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Derived category of moduli of pointed curves. I

Ana-Maria Castravet and Jenia Tevelev

Abstract

This is the first paper in a sequence devoted to the derived category of moduli spaces of curves of genus 0 with marked points. We develop several approaches to describe it equivariantly with respect to the action of the symmetric group permuting marked points. We construct an equivariant full exceptional collection on the Losev–Manin space which categorifies derangements.

1. Introduction

The special feature of moduli spaces of curves with marked points is the action of the symmetric group permuting marked points, and our goal is to exhibit this action in the description of the derived category. One can think about the derived category as an enhanced cohomological invariant. Although there are many papers in the literature computing cohomology of $\overline{M}_{0,n}$, the moduli space of curves with n marked points, as a module over the symmetric group (for example, [Get95, BM13]), the equivariant Euler–Poincaré polynomial is expressed as an alternating sum, which therefore has no obvious categorification. On the other hand, it is often easy to get some description of the derived category which, however, does not respect the group action. For example, it is obvious that $D^b(\overline{M}_{0,n})$ has a full exceptional collection. Indeed, $\overline{M}_{0,n}$ has a Kapranov model as an iterated blow-up of $\mathbb{P}^{n-3}$ in $n-1$ points followed by the blow-up of $\binom{n-1}{2}$ proper transforms of lines connecting points, etc. With a little work, Orlov's theorem on the derived category of the blow-up (see Section 3) gives a full exceptional collection. However, Kapranov's model is not unique: it depends on the choice of the ψ class, that is, the choice of a marking, and therefore this collection is not preserved by S_n (it is preserved only by S_{n-1}). The derived categories of $\overline{M}_{0,n}$ and related Hassett spaces and GIT quotients have been studied in [BFK19] and [MS13], although not from the equivariant perspective.

Question 1.1. Is there a full exceptional S_n-invariant collection on $\overline{M}_{0,n}$?

This question of D. Orlov, communicated to us by A. Kuznetsov, will be investigated in detail in the second paper in the series. Note that a striking and unexpected corollary of its existence is that the K-group $K_0(\overline{M}_{0,n})$ is a *permutation* representation of S_n. As a motivation, one can argue that since $\overline{M}_{0,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$, maybe it is somehow "defined over $\mathbb{F}_1$",

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and therefore the same should be true of its K-theory as an S_n-module, and so perhaps it should be a permutation representation.

In this paper, we suggest two general strategies which may have other applications and provide an answer for the Losev–Manin space [LM00].

One approach, which justifies why we consider the case of Losev–Manin spaces, is based on an equivariant version of Orlov's theorem on blow-ups (Section 2) and inspired by the work of Bergström and Minabe in [BM13].

Let X be a smooth projective variety, and let $Y_1, \dots, Y_n \subseteq X$ be smooth transversal subvarieties of codimension l. For any subset $I \subseteq \{1, \dots, n\}$, we denote the intersection $\cap_{i\in I} Y_i$ by Y_I. In particular, $Y_\emptyset = X$. Let $q\colon \tilde{X} \to X$ be an iterated blow-up of (proper transforms of) $Y_1, \dots, Y_n$. In addition, let G be a finite group acting on X permuting $Y_1, \dots, Y_n$. Then G also acts on $\tilde{X}$, and the morphism q is G-equivariant. Let $G_I \subseteq G$ be a normalizer of Y_I for each subset $I \subseteq \{1, \dots, n\}$ (in particular, $G_\emptyset = G$). We show in Lemma 2.3 that if $D^b(Y_I)$ admits a full G_I-equivariant exceptional collection for every subset I, then $D^b(\tilde{X})$ admits a full G-equivariant exceptional collection.

Next we generalize an inductive computation given in [BM13] of the equivariant Euler–Poincaré polynomial of $\overline{M}_{0,n}$. In the derived category setting, it gives the following theorem. Fix integers $l \geqslant 1$ and $0 \leqslant k \leqslant n$. For a weight

$$\mathbf{a} = \left(1, \dots, 1, \frac{1}{l}, \dots, \frac{1}{l}\right)$$

(with k copies of 1 and $n-k$ copies of $1/l$), let $\overline{M}^n_{k,l}$ be the Hassett moduli space [Has03] of $\mathbf{a}$-weighted stable rational curves. For example, $\overline{M}^n_{0,1} \simeq \overline{M}_{0,n}$, and $\overline{M}^n_{0,\lfloor (n-1)/2 \rfloor}$ is a symmetric GIT quotient $(\mathbb{P}^1)^n /\!\!/ \mathrm{PGL}_2$ if n is odd and its Kirwan resolution if n is even.

