.3
Let $x \in \overline{X}(r, n)$ be the limit point for the one-parameter family given by $\mathscr{F}$. Assume that x belongs to a stratum given by the matroid decomposition $\underline{P}$. Then the maximal-dimensional polytopes of $\underline{P}$ are precisely the P_M for which M is connected.

Proof
By [K1] and the theory of VGIT, the matroid decomposition is obtained as follows. GIT equivalence determines a polyhedral decomposition of $\Delta(r, n)$. Chambers (interiors of maximal-dimensional polytopes in the decomposition) correspond to generic polarisations. For each such chamber, there is a corresponding GIT quotient that is a fine moduli space for configurations of hyperplanes stable for this polarisation. The one-parameter family has a unique limit in each such quotient, and, in particular, associated to each chamber, we have a limiting configuration. Associated to the configuration is its matroid polytope, and the polytopes obtained in this way are precisely the facets of $\underline{P}$. Now by Definition 8.1, if L is a polarisation in a chamber, then the limiting configuration is C_{M_L} for a unique $[M_L] \in [\mathscr{F}]$. Conversely, if we take $[M] \in [\mathscr{F}]$ so that C_M has no automorphisms, the polytope $P_M \subset \Delta(r, n)$ is maximal dimensional (see [L2, Theorem 1.11]). General $L \in P_M$ will be generic (in the sense of Definition 8.1), and it is clear that C_M is L-stable, so $M = M_L$. □

8.4. Stratification
The membrane $[\mathscr{F}]$ is, by Lemma 4.9, a union of apartments. We have an apartment for each $I \subset N$, $|I| = r$, and thus for each vertex of $\Delta(r, n)$. We stratify $[\mathscr{F}]$ by apartments—with one stratum for each collection of vertices—those points that lie in these, but no other, apartments. It follows easily from Theorem 8.3 that the stratum is nonempty if and only if the collection is the vertices of some $P \in \underline{P}$, in which case the stratum consists of those $[M] \in [\mathscr{F}]$ (or, more generally, rational points of the realization; see §4.10) with $P_M = P$. The dimension of the stratum is the codimension of the polytope P in $\Delta(r, n)$. We write $[\mathscr{F}]^k$ for the union of k-strata. Note that $[\mathscr{F}]^0$ is precisely the union of connected lattices.

It is easy to describe the stratification in terms of the power series $\mathscr{F}$. For any $a_1, \dots, a_r$, let $S(a_1, \dots, a_r)$ be the stratification of $[f_1, \dots, f_r]$ by cones spanned by

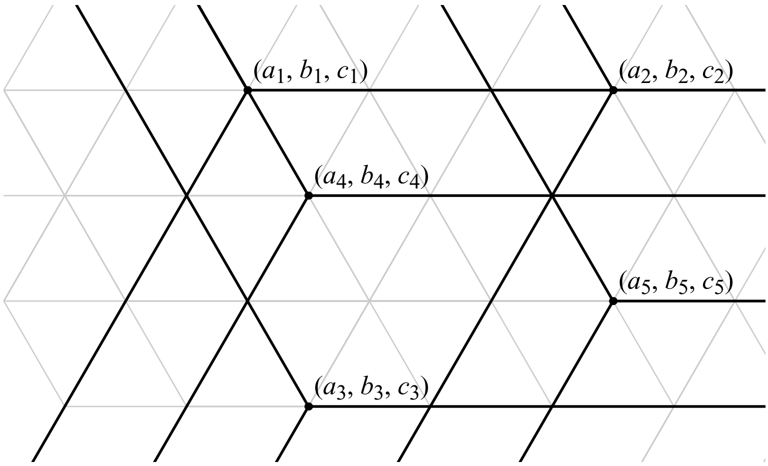

Figure 7

rays $R_0, \dots, R_r$, where

$$R_i = [z^{a_1} f_1, \dots, z^{a_{i-1}} f_{i-1}, z^{a_i+p} f_i, z^{a_{i+1}} f_{i+1}, \dots, z^{a_r} f_r], \quad p \geq 0.$$

LEMMA 8.5
Let $f_i = p_i^1 z^{a_i^1} f_1 + \cdots + p_i^r z^{a_i^r} f_r$ for $i = r+1, \dots, n$, where $p_i^k \in R^*$. Then the induced stratification of an apartment $[f_1, \dots, f_r]$ is defined by intersections of the $S(a_i^1, \dots, a_i^r)$.

Figure 7 illustrates the case $r = 3$, where $(a_2, b_2, c_2) = (a - 3, b, c)$, $(a_4, b_4, c_4) = (a, b + 1, c)$, and so on.

