Seshadri Constants on Irrational Surfaces

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1 Introduction

Algebraic geometry is the branch of mathematics interested in studying the vanishing of sets of polynomials, which we call algebraic varieties. Many other tools have been developed to capture additional data, such as divisors, which are used to organize the zeros and poles of meromorphic functions. This leads to a fundamental invariant: the cone of effective divisors. The effective cone is a set in Euclidean space defined using divisors, which carries useful information about the variety. To understand the effective cone, it suffices to describe so-called extremal rays; the effective cone is determined by these, in the same way that a pyramid is determined by its edges.

An *interpolation problem* is a game where you want to find some geometric object that passes through a collection of sufficiently general points with certain multiplicities. Such questions date all the way back to the beginning, when Euclid postulated that there is a line that passes through any two points. For being such a fundamental question, we know very little about interpolation in general. We have known what happens in the one dimensional case for over two hundred years through Lagrangian polynomials. However, past the most basic case we are in the dark. In $\mathbb{P}^2$, we have a good prediction of what will happen through Nagata's Conjecture, although we are very far from being able to prove it. In investigating interpolation on more complicated varieties, an interesting problem is computing *Seshadri constants*, which are the maximum multiplicity an irreducible curve can have at a given general point. We call the curves that pass through a point with maximal multiplicity *Seshadri exceptional curves* [5]. These definitions will be made precise in the next section. There is also Gromov-Witten theory, which comes from studying mirror symmetry in string theory, can also be used to investigate such questions by counting the number of curves imposed with tangency conditions. Techniques from interpolation are often more versatile though, and can be used when Gromov-Witten can't.

In the paper *Seshadri constants on elliptic ruled surfaces* by Luis García Fuentes [2], we see that on the surface S_{-1}, which is defined as the indecomposable elliptic ruled surface with invariant $e = -1$ and is isomorphic to the symmetric square of an elliptic curve, there exists a Seshadri exceptional curve $\mathrm{C}_n \sim n(n+1)X_0 - 2nf$ with multiplicity $2n^2 - 1$ at some sufficiently general point, for any n. However, the proof that these curves exist essentially just shows that there is a divisor with more global sections than the number of conditions necessary to impose a high multiplicity at a point, and does not tell us anything about what these curves are or what other properties they may have.

In interpolation problems, we not only want to know that there exists some object meeting the conditions that we impose, but we want to understand what they are. I want to understand what these special curves are and what other interesting properties they may have. In particular, I am interested in the surface S given by

$$\pi : S = Bl_x S_{-1} \to S_{-1}$$

where $x \in S_{-1}$ is the sufficiently general point that the curves C_n are based at. The main theorem to be proved is as follows:

Theorem 1.1. *The convex cone of effective divisors on S, $\overline{Eff}(S)$, is generated by the curves*

$$\{\widetilde{C_g} \sim g(g+1)X_0 - 2gf + (2g^2-1)E\}_{g>0}$$

where $\widetilde{C_g}$ is a smooth genus g curve, for each g.

This would give us an exceptionally beautiful effective cone for S_{-1}, since it would be generated by infinitely many curves C_0, C_1, ... where each C_g is a genus g curve.

In addition to simply finding the geometry object that satisfies an interpolation problem, geometers usually want to find out how many such objects exist and the relations between them. In the paper referenced above, Fuentes never mentions the base elliptic curve for S_{-1} or how it affects the C_n curves. These C_n can be seen to have two parameters, the j-invariant of the base elliptic curve and the point p where we impose high multiplicity. I am interested in varying these parameters and studying the resulting surface cut out in $\mathcal{M}_n$, the moduli space of genus n curves. That is, I want to see how changing the base elliptic curve up to isomorphism and the point that has high multiplicity affects the resulting curves C_n up to isomorphism. This leads to my first two questions:

(Q1): *Are the Seshadri exceptional curves, C_n, on the elliptic ruled surface S_{-1}*

smooth after blowing up the point of high multiplicity?

(Q2): *Assuming these curves are smooth after the blow up, how does varying the j-invariant of our base elliptic curve and point of high multiplicity change the resulting C_n.*

After answering (Q2) it is natural to consider what happens when constructing a surface similar to S_{-1} over nodal and cuspidal cubics so that we can extend our results to $\overline{\mathcal{M}}_g$, the compactification of $\mathcal{M}_g$, which also includes singular cubics. Since $S_{-1} \cong \mathrm{Sym}^2E$, we should consider the surface Sym^2C where C is a singular cubic. This leads to the question:

(Q3): *What is the surface Sym^2C where C is a singular cubic curve. What are the Seshadri exceptional curves C_n on this surface? Are these curves smooth? How are they related to our results on the smooth elliptic ruled surface?*

These questions posed after Theorem 1.1 remain unanswered and are open for further investigation.

2 Introduction to Divisors

The study of divisors provides a lot of information about algebraic varieties that we are interested and are essential in the study of algebraic geometry. We begin by defining what a (Weil) divisor is. Let X be a variety throughout this section.