Theorem 1.2. *If $\overline{M}^n_{k,r(n,k)}$ admits a full $(S_k \times S_{n-k})$-equivariant exceptional collection for every n and every $0 \leqslant k \leqslant n-3$, then $\overline{M}_{0,n}$ admits a full S_n-equivariant exceptional collection for every n. Here*

$$r(n,k) := \begin{cases} \lfloor (n-1)/2 \rfloor & \text{if } k = 0\,, \\ n-2 & \text{if } k = 1\,, \\ n-k & \text{if } k \geqslant 2\,. \end{cases}$$

Concretely, we need the following spaces:

- the symmetric GIT quotient and its Kirwan resolution, which will be studied in the sequel to this paper
- $\overline{M}^n_{1,n-2}$, which is isomorphic to $\mathbb{P}^{n-3}$ via the Kapranov map (we can take any standard exceptional collection on $\mathbb{P}^{n-3}$, for example $\mathcal{O}, \dots, \mathcal{O}(n-3)$)
- $\overline{M}^n_{2,n-2}$ (this is the Losev–Manin space studied in this paper)
- spaces $\overline{M}^n_{k,n-k}$ for $k > 2$ (these spaces are still too complicated for the calculations of the derived category, and in the sequel to this paper, we will investigate their further equivariant reductions).

We now discuss another strategy, which is the one we will use in this paper for the case of the Losev–Manin spaces $\overline{LM}_n$. We start with an example.

Example 1.3. Unlike $\overline{M}_{0,5}$, which has five Kapranov models and therefore five Orlov-style exceptional collections, the 2-dimensional Losev–Manin space, which we denote by $\overline{LM}_3$ in this paper (see below), has only two non-trivial ψ-classes ψ_0 and ψ_∞, realizing it as $\mathbb{P}^2$ blown-up at three points p_1, p_2, p_3 in two ways, related by the Cremona involution. The corresponding exceptional collection invariant under all automorphisms has three blocks and consists of line bundles

$$\{-\psi_0, -\psi_\infty\}, \quad \{\pi_1^*\mathcal{O}(-1), \pi_2^*\mathcal{O}(-1), \pi_3^*\mathcal{O}(-1)\}, \quad \mathcal{O}, \tag{1.1}$$

where $\pi_i\colon \overline{LM}_3 \to \overline{LM}_2 \simeq \mathbb{P}^1$ is a forgetful map, which can be thought of as a linear projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from the point p_i.

The last four line bundles in (1.1) are pull-backs under forgetful maps, but the first two have a trivial derived pushforward by any forgetful map. To study situations of this sort more systematically, we introduce an inclusion-exclusion principle in triangulated categories (see Lemma 3.6) and its application in the following set-up.

Definition 1.4. Given a collection of morphisms of smooth projective varieties $\pi_i\colon X \to X_i$ for $i \in I$, we call an object $E \in D^b(X)$ *cuspidal*[1] if

$$R\pi_{i*}E = 0 \quad \text{for every } i \in I\,.$$

The *cuspidal block* is the full triangulated subcategory of cuspidal objects

$$D^b_{\text{cusp}}(X) \subset D^b(X)\,.$$

Philosophically, the cuspidal block captures information about the derived category not already encoded in $D^b(X_i)$ for $i \in I$. We show in Theorem 3.5 that under quite general assumptions, $D^b_{\text{cusp}}(X)$ is an admissible subcategory and in fact the first block in the "inclusion–exclusion" semi-orthogonal decomposition of $D^b(X)$. In our applications, morphisms π_i are forgetful maps such as $\overline{M}_{0,n} \to \overline{M}_{0,n-1}$, and thus an S_n-equivariant description of $D^b(X)$ can be reduced to an S_n-equivariant description of $D^b_{\text{cusp}}(X)$.

Question 1.5. Find a full S_n-invariant exceptional collection in the cuspidal block $D^b_{\text{cusp}}(\overline{M}_{0,n})$ with respect to all the forgetful maps $\overline{M}_{0,n} \to \overline{M}_{0,n-1}$.

An answer to Question 1.5 together with Proposition 1.6 (an application of Theorem 3.5) will therefore answer Question 1.1.

Proposition 1.6. *We write $\overline{M}_N \simeq \overline{M}_{0,n}$ for the moduli space of stable rational curves with points marked by any n-element set N. Then $D^b(\overline{M}_N)$ admits a semi-orthogonal decomposition*

$$D^b(\overline{M}_N) = \left\langle D^b_{\text{cusp}}(\overline{M}_N), \{\pi_K^* D^b_{\text{cusp}}(\overline{M}_{N\setminus K})\}_{K\subset N}, \mathcal{O}\right\rangle, \tag{1.2}$$

where K runs over subsets with $1 \leqslant |K| \leqslant n-4$ in the order of increasing cardinality $|K|$ and $\pi_K\colon \overline{M}_N \to \overline{M}_{N\setminus K}$ is the map that forgets markings in K.

We mention the answer to Question 1.5 in the first few small n cases.

Example 1.7. Let $T(-\log)$ be the rank $n-3$ vector bundle on $\overline{M}_{0,n}$ of vector fields tangent to its (normal crossing) boundary divisor. It is easy to deduce from the results of [KT09] that $T(-\log)$ is an exceptional vector bundle and an element of $D^b_{\text{cusp}}(\overline{M}_{0,n})$ for every n. This fact, which we view as a manifestation of rigidity of $\overline{M}_{0,n}$, was one of our original motivations for writing this paper. For small n, the category $D^b_{\text{cusp}}(\overline{M}_{0,n})$ has the following full S_n-equivariant exceptional collection:

[1] The terminology (suggested to us by A. Oblomkov) comes from cuspidal representations of representation theory. When considering a single morphism, the cuspidal block is sometimes known as the *null-category*.