Proof
The proof is immediate from the definitions. □

Now consider the limit pair (S, B). The irreducible components correspond to $[\mathscr{F}]^0$; for $[M] \in [\mathscr{F}]^0$, the corresponding component is the log canonical model of $\mathscr{P}(\overline{M}) \backslash C_{[M]}$, the complement to the union of limiting hyperplanes. We note by [L2, Theorem 5.3] that a collection of components has a common point of intersection if and only if the corresponding matroids have a common face, which is if and only if the points in $[\mathscr{F}]^0$ all lie on the boundary of the corresponding stratum. In particular, if they have a nonempty intersection, they all lie in a single apartment.

From now on, we assume that $r = 3$.

LEMMA 8.6
A configuration of lines in $\mathbb{P}^2$ is connected if and only if it contains four lines in linear general position. A configuration is not connected if and only if there is a point in the configuration which is in the complement of at most one of the lines. In this case, the automorphism group is positive dimensional.

Proof
The proof is easy linear algebra. □

Remark 8.7
By Lemma 8.6, *stable* is the same as *connected* if $r = 3$. This fails in higher dimensions: the configuration of planes

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0, \qquad x_4 = 0, \qquad x_1 + x_2 + x_3 = 0,$$
$$x_2 + x_3 + x_4 = 0$$

in $\mathbb{P}^3$ is connected but not stable.

LEMMA 8.8
Let C be a stable configuration of lines indexed by N. Let $\tilde{S} \to \mathbb{P}^2$ be the blowup of all points of multiplicity at least 3. *Let $B \subset \tilde{S}$ be the reduced inverse image of the lines. Then $K_{\tilde{S}} + B$ is ample, and*

$$\mathbb{P}^2 \backslash C \subset \tilde{S}$$

is the log canonical compactification except in one case: if there are points a, b on a line L of C such that any other line of C meets L in either a or b. In this case, (the strict transform of) $L \subset \tilde{S}$ is a (-1)-curve, the blowdown is $\mathbb{P}^1 \times \mathbb{P}^1$, and

$$\mathbb{P}^2 \backslash C \subset \mathbb{P}^1 \times \mathbb{P}^1$$

is the log canonical model with boundary a union of fibres for the two rulings.

We refer to this exceptional case as a *special stable configuration* (see Figure 8).

Proof
We induct on n. When $n = 4$, then $\tilde{S} = \mathbb{P}^2$ and the result is obvious. So we assume that $n > 4$. If the configuration is special, the result is clear, so we assume that it is not. By Lemma 8.6, we can drop a line, M, so the resulting configuration C' is stable. If C' is special, with special line L, then since C is not special, it follows that if we instead drop L, the resulting configuration is stable and nonspecial. So we may assume that

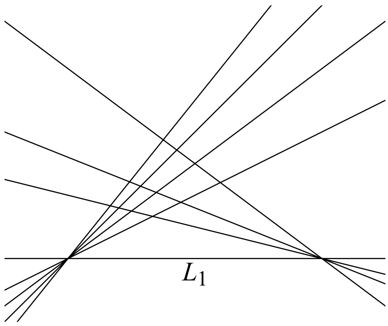

Figure 8

C' is not special. Add primes to the notation to indicate analogous objects for C'. We have $q : \tilde{S} \to \tilde{S}'$, the blowup along the points of M where C has multiplicity exactly 3. (Note that $\tilde{S}' \to \mathbb{P}^2$ is an isomorphism around these points.) Thus

$$K_{\tilde{S}} + B = q^*(K_{\tilde{S}'} + B') + M$$

(where we use the same symbol for a curve and for its strict transform). It is clear that $K_{\tilde{S}} + B$ is q-ample. As $K_{\tilde{S}'} + B'$ is ample, the only curve on which $K_{\tilde{S}} + B$ can have nonpositive intersection is M. But $(K_{\tilde{S}} + B) \cdot M > 0$ by adjunction since C is not special. It follows that $K_{\tilde{S}} + B$ is ample. □

8.9. The limit surface

Now we describe the limit pair (S, B) precisely. The irreducible components are smooth and described by Lemma 8.8. We write S_M for the component corresponding to $[M] \in$ Stab. Unbounded 1-strata—rays in some apartment—correspond to irreducible components of B, and bounded 1-strata correspond to irreducible components of $\operatorname{Sing}(S)$. For each $[M] \in$ Stab, the 1-strata that bound $[M]$ correspond to boundary components of S_M (components of the complement to $\mathscr{P}(\overline{M}) \setminus C_M$). S_M and S_N have a one-dimensional intersection if and only if $[M], [N]$ comprise the boundary of a 1-stratum, and in that case they are glued along the corresponding boundary component (a copy of $\mathbb{P}^1$). Triple points of S (points on three or more components) correspond to bounded 2-strata. The local analytic singularities of (S, B) are described by the following theorem, which was discovered independently by Hacking.