Definition 2.1. *A* ***prime divisor*** *on X is an irreducible one dimension subspace.*

Definition 2.2. *Let Div(X) be the free Abelian group generated by all prime divisors on X. If $D \in Div(X)$, then we call D a* ***(Weil) divisor****. That is, $D = \sum n_i D_i$ where $n_i \in \mathbb{Z}$ and D_i is an irreducible one dimensional subspace.*

Definition 2.3. *We call a divisor $D = \sum n_i D_i$ an* ***effective divisor*** *if $n_i \geq 0$ for all i.*

Definition 2.4. *We say that $D \in Div(X)$ is a* ***principal divisor*** *if it is a divisor of zeros and poles of a rational function on X.*

Definition 2.5. *Let $D_1, D_2 \in Div(X)$. We say that D_1 is* ***linearly equivalent*** *to D_2, denoted $D_1 \sim D_2$, if $D_1 - D_2$ is a principal divisor. Notice that linear equivalence is and equivalence relation.*

Example 2.1. *Consider $X = \mathbb{P}^2$. Then, if $D \in Div(X)$ we have that $D \sim dL$ where L is a line in $\mathbb{P}^2$, and D is a degree d curve. This is because if D and D' are any two degree d curves, then they are given by the vanishing of f(x,y,z) and f'(x,y,z) two homogeneous degree d functions. We then have that $\frac{f(x,y,z)}{f'(x,y,z)}$ is a rational function on $\mathbb{P}^2$ with zeroes being the zero locus of f(x,y,z), which is D, and poles being the zero locus of f'(x,y,z), which is D'. Thus D-D' is principal, so D and D' are linear equivalent.*

Definition 2.6. *We define the* ***complete linear system*** *of a divisor D, denoted $|D|$ as follows:*

$$|D| = \{D' \in Div(X) | D' \text{ is effective and } D \sim D'\}$$

Example 2.2. *Let $\pi : S = Bl_3\mathbb{P}^2 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at 3 sufficiently general points $p_1, p_2,$ and p_3. Let $E_1, E_2,$ and E_3 be the exceptional divisors on S for the 3 pointes respectively. Let L be the preimage of general line in $\mathbb{P}^2$. Then every divisor* *on S is linearly equivalent to a divisor of the form*

$$dL - m_1E_1 - m_2E_2 - m_3E_3$$

To see this, take a degree d curve on $\mathbb{P}^2$, passing through points p_1, p_2, p_3 with multiplicities m_1, m_2, m_3 respectively. Since $dL \sim C$ and L does not pass through any of the 3 points, we also have

$$dL \sim \pi^{-1}(C) = \tilde{C} + m_1E_1 + m_2E_2 + m_3E_3$$

where $\tilde{C} = \pi^{-1}(C \setminus \{p_1, p_2, p_3\})$. Then by the definition of linear equivalence,

$$dL - \tilde{C} - m_1E_1 - m_2E_2 - m_3E_3$$

is a principal divisor. Rearranging the terms, we have:

$$(dL - m_1E_1 - m_2E_2 - m_3E_3) - \tilde{C}$$

$$\implies \tilde{C} \sim dL - m_1E_1 - m_2E_2 - m_3E_3$$

Now consider a degree d curve C' passing through points p_1, p_2, p_3 with multiplicities $m_1' \geq m_1, m_2' \geq m_2, m_3' \geq m_3$. Then from the above discussion we have

$$\tilde{C}' \sim dL - m_1'E_1 - m_2'E_2 - m_3'E_3$$

We can then see that

$$\tilde{C}' + (m_1' - m_1)E_1 + (m_2' - m_2)E_2 + (m_3' - m_3)E_3 \in |dL - m_1E_1 - m_2E_2 - m_3E_3|$$

Since this divisor is effective by the way we chose C', and we get

$$dL - m_1E_1 - m_2E_2 - m_3E_3$$

when we substitute $\tilde{C}'$ for its linear equivalence. This is a complete description of the linear system $|dL - m_1E_1 - m_2E_2 - m_3E_3|$.

Now, if D is a divisor, we define

$$\mathcal{L}(D) = \{f \in k(S)|(f) + D \text{ is effective } \}$$

We have $h^0(D) = dim\mathcal{L}(D)$. If $\mathcal{L}(D)$ has a basis $f_1(x), ..., f_l(x)$ then we can define a rational map for the divisor D as follows:

$$\varphi_D : X \dashrightarrow \mathbb{P}^{h^0(D)-1} \text{ by } x \mapsto [f_1(x) : ... : f_l(x)]$$

Definition 2.7. *Let D be a divisor. If φ_D is an embedding, then we call D* ***very ample****. If $\varphi_m D$ is an embedding for some integer m, then we call D* ***ample****. If φ_D is a morephism, then we call D* ***globally generated****.*

Now let S be a smooth projective surface. The equivalence classes of Div(S) under linear equivalence form a group.

