This homework is due before the class on Monday October 12. These problems will be discussed during the review section on Monday at 4pm. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. Please make sure that all solutions are complete and accurately written.

There is a "bail-out" provision: you can ask the grader not to grade two of the problems. Please indicate clearly in the beginning of your homework which problems you don’t wish to be graded.

1. Let C be a category. (a) Prove that an identity morphism \( A \to A \) is unique for each object \( A \in \text{Ob}(C) \). (b) Prove that each isomorphism in \( C \) has a unique inverse.

2. Let \( C \) be a category. An object \( X \) of \( C \) is called an initial object (resp. a terminal object) if, for every object \( Y \) of \( C \), there exists a unique morphism \( X \to Y \) (resp. a unique morphism \( Y \to X \)). (a) Decide if the following categories contain initial objects, and if so, describe them: the category of vector spaces, the category of groups, the category of commutative rings (with 1). (b) Prove that a terminal object (if exists) is unique up to a canonical isomorphism (and what exactly does it mean?).

3. Let \((I, \leq)\) be a poset (partially ordered set) and let \( C_I \) be the corresponding category (defined in class). Unwind definitions (i.e. give definitions in terms of the poset, without using any categorical language) of (a) terminal and initial objects in \( C_I \) (if they exist); (b) product and coproduct in \( C_I \) (if they exist).

4. Let \( X \) be a fixed object of a category \( C \). We define a new category \( C/X \) of objects of \( C \) over \( X \) as follows: an object of \( C/X \) is an object \( Y \) of \( C \) along with some morphism \( Y \to X \). In other words, an object of \( C/X \) is an arrow \( Y \to X \). A morphism from \( Y \to X \) to \( Y' \to X \) is a morphism from \( Y \) to \( Y' \) that makes an obvious triangle commutative. Prove that \( C/X \) is indeed a category and that \( 1_X : X \to X \) is its terminal object.

5. In the notation of Problem 3, let \( C_I \) be the category associated with a poset \( I \) and let \( \text{Ab} \) be the category of Abelian groups. A contravariant functor \( C_I \to \text{Ab} \) is called an inverse system of Abelian groups indexed by a partially ordered set \( I \). (a) Reformulate this definition without using categorical language. (b) Consider Abelian groups \( \mathbb{Z}/2^n\mathbb{Z} \) for \( n = 1, 2, \ldots \) and natural homomorphisms \( \mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^m\mathbb{Z} \) for \( n \geq m \). Show that this is an inverse system. (c) Let \( C \) be an arbitrary category. Give a definition of an inverse system of objects in \( C \) indexed by a poset \( I \). Show that (b) is an inverse system of rings.
6. In the notation of Problem 4, fix some inverse system \( F : C_I \to \text{Ab} \). Also, let’s fix an Abelian group \( A \) and consider an inverse system \( F_A : C_I \to \text{Ab} \) defined as follows: \( F_A(i) = A \) for any \( i \in I \) and if \( i \leq j \) then the corresponding morphism \( A \to A \) is the identity. (a) Prove that \( F_A \) is indeed an inverse system. (b) Show that the rule \( A \to F_A \) can be extended to a functor from the category \( \text{Ab} \) to the category of inverse systems \( C_I \to \text{Ab} \) (with natural transformations as morphisms). (c) Unwind definitions to describe what it means to have a natural transformation from \( F_A \) to \( F \) without categorical language.

7. In the notation of Problem 6, an Abelian group \( A \) is called an inverse limit of an inverse system \( F : C_I \to \text{Ab} \) if for any Abelian group \( B \), and for any natural transformation \( F_B \to F \), there exists a unique homomorphism \( B \to A \) such that \( F_B \) factors through \( F_A \). (a) Unwind definitions to describe the inverse limit without categorical language. (b) Show that the inverse system of rings in Problem 5(b) has an inverse limit (called the ring of 2-adic numbers).

8. Let \( F : \text{Sets} \to \text{Sets} \) be a contravariant functor that sends any set \( S \) to the set of subsets of \( S \) and any function \( f : S \to S' \) to a function that sends \( U \subset S' \) to \( f^{-1}(U) \subset S \). (a) Show that \( F \) is representable by a two-element set \( \{0, 1\} \). (b) Describe a contravariant functor representable by a three-element set \( \{0, 1, 2\} \).

9. Let \( F : \text{Ab} \to \text{Sets} \) be an obvious forgetful covariant functor. Is it representable?

10. Let \( V \) be a real vector space. Prove that its complexification \( V_C \) represents the covariant functor \( F : \text{Vect}_C \to \text{Sets} \). Namely, for any complex vector space \( U \), \( F(U) \) is the set of \( \mathbb{R} \)-linear maps \( V \to U_\mathbb{R} \) (where \( U_\mathbb{R} \) is \( U \) considered as a real vector space).

11. Let \( C \) and \( D \) be categories and let \( F : C \to D \) and \( G : D \to C \) be functors. Then \( F \) is called a left adjoint of \( G \) (and \( G \) is called a right adjoint of \( F \)) if, for each pair of objects \( X \in C \) and \( Y \in D \), there exist bijections of sets

\[
\tau_{X,Y} : \text{Mor}_D(F(X), Y) \to \text{Mor}_C(X, G(Y))
\]

that are natural transformations in \( X \) for fixed \( Y \) and in \( Y \) for fixed \( X \). (a) Explain what this last condition means explicitly. (b) Show that complexification \( \text{Vect}_\mathbb{R} \to \text{Vect}_C \) and restriction of scalars \( \text{Vect}_C \to \text{Vect}_\mathbb{R} \) are adjoint functors.

12. Let \( G : \text{Vect}_k \to \text{Sets} \) be a forgetful functor. Describe its left-adjoint.

13. Let \( C \) be a category and let \( X, Y \in \text{Ob}(C) \). Consider representable functors \( C \to \text{Sets} \) given by \( X \) and \( Y \), i.e. \( h_X = \text{Mor}(\cdot, X) \) and \( h_Y = \text{Mor}(\cdot, Y) \). Show that there is a natural bijection between morphisms \( X \to Y \) and natural transformations \( h_X \to h_Y \). More precisely, let \( D \) be a category of functors \( C \to \text{Sets} \) (with natural transformations as morphisms). Show that the rule \( X \to h_X \) extends to a fully-faithful functor \( C \to D \).