VARIATIONS OF HODGE STRUCTURES - II

EDUARDO CATTANI

X → S a family of smooth complex manifolds (submersion, proper map).

0.1. Theorem (Ehresmann). f is a $C^\infty$ fibration.

Proof. Pick $s_0 \in S$. Think about S as a small disk. Let $X_{s_0}$ is a fiber. Locally $f$ is a projection. We can lift any vector field $v$ from the base to the vector field $\hat{v}$ on $X$. Can start the lift at any point of $X_{s_0}$. By compactness we can do it in the neighborhood of $s_0$. So the flow of $\hat{v}$ for $|s| < \delta$ gives the diffeomorphism $\phi_v : X_{s_0} \simeq X_s$ (that depends on the vector field $v$).

The lift depends on many choices. Let $\hat{v}_1, \hat{v}_2$ be two lifts of the same vector field.

Then $df(\hat{v}_1 - \hat{v}_2) = 0$. This gives the map $T_{s_0}(S) \to C^1(X_{s_0}, T^1_{X_{s_0}})$ (the connecting homomorphism in the relative tangent sequence) and the canonical Kodaira-Spencer map

$$K_v : T_{s_0}S \to H^1(X_{s_0}, T^1_{X_{s_0}}).$$

Another use of lifting these vector fields: given a curve $\gamma : [0, 1] \to S, \gamma(0) = s_0$, we get a map $\phi^* : X_{s_0} \to X_{\gamma(1)}$ that depends on many choices. But in fact it is well-defined in cohomology:

$$[\phi^*]^{-1} : H^k(X_{\gamma(1)}, \mathbb{Z}) \to H^k(X_{s_0}, \mathbb{Z}).$$

It only depends on a homotopy class of $\gamma$, so gives a map

$$\pi_1(S, s_0) \to \text{End}_\mathbb{Z}(H^k(X_{s_0}, \mathbb{Z}))$$

(the monodromy representation).

This gives a locally constant sheaf of $\mathbb{Z}$-modules on $Z$. But a locally constant sheaf of $\mathbb{C}$-vector spaces is the same thing as a vector bundle with flat connection. The locally constant sheaf is $H^k := R^k f_* \mathbb{C}$, and the connection is the Gauss–Manin connection $\nabla$.

If $X_s$ is a smooth projective variety (or just a compact Kähler manifold), the cohomology carries a lot of structure. We want to transport these structures along the Gauss–Manin connection.

0.2. Theorem (Hodge decomposition).

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{p,q} = \overline{H^{q,p}}.$$

$H^{p,q}$ are elements of $H^k(X, \mathbb{C})$ that have a harmonic representative of bidegree $(p, q)$ (harmonic relative to the Kähler metric, i.e. killed by the Laplacian).

$H^{p,q}$ is isomorphic to $H^q(X, \Omega^p)$ and also to $H^{p,q}_\partial(X)$ (Dolbeaut complex). In particular, $X^{1,0} \simeq H^0(X, \Omega^1)$. For example, if $\dim X = 1$ then

$$H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

and $H^{1,0}$ are “abelian differentials”. The LHS is independent on the fiber but the decomposition on the RHS depends on the Kähler structure. Fix the basis $\omega_1, \ldots, \omega_g$ of $H^{1,0}$. Classically, fix the basis $a_i, b_j$ in 1-homology $H_1(X, \mathbb{Z})$ and integrate $\omega_i$’s. This gives a matrix of periods $\Lambda = [\int_{a_i} \omega_j, \int_{b_i} \omega_j]$. 

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Griffiths (68): do the same thing for any $X$.
Let $\omega \in H^{1,1} \cap H^2(\mathbb{R})$, $n = \dim X$, $k = n - l$. If $\alpha, \beta \in H^k(X, \mathbb{C})$, consider
$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_X \alpha \cup \beta \cup \omega^l$$

Then morally $\partial^{p,q} Q(\alpha, \alpha) > 0$ if $\alpha \in H^{p,q}$. (More precisely: $\alpha$ should also be in the primitive cohomology $\text{Ker}(L_{1,1})$ ($L_{1,1}$ is a left multiplication by $\omega$). Primitive cohomology if flat if, for example, the Kähler structure on fibers $X_s$ comes from the Kähler structure on the total space $X$).

We have a decomposition of vector bundles $\mathcal{H}^k = \oplus_{p+q=k} H^{p,q}$ but only as $C^\infty$ vector bundles! E.g., take a 3-fold. Then
$$H^1 = H^{3,0}(+) \oplus H^{2,1}(-) \oplus H^{1,2}(+) \oplus H^{0,3}(-),$$
where $\pm$ is the signature of the Hermitian form. $H^{3,0} \oplus H^{1,2}$ is the “Weil’s intermediate Jacobian” but it is not a holomorphic bundle!

Griffiths realized that we should look not at the decomposition but at the filtration associated to this decomposition
$$\mathcal{F}^p = \oplus_{a \geq p, a+b=k} \mathcal{H}^{a,b}.$$

0.3. Theorem (Griffiths). $\mathcal{F}^p$ are holomorphic subbundles.

We are looking for a classifying space for Hodge structures.

0.4. Definition. A Hodge structure is the following datum: $H = H_2 \otimes \mathbb{C}$ (a fixed vector space), a form $Q$ (“polarization”), $k$ (the “weight”). A polarized HS of weight $k$ is the decomposition $H = \oplus_{p+q=k} H^{p,q}$, $H^{p,q} = \overline{H^{q,p}}$, $Q$ has parity $(-1)^k$ such that $Q(H^{p,q}, H^{p', q'}) = 0$ if $q' \neq p$ and $i^{p-q} Q(\alpha, \alpha) > 0$ if $\alpha \in H^{p,q}$.

Now define $F^p = \oplus_{a \geq p} H^{a,b}$. Then
$$H = F^p \oplus \overline{F^{k-p+1}}$$
And conversely, consider the flag $F^k \subset F^{k-1} \subset \ldots \subset F^0 = H$. If (1) is satisfied then it is a Hodge structure with $H^{p,q} = F^p \cap F^{k-p}$.

Define the (closed subset of) flag variety $D^\nu$ of flags with $Q(F^p, F^{k-p+1}) = 0$ and inside it the open subset $D$ of polarized Hodge structures with numbers $h^{p,q}$. Given a family $X \to S$, we have a map $S \to D/(\text{monodromy group})$ induced by the map that sends $s$ to $[s^*]^{-1}(H^{p,q}(X_s))$. This is the general period map.

Differential of the period map. What is the tangent space to the space of flags $\mathcal{F}$? We have bundles $\mathcal{E}^p \to \mathcal{F}$ that to each flag associate $F^p$. Note: $\mathcal{E}_0$ is a constant bundle $H$. Then $T \mathcal{F} \simeq \oplus_p \text{Hom}(\mathcal{E}_p, \mathcal{E}_0/\mathcal{E}_p)$ “with some compatibility condition” (should be lower-triangular matrices).

0.5. Theorem (Griffiths’ Transversality). In fact the image is in
$$\oplus_p \text{Hom}(H^{p,q}, H^{p-1,q+1}).$$

How do you prove this? $H^{p,q} = H^q(X_s, \Omega^p)$. In fact the corresponding component of the direct sum is just a cup product with the Kodaira-Spencer class of $v$ (a vector from $T_{s_0}S$).