Definition 2.8. *We call the group of linear equivalence classes the* ***divisor class***

group *denoted Cl(S). That is,*

$$Cl(S) = \frac{Div(S)}{\sim}$$

We can then define the following product on Cl(S), known as the *intersection product*: Let D_1 and D_2 be curves with no common components. Then

$$D_1 \cdot D_2 = \text{ the number of points in } D_1 \cap D_2 \text{ counting multiplicity} \in \mathbb{Z}$$

Theorem 2.1. *The intersection product on Cl(S) gives a well defined inner product.*

Example 2.3. *Consider $\mathbb{P}^2$. Earlier, we showed that all degree d curves are in the same linear equivalence classes. Thus,*

$$Cl(\mathbb{P}^2) = \mathbb{Z}$$

Consider L a line in $\mathbb{P}^2$. Then we have $L \cdot L = 1$ since $L \sim L'$ for any other line L', and $L \cdot L = L \cdot L' = 1$. Then if C is a degree d curve and C' is a degree d' curve,

$$C \cdot C' = (dL) \cdot (d'L) = (dd')(L \cdot L) = dd'$$

Theorem 2.2.

Definition 2.9. *We say divisors D_1 and D_2 are* ***numerical equivalent*** *if $D_1 \cdot C = D_2 \cdot C$ for all curves C.*

Definition 2.10. *We say a divisor D is a* ***nef divisor*** *if $D \cdot C \geq 0$ for all curves C.*

We can then consider the quotient group resulting from quotienting Cl(S) by numerical equivalence. The resulting group is a free Abelian group of rank ρ, which is called the *Neron-Severi group* and is denoted by NS(S). We call the rank of this group, ρ, the *Picard number* of the surface S. Note that if a surface has Picard number $\rho = 1$ then numerical equivalence and linear equivalence are the same. We can turn NS(S) into a ρ-dimesnional real vector space as follows:

$$NS_{\mathbb{R}}(S) = NS(S) \otimes \mathbb{R}$$

Now we will define the nef and effective cones, which are convex cones of divisors and provide a lot of information about the surface of interest. We define the *effective cone on S* denoted Eff(S) as:

$$Eff(S) = \{D \in NS(S) | D \equiv D' \text{ for some } D' \text{ effective divisor}\} \subset NS(S)$$

and the *pseudo effective cone* as

$$\overline{Eff}(S) = \overline{Cone(Eff(S))} \subset NS_{\mathbb{R}}(S).$$

We now define the *nef cone* as

$$Nef(S) = \{D | D \cdot C \geq 0 \forall C\} \subset \overline{Eff}(S)$$

Theorem 2.3. *If S is a surface, then $\overline{Eff}(s)$ is the dual cone (in the sense of convex geometry) of $Nef(S)$ the convex Nef cone.*

3 Introduction to Seshadri Constants

Seshadri Constants were first introduced by Demailly in an attempt to use them to prove the Fujita Conjecture [1]. Although this feat was unsuccessful, they became interesting in their own right because of their use in measuring the local positivity of ample line bundles on algebraic varieties [5]. In general, Seshadri constants are very hard to compute. It is even unknown whether or not they are always rational; there are currently no known examples of irrational Seshadri Constants, but there are very few examples where the Seshadri constants are known at all. Recall the following theorem seen in Hartshorn [4]:

Theorem 3.1 (Seshadri's Criterion)**.** *Let X be a smooth projective variety and L be a line bundle on X. Then L is ample if and only if there exists a positive number ϵ such that for all points x on X and all (irreducible) curves C passing through x one has*

$$L \cdot C \geq \epsilon \cdot mult_x C$$

It is natural to ask what the optimal ϵ is. This question is what led to the following definition:

Definition 3.1. *Let S be a smooth surface. Let A be a nef divisor on S. Let $x \in S$ any point. We define the* ***Seshadri Constant of A at x*** *as:*

$$\epsilon(A, x) \coloneqq inf_{C \ni x} \left\{ \frac{A \cdot C}{mult_x(C)} \middle| C \text{ irred curve passing through } x \right\}$$

We can see that this is equivalent to the following alternative definition:

Definition 3.2. *Let S be a smooth surface. Let A be a nef divisor on S. Let*

*$f : Bl_xS = \tilde{S} \to S$ be blow up of S at the point $x \in S$. We define the **Seshadri Constant of A at x** as:*

$$\epsilon(A,x) \coloneqq sup\{\epsilon \in \mathbb{R} \mid f^\star A - \epsilon E \text{ is nef }\}$$

We have the following easy upper bound for Seshadri constants:

Lemma 3.2. *For a smooth surface S, nef divisor A on S, and point $x \in S$ we have*

$$\epsilon(A,x) \leq \sqrt{A^2}$$

Proof. By definition 3.2 we know that $f^\star A - \epsilon(A,x)E$ is nef, so we have that

$$(f^\star A - \epsilon(A,x)E)^2 \geq 0$$

By simple intersection theory we then have

$$A^2 - \epsilon(A,x)^2 \geq 0 \implies \sqrt{A^2} \geq \epsilon(A,x) \quad \square$$