THEOREM 8.10

Let $p \in S$ be a point where the pair (S, B) fails to have normal crossings. There are two possibilities for the germ of (S, B) in an analytic neighbourhood $p \in U_p$:

$$U_p = \langle e_1, e_2 \rangle \cup \langle e_2, e_3 \rangle \cup \langle e_3, e_4 \rangle \subset \mathbb{A}^4, \tag{8.10.1}$$

$B \cap U_p$ is the union of $\langle e_1 \rangle$ and $\langle e_4 \rangle$, and these are components of a single B_i;

$$U_p = \langle e_1, e_2 \rangle \cup \langle e_2, e_3 \rangle \cup \cdots \cup \langle e_n, e_1 \rangle \subset \mathbb{A}^n, \qquad (8.10.2)$$

$n = 3, 4, 5, 6$, and $B \cap U_p = \emptyset$.

Here $e_1, \ldots, e_n$ are coordinate axes in $\mathbb{A}^n$, and $\langle \cdot \rangle$ is the linear span.

Proof

It is simple to classify bounded 2-strata using Lemma 8.5. Then the gluing among components is described by [L2, Theorem 5.3]. □

9. The bundle of relative log differentials

Here we recall a general construction that is used at various points throughout the article. Let

$$p : (\mathscr{S}, \mathscr{B}) \to C, \qquad \mathscr{B} = \sum_{i=1}^{n} \mathscr{B}_i$$

be a pair of a nonsingular variety with normal crossing divisor, semistable over the curve C, in a neighbourhood of $0 \in C$ (i.e., $(\mathscr{S}, F + \mathscr{B})$ has normal crossings, where F is the fibre over zero). We assume that the general fibre is projective but not necessarily the special fibre. We define the bundle of relative log differentials $\Omega^1_p(\log \mathscr{B})$ by the exact sequence

$$0 \longrightarrow \Omega^1_{C/k}(\log 0) \to \Omega^1_{\mathscr{S}/k}(\log F + \mathscr{B}) \to \Omega^1_p(\log \mathscr{B}) \to 0. \qquad (9.0.1)$$

Assume on the generic fibre S that the restrictions of the boundary components, B_i, B_j, are linearly equivalent. Then we can choose a rational function f on $\mathscr{S}$ so that

$$(f) = \mathscr{B}_i - \mathscr{B}_j + E$$

for E supported on F. Then $\operatorname{dlog}(f)$ gives a global section of $\Omega^1_{\mathscr{S}/k}(\log F + \mathscr{B})$. Note that f is unique up to multiplication by a unit on $C \backslash 0$, and thus the image of $\operatorname{dlog}(f)$ in $\Omega^1_p(\log \mathscr{B})$, which we denote by $\operatorname{dlog}(\mathscr{B}_i/\mathscr{B}_j)$, is independent of f.

From now on, we assume that for the general fibre $H^0(S, \Omega^1_S) = 0$, $\operatorname{dlog}(\mathscr{B}_i/\mathscr{B}_j)$ is characterized as the unique section whose restriction to the general fibre has residue 1 along B_i, -1 along B_j, and is regular off of $B_i + B_j$.

In this way, we obtain a canonical map

$$V_n \to H^0\big(\mathscr{S}, \Omega^1_p(\log \mathscr{B})\big) \qquad (9.0.2)$$

(V_n is the standard k-representation of the symmetric group S_n) which is easily seen to be injective (e.g., by the description of the residues on the general fibre).

The restriction

$$\Omega^1_{S/k}(\log B) := \Omega^1_p(\log \mathscr{B})|_S \tag{9.0.3}$$

for (S, B), the special fibre of $(\mathscr{S}, \mathscr{B})$, is canonically associated to (S, B), that is, is independent of the smoothing (see, e.g., [Fr, §3] or [KN]: these authors treat normal crossing varieties without boundary, but the theory extends to normal crossing pairs in an obvious way). Finally, there is a canonical residue map (e.g., induced via (9.0.1) by ordinary residues on $\mathscr{S}$)

$$\mathrm{Res} : \Omega^1_{S_Y/k}(\log B)|_{B_i} \to \mathscr{O}_{B_i}. \tag{9.0.4}$$

Together with (9.0.2), this gives a canonically split inclusion

$$V_n \subset H^0\big(S, \Omega^1_S(\log B)\big). \tag{9.0.5}$$

Definition 9.1
Let (S, B) be a pair of a normal variety S with a reduced Weil divisor $B \subset S$. Assume for an open subset $i : U \subset S$ with complement of codimension at least two that U is nonsingular and $B|_U$ has normal crossings. Define

$$\Omega^1_{S/k}(\log B) := i_*\big(\Omega^1_{U/k}(\log B|_U)\big).$$

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