$\square$

When this upper bound is not reached we know that there must exist a curve C such that

$$\epsilon(A,x) = \frac{A \cdot C}{mult_x C}$$

We will call such a curve a *Seshadri Exceptional Curve.* In particular the curve

C will satisfy

$$C^2 < (mult_x C)^2$$

This leads us to the following definition:

Definition 3.3. *An irreducible curve C passing through a point $x \in S$ with multiplicity $m \geq 1$ that satisfies $C^2 < m^2$ is called a* ***Seshadri exceptional curve based at x****.*

Given a Seshadri exceptional curve C we can then define the following continuous map:

$$q_C : Nef(S) \to \mathbb{R} \quad \text{by} \quad q_C(A,x) = \frac{A \cdot C}{mult_x C}$$

This discussion leads us to a third equivalent definition of a Seshadri constant:

Definition 3.4. *Let S be a smooth surface. Let A be a nef divisor on S. Let $x \in S$ any point. We define the* ***Seshadri Constant of A at x*** *as:*

$$\epsilon(A,x) \coloneqq min\{\{q_C(A,x) \mid C \text{ is a Seshadri exceptional curve at } x\} \cup \{\sqrt{A^2}\}\}$$

Definition 3.5. *Let C be a Seshadri exceptional curved based at x. We can then consider the open set of Nef divisors A satisfying*

$$q_C(A,x) < \sqrt{A^2}$$

This set is called the ***influence area of C****, and is denoted by Q_C.*

Lemma 3.3. *If C is a Seshadri exceptional curve based at x, then C is the unique Seshadri exceptional curve in $\overline{Q_C}$*

Proof. First we will show that $C \in Q_C$. Since C is Seshadri exceptional we know

$$C^2 < m^2 \implies \sqrt{C^2} < m \implies \frac{1}{m} < \frac{1}{\sqrt{C^2}}$$

so it follows that

$$q_C(C,x) = \frac{C^2}{m} < \frac{C^2}{\sqrt{C^2}} = \sqrt{C^2}$$

so $C \in Q_C$. Now suppose that there exists another Seshadri exceptional curve D that is different from C such that $D \in \overline{Q_C}$. Then we will have

$$\frac{D \cdot C}{mult_x C} < \sqrt{D^2} \implies D \cdot C < mult_x C \cdot \sqrt{D^2}$$

However, since D is a Seshadri exceptional curve, we have that $\sqrt{D^2} < mult_x D$ giving us

$$D \cdot C < mult_x C \cdot mult_x C$$

which is impossible since and C and D are distinct curves. □

Corollary 3.1. *If C is a nef Seshadri exceptional curve based at x, then*

$$\epsilon(C,x) = q_C(C,x) = \frac{C^2}{mult_x C}$$

4 Finding equations of curves C_n

Recall from the introduction, we would like to prove theorem 1.1 regarding the curves

$$C_n \sim 2n(n+1)X_0 - 2nf$$

where X_0 is a section on S_{-1} and f is a fiber. To show that the curves C_n are smooth, I begin by using Macaulay2 to explicitly find equations for the curves for small n.

I have broken down the problem of constructing explicit equations for the Seshadri exceptional curves into several subproblems:

1. Find the homogenous coordinate ring $R(S_{-1}) := k[x_0, ..., x_n]/I(S_{-1})$ [4]

2. Construct the linear system $\mathcal{L}(2n(n+1)X_0 - 2nf)$ in Macaulay2

3. Find the equation of the Seshadri exceptional curves C_n

To construct equations for the curves C_n I simply need to find the generators of the linear system $\mathcal{L}(2n(n+1)X_0 - 2nf)$. The idea is to find a way of representing the linear system $\mathcal{L}(2n(n+1)X_0 - 2nf)$ as a vector space of polynomials in $R(S_{-1})$, and then finding a linear combination of the generating polynomials which has multiplicity $2n^2 - 1$ at some general point.

4.1 Finding the homogenous coordinate ring of S_{-1}

In finding $R(S_{-1})$ it is useful to use the following description of S_{-1}, where E is an elliptic curve:

$$S_{-1} \simeq \mathrm{Sym}^2 E \simeq {}^{E \times E} \diagup_{(P,Q) \sim (Q,P)}$$

If we consider this quotient as the quotient by the S_2 action on $E \times E$ that maps $(P,Q) \mapsto (Q,P)$, then we have:

$$R(S_{-1}) \simeq R(E \times E)^{S_2}.$$

We can consider $E \times E$ as being embedded in $\mathbf{P}^2_{[x_1:y_1:z_1]} \times \mathbf{P}^2_{[x_2:y_2:z_2]}$. So

$$R(E \times E) \simeq {k[x_1 : y_1 : z_1; x_2 : y_2 : z_2]} \diagup {(F_1, F_2)}$$

where

$$F_1(x_1, y_1, z_1, x_2, y_2, z_2) = y_1^2 z_1 - x_1^3 - a x_1 z_1^2 - b z_1^3;$$

$$F_2(x_1, y_1, z_1, x_2, y_2, z_2) = y_2^2 z_2 - x_2^3 - a x_2 z_2^2 - b z_2^3.$$

Now the S_2 action on $E \times E$ is realized in $R(E \times E)$ by

$$[x_1 : y_1 : z_1; x_2 : y_2 : z_2] \mapsto [x_2 : y_2 : z_2; x_1 : y_1 : z_1].$$

Thus $R(E \times E)^{S_2}$ is exactly the set of polynomials in ${k[x_1 : y_1 : z_1; x_2 : y_2 : z_2]} \diagup {(F_1, F_2)}$ that are bihomogenous and multisymmetric. Since

$$x_1 x_2, \ \ y_1 y_2, \ \ z_1 z_2, \ \ x_1 y_2 + y_1 x_2, \ \ x_1 z_2 + z_1 x_2, \ \ y_1 z_2 + z_2 y_2$$

generate all such polynomials in $k[x_1 : y_1 : z_1; x_2 : y_2 : z_2]$ let's consider the ring map

$$\varphi : k[e_1, e_2, e_3, e_4, e_5, e_6] \to {k[x_1 : y_1 : z_1; x_2 : y_2 : z_2]} \diagup {(F_1, F_2)}$$

where each e_i maps to one of the six generators listed above. Then we have

$$R(S_{-1}) = R(E \times E)^{S_2} \simeq {k[e_1, e_2, e_3, e_4, e_5, e_6]} \diagup {ker\, \varphi}$$

since $ker\,\varphi$ is the ideal consisting of all relations between the generators and the preimages of F_1 and F_2.

Macaulay2 code for generating $R(S_{-1})$:

```
P2xP2= QQ[x,y,z,x',y',z']; S = QQ[f1,f2,f3,f4,f5,f6];
e1=x*x'; e2=y*y'; e3=z*z'; e4=x*y'+y*x';
e5=x*z'+z*x'; e6=y*z'+z*y';
F1 = z*y^2-x^3-x*z^2-z^3; F2 = z'*y'^2-x'^3-x'*z'^2-z'^3;
ExEinP2xP2 = P2xP2/ideal(F1,F2);
F = map(ExEinP2xP2, S, {e1,e2,e3,e4,e5,e6});
kerF = trim kernel F;
Sym2E = S/kerF;
```

Lemma 4.1. *The embedding described above to embed S_{-1} into $\mathbb{P}^5$ is given by the divisor $|3X_0|$ on S_{-1}. That is, it is the map*

$$\varphi_{3X_0} : S_{-1} \dashrightarrow \mathbb{P}^5$$

4.2 Constructing the linear system $\mathcal{L}(2n(n+1)X_0 - 2nf)$

Recall that S_{-1} is an elliptic ruled surface given by

$$\sigma : {}^{E\times E}\!\big/_{(P,Q)\sim(Q,P)} \to E\,; \qquad (P,Q) \mapsto P+Q.$$

I want to construct the linear system $\mathcal{L}(2n(n+1)X_0 - 2nf)$ in Macaulay2, where X_0 is the section of minimum self intersection and f is a fiber of σ that is invariant under the involution $(P,Q) \mapsto (Q,P)$.

it is clear that the fiber above "∞" will be given by the ideal $(y_1 + y_2)$, since $P + Q =$ "∞" $\iff$ P and Q are reflections of each other about the y-axis. This ideal is not defined in $R(S_{-1})$ since it is not bihomogenous. However, notice that if P and Q are points of E that are reflections of each other across the y-axis, then we also have that $x_1 = x_2$. Thus, we can fix the issue of $(y_1 + y_2)$ not being in our ring by instead using the ideal $(x_2y_1 + x_1y_2)$ which is in $R(S_{-1})$, and give the same locus of S_{-1}. However, this gives a reducible divisor, so we will use the irreducible component that corresponds to the desired divisor, which in our case is the one with self intersection 0, since one of the two irreducible components does not.

We can let X_0 be the set of points (P,Q) where $P =$ "∞" or $Q =$ "∞", so it is given by the ideal $(x_1x_2, x_1z_2, z_1x_2, z_1z_2)$ since all four of these polynomials are zero $\iff$ $P = [0 : 1 : 0]$ or $Q = [0 : 1 : 0]$. I will now outline a method of constructing linear systems in Macaulay2.

Constructing linear systems in Macaulay2 using the Divisor package

In the "Divisor" package provided by Macaulay2 [**?**], there is a function called *mapToProjectiveSpace*, which computes the map to projective space associated with the global sections of a Cartier divisor. This function outputs the explicit equations of the map, which we know also forms a basis of the linear system. All divisors on S_{-1} are Cartier divisors since it is smooth, so this method will produce the desired basis. The following code finds a basis for the linear system:

```
i16 : loadPackage "Divisor";
i17 : f = divisor ideal(f4)
o17 = 1*Div(f6, f4, 2*f2*f3+2*f3^2+f1*f5+f3*f5, 4*f1*f3-f5^2,
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```
        4*f1^2+2*f2*f5+2*f3*f5+f5^2) + 1*Div(f4, f2*f3-f3^2+f1*f5,
        f2^2+3*f3^2-3*f1*f5+f2*f5-f6^2, f1*f2-f1*f3+f1*f5+f5^2,
        f1^2-f1*f3-f3*f5) of Sym2E
o17 : WDiv
        //notice that the above is a reducible divisor.
        //We will let f be the first of the two, which is the desired one.
i17 : f = divisor ideal(f6, f4, 2*f2*f3+2*f3^2+f1*f5+f3*f5, 4*f1*f3-f5^2,
        4*f1^2+2*f2*f5+2*f3*f5+f5^2);
o17 = 1*Div(f6, f4, 2*f2*f3+2*f3^2+f1*f5+f3*f5, 4*f1*f3-f5^2,
        4*f1^2+2*f2*f5+2*f3*f5+f5^2) of Sym2E
o17 : WDiv
i18 : X = divisor ideal(f5,f3,f1,f4^3-f2^2*f6+f4*f6^2+f6^3)
o18 = 1*Div(f5, f3, f1) of Sym2E
o18 : WDiv
i19 : D = X + bf
o19 = 1*Div(f5, f3, f1) + 1*Div(f4, f2*f3-f3^2+f1*f5,
        f2^2+3*f3^2-3*f1*f5+f2*f5-f6^2, f1*f2-f1*f3+f1*f5+f5^2,
        f1^2-f1*f3-f3*f5) of Sym2E
o19 : WDiv
i20 : mapToProjectiveSpace D
o20 = map(Sym2E,QQ[Y1 , Y2 , Y3],
         {f4*f5 - 2f1*f6, 2f3*f4 - f5*f6, 4f1*f3 - f5^2})
o20 : RingMap Sym2E <--- QQ[Y1 , Y2 , Y3]
```

This method produces a dimension 3 vector space, as desired. As a sanity check, we can do Riemann-Roch calculations to confirm that it has the correct dimension for other cases as well.

Another method of constructing linear systems: We can use lemma 1.1 to assist us in constructing linear systems. Since our embedding is given by φ_{3X_0} we can use this to construct any linear system that has nX_0 where $3|n$. That is, if our linear system is $\mathcal{L}(3kX_0 - af - mE)$ then we can start by finding $\mathcal{L}(3kX_0)$ and imposing restrictions on the system of polynomials. Now since our embedding is φ_{3X_0}, the linear system $\mathcal{L}(3kX_0)$ is then just all degree k hypersurfaces in $R(S_{-1})$, or, is generated by all degree k monomials in 6 variables modulo $I(S_{-1})$. We can then, using the method outlined in the next section, find a basis for the vector subspace which contains $I(f)^a$ and $I(p)^{m-a}$. This subspace is exactly the linear system $\mathcal{L}(3kX_0 - af - mE)$. Since the linear systems we are working with are described by $\mathcal{L}(2n(n+1)X_0 - 2nf - (2n^2-1)E)$, this method will work when $3|2n(n+1)$, or when $n \equiv 0, 2 (mod 3)$. One issue that may restrict the usage of this method is the fact that our initial linear system $\mathcal{L}(3kX_0)$ will contain $\binom{k+5}{5}$, which will be too large to work for larger n.

4.3 Finding Seshadri exceptional curves

Once we have a method for finding the basis of the vector space $\mathcal{L}(aX_0 + \mathfrak{b}f)$ we can finally find equations for the Seshadri exceptional curves C_n. We know that $C_n \equiv 2n(n+1)X_0 - 2nf$, and on elliptic ruled surfaces numerical equivalence is the same as taking a point on another fiber. However this is equivalent to fixing a point on our fiber and taking divisors linear equivalent to $2n(n+1)X_0 - 2nf$. We know that a divisor is linear equivalent to our divisor if it is some linear combination of the generators for the vector space $\mathcal{L}(2n(n+1)X_0 - 2nf)$, and a linear equivalent divisor is the Seshadri exceptional curve C_n if it has multiplicity $2n^2 - 1$ at a some general point on the fiber. i.e. if x is some general point and $f_1, ..., f_k$ are the generators of $|2n(n+1)X_0 - 2nf|$ then we want to find $a_1, ...a_n$ constants such that $a_1 f_1 + ... + a_n f_n (\text{mod I}(x)^{2n-1}) \equiv 0$ in $R(S_{-1})$.

Finding the constants a_i is equivalent to finding the null space of the matrix of coefficient vectors for $f_i (\text{mod I}(p)^{2n-1})$. Once we have found the linear combination which gives this curve of high multiplicity in the linear system we have found an equation for our C_n curve, however this equation we get will not be an irreducible curve. It will reduce to a union of some powers of X_0 and f along with C_n, so we must decompose it to find the equation for the irreducible C_n curve that we are looking for. I currently do not have a good way of doing this because the decompose function for ideals in Macaulay2 is too computationally expensive.

4.4 Issues with methods

- Takes really long time to compute linear systems, currently cannot compute any for $n > 3$

5 Low genus curves

5.1 Genus 1 curves

From using the method outlined in section 1 we have found an equation for C_1 confirming that C_1 is a smooth curve. Thus from the above result, C_1 has genus 1 and is therefore an elliptic curve.

Lemma 5.1. *The C_1 curves are given by a pencil of elliptic curves which are isomorphic to the elliptic curve that S_{-1} is ruled over.*

From [2] we know that

$$C_1 \sim 4X_0 - 2f.$$

From a simple Riemann-Roch calculation we have that $h^0(S_{-1}, 4X_0 - 2f) = 2$ so this divisor gives a map to $\mathbb{P}^1$.

Lemma 5.2. *The map $\phi : S_{-1} \to \mathbb{P}^1$ given by the linear system $|4X_0 - 2f|$ is given by $(P, Q) \mapsto \pm(P - Q)$ where $(P, Q) \in Sym^2 E$ and "$-$" is the inverse of the group operation on E.*

5.2 Genus 2 curves

From the paper we know that $C_2 \sim 12X_0 - 4f$ and has multiplicity 7 at some point. I currently have 2 approaches in progress for finding these curves:

First approach When using the divisor package mentioned above along with the linear algebra method for finding the curve of high multiplicity, I am able to find an ideal which contains C_2, however, it takes too long to decompose the ideal so I can not find what the irreducible components are. I do know that one of the irreducible components will be the fiber, and will have multiplicity four. I can then use the saturation method to remove the four fibers from the ideal, which results in the ideal of an irreducible curve. This curve is our C_2. However, the ideal for the curve is extremely large so Macaulay cannot tell us if it is smooth. We can work around this by showing that the minors of the Jacobian of our ideal intersect to give the entire ring. That is, we can find the determinant of a few minors and compute their intersection and hope that we get (1). These computations also take a very long time and I have, so far, been unable to show that C_2 is smooth.

Second approach Note that the C_2 curve is a hyperelliptic curve of genus 2. We have the short exact sequence of sheaves:

$$0 \to \mathcal{O}_{BlS_{-1}}(K_{BlS_{-1}}) \to \mathcal{O}_{BlS_{-1}}(K_{BlS_{-1}} - C_2) \to \mathcal{O}_{C_2}(K_{C_2}) \to 0$$

Which then gives us the long exact sequence of cohomology groups:

$$0 \to H^0(BlS_{-1}, K_{BlS_{-1}}) \to H^0(BlS_{-1}, K_{BlS_{-1}}{+}C_2) \to H^0(C_2, K_{C_2}) \to H^1(BlS_{-1}, K_{BlS_{-1}}) \to \ldots$$

We know that $h^0(C_2, K_{C_2}) = 2$ since $|K_{C_2}|$ gives a double cover of $\mathbb{P}^1$. Further, from looking at the ramified points of this map we can reverse engineer the hyperelliptic curve C_2. From a Macaulay2 calculation we know $h^0(BlS_{-1}, K_{BlS_{-1}} + C_2) = 2$, so from the long exact sequence we get that these two vector spaces are isomorphic. Thus, to understand the map $|K_{C_2}|$ we only need to understand $|K_{BlS_{-1}} + C_2|$ restricted to

C_2.

5.3 Genus n curves

Lemma 5.3. *Let $\widetilde{C_n} \subset Bl_p(S_{-1})$ be the proper transform of C_n. Then $\widetilde{C_n}$ is a genus n curve.*

Proof. By the above lemma, blowing up S_{-1} at x will resolve the singularity so $\widetilde{C_n}$ is a smooth curve. We can then apply Riemann-Roch, thus we can use the genus formula

$$2g-2 = \widetilde{C_n}.(\widetilde{C_n} + K_{Bl_p(S_{-1})})$$

where g is the genus of $\widetilde{C_n}$ and $K_{Bl_p(S_{-1})})$ is the canonical divisor on $Bl_p(S_{-1})$. We have

$$\widetilde{C_n} \equiv 2n(n+1)X_0 - 2nf + (2n^2-1)L \quad \text{and} \quad K_{Bl_p(S_{-1})}) \equiv -2X_0 + f - E.$$

Where L is the exceptional divisor. Thus,

$$g = 1 + {}^1\!/_2(4n^2(n+1)^2 - 8n^2(n+1) - (2n^2-1)^2 - 4n(n+1) + 4n + 2n(n+1) + 2n^2 - 1)$$

$$= 1 + {}^1\!/_2(2n-2) = n$$

$\square$

6 The convex cone of effective divisors on S

Theorem 6.1. *The effective cone Eff(S) is generated by the curves $\tilde{C}_n$, where $\tilde{C}_n$ are the proper transforms of the Seshadri exceptional curves C_n on S_{-1}.*

Proof. Suppose C is an irreducible curve on S_{-1} passing through a point x with multiplicity $m > 0$, and $C^2 < m^2$. Then we can define the continuous function

$$q_C : Nef(S) \to \mathbb{R} \quad \text{by} \quad q_C(A, x) = \frac{A \cdot C}{m}$$

We then define Q_C, which we call the influence area of C, to be the open set of nef divisors satisfying

$$q_C(A, x) < \sqrt{A^2}$$

In the case of S_{-1}, we have the curves $C_n \equiv 2n(n+1)X_0 - 2nf$ with $mult_x C_n = 2n^2 - 1$, so $C_n^2 = mult_x C_n - 1$. Thus the map q_{C_n} is defined and gives us the influence area:

$$Q_{C_n} = \{A \equiv aX_0 + bf \in Nef(S_{-1}) | (1 + \frac{1}{n})^2 < \frac{a}{2b + a} < (1 + \frac{1}{n-1})^2\}$$

Recall that the nef cone of S_{-1} is generated by the divisors $2X_0 - f$ and f. We can then rewrite the influence area in terms of linear combinations of these generators as follows:

$$Q_{C_n} = \{A \equiv a(2X_0 - f) + bf \in Nef(S_{-1}) | (1 + \frac{1}{n})^2 < \frac{a}{b} < (1 + \frac{1}{n-1})^2\}$$

Notice that these sets cover all divisors $A \equiv a(2X_0 - f) + bf$ where $\frac{a}{b} > 1$, or where $a > b$. Now we also know that f has multiplicity 1 through any point lying on the fiber, and $f^2 = 0$, so q_f is defined. We can now calculate the influence area of f:

$$f \cdot (2aX_0 + (b-a)f) = 2a \quad \text{and} \quad (2aX_0 + (b-a)f)^2 = 4ab$$

So a nef divisor A satisfies $q_f(A, x) < \sqrt{A^2}$ if $b > a$. That is,

$$Q_f = \{A \equiv a(2X_0 - f) + bf \in Nef(S_{-1}) | \frac{a}{b} < 1\}$$

We now have all nef divisors on S_{-1} covered by one of the above influence areas, except for the divisors $A \equiv 2aX_0$. All nef divisors on S will be of the form $A - mE$. Since we are blowing up at a point away from X_0, $2aX_0 - mE$ will never be nef on S, so this is not a problem. Let A be a nef divisor in the influence area Q_C. We can now define the basins

$$B_C = \{\tilde{A} = A - mE | A \in Q_C \text{ and } m \leq q_C(A)\}$$

I claim that $Nef(S) = (\cup_n B_{C_n}) \cup B_f$. To see this, first notice that any divisor outside of these basins are not nef. For if $A \in Q_C$ and $m > q_C(A)$ then

$$\tilde{A} \cdot \tilde{C} = A \cdot C - m \cdot mult_x C < 0$$

Now Suppose $\tilde{A} = A - mE \in B_C$ is not nef. So there is some divisor $\tilde{D} = D - m_D E$ such that $\tilde{A} \cdot \tilde{D} < 0$. Now $\tilde{A}^2 = A^2 - m^2 > 0 \implies \tilde{A}$ is effective, so $\tilde{A} = \tilde{A}_1 + \tilde{A}_2 + \ldots + \tilde{A}_r$ where $\tilde{A}_i$ are irreducible divisors. Since $\tilde{A} \cdot \tilde{D} < 0$, there is some $\tilde{A}_i = A_i - mult_x A_i E$ which has $\tilde{A}_i \cdot \tilde{D} < 0$. But then $A_i \cdot D < mult_x D \cdot mult_x A_i$, which is impossible unless

$A_i = D$. However, if this were the case, then we would have $D^2 < mult_x D^2$ which means that D is a Seshadri exceptional curve based at x. However, we know that if $\tilde{A}_i \in Q_C$ then $q_D(A_i) > q_C(A_i)$ since $A_i \in Q_C$ if and only if $q_C(A_i) < \sqrt{A_i^2}$, and all of the influence areas are disjoint, so we must then have $q_D(A_i) \geq \sqrt{A_i^2} \implies q_C(A_i) < q_D(A_i)$. From this, we know that

$$\tilde{A}_i \cdot \tilde{D} = A_i \cdot D - mult_x A_i \cdot mult_x D = mult_x D(q_D(A_i) - mult_D) > 0$$

Thus all of the divisors in the basins B_C are nef and these are all of the nef divisors.

Now we know that $Eff(S) = Nef(S)^*$ dual cones. So

$$Eff(S) = \{\phi \in V^* | \phi(x) \geq 0 \ \ \forall x \in Nef(S)\}$$

Since the faces of $Nef(S)$ were given by $\tilde{C}_n \cdot \tilde{A} = 0$, these will be the generators of the dual cone, so $Eff(s)$ is generated by $f, C_1, C_2, C_3, ...$ □

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