- $n = 4$: $T(-\log)$ (one object)
- $n = 5$: $T(-\log)$ (one object)
- $n = 6$: $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$, $\mathcal{L}^\vee$, $T(-\log)$ (twelve objects).

Here $\mathbb{P}^1 \times \mathbb{P}^1 \subset \overline{M}_{0,6}$ are boundary divisors of type $(3,3)$, and $\mathcal{L}$ is a pull-back of the symmetric GIT polarization (the Segre cubic).

We apply this approach to the Losev–Manin moduli space [LM00]. For an n-element set N, we let $\tilde{N} = \{0,\infty\} \sqcup N$. We write $\overline{LM}_N$ for the moduli space of nodal linear chains of projective lines $\mathbb{P}^1$ marked by $\tilde{N}$ with 0 on the left tail and ∞ on the right tail of the chain. This is a “simplified” version of $\overline{M}_{0,n}$, with linear chains replacing arbitrary trees. The stability conditions are as follows:

- Marked points are never at the nodes.
- Only points marked by N are allowed to coincide with each other.
- Every $\mathbb{P}^1$ has at least three special points (marked points or nodes).

The space $\overline{LM}_N$ has an action by the group $S_2 \times S_N$ permuting markings. The action of S_2, which we call the *Cremona action*, interchanges ∞ and 0. Both ψ-classes ψ_0 and ψ_∞ induce birational morphisms $\overline{LM}_N \to \mathbb{P}^{n-1}$, “Kapranov models”, which realize $\overline{LM}_N$ as an iterated blow-up of $\mathbb{P}^{n-1}$ in n points (standard basis vectors) followed by blowing up $\binom{n}{2}$ proper transforms of lines connecting points, etc. (We note that the other ψ-classes of $\overline{LM}_N$ are trivial.) In these coordinates, the Cremona action is given by the standard Cremona involution

$$(x_1 : \cdots : x_n) \to \left(\frac{1}{x_1} : \cdots : \frac{1}{x_n}\right).$$

The Losev–Manin space $\overline{LM}_N$ is a toric variety of dimension $n-1$. Its toric orbits (or their closures, the boundary strata of the moduli space) can be described as follows. Every non-trivial bipartition $N = N_1 \sqcup N_2$ corresponds to the boundary divisor, which we denote by δ_{N_1}, parametrizing (degenerations of) chains of two lines $\mathbb{P}^1$, one with markings $N_1 \cup \{0\}$ and another with markings $N_2 \cup \{\infty\}$. This notation is different from the standard notation for $\overline{M}_{0,n}$ (where an analogous divisor is denoted by $\delta_{N_1\cup\{0\}}$) but more convenient for us. More generally, every partition $N = N_1 \sqcup \cdots \sqcup N_k$ with $|N_i| > 0$ for every i corresponds to the boundary stratum

$$Z_{N_1,\ldots,N_k} = \delta_{N_1} \cap \delta_{N_1\cup N_2} \cap \cdots \cap \delta_{N_1\cup\cdots\cup N_{k-1}},$$

which parametrizes (degenerations of) linear chains of lines $\mathbb{P}^1$ with points marked by, respectively, $N_1 \cup \{0\}, N_2, \ldots, N_{k-1}, N_k \cup \{\infty\}$. We can identify

$$Z_{N_1,\ldots,N_k} \simeq \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_k},$$

where the left node of every $\mathbb{P}^1$ is marked by 0 and the right node by ∞.

We have a collection of forgetful maps

$$\pi_K\colon \overline{LM}_N \to \overline{LM}_{N\setminus K}$$

for every subset $K \subset N$ with $1 \leqslant |K| \leqslant n-1$. The map π_K is given by forgetting points marked by K and stabilizing. In particular, we can define the cuspidal block $D^b_{\mathrm{cusp}}(\overline{LM}_N)$, and applying Theorem 3.5, we show that we have a similar statement as for $\overline{M}_{0,n}$ (Proposition 1.6).

PROPOSITION 1.8. *The derived category $D^b(\overline{LM}_N)$ admits the semi-orthogonal decomposition*

$$D^b(\overline{LM}_N) = \langle D^b_{\text{cusp}}(\overline{LM}_N), \{\pi_K^* D^b_{\text{cusp}}(\overline{LM}_{N\setminus K})\}_{K\subset N}, \mathcal{O}\rangle,$$

where subsets K with $1 \leqslant |K| \leqslant n-2$ are ordered by increasing cardinality.

Next we construct a collection $\hat{\mathbb{G}}$ of sheaves on $\overline{LM}_N$. We note that in this definition, and in the rest of the paper, we do not always distinguish notationally between divisors and line bundles.

DEFINITION 1.9. Let $\mathbb{G}_N = \{G_1^\vee, \dots, G_{n-1}^\vee\}$ be the set of the following line bundles on $\overline{LM}_N$:

$$G_a = a\psi_0 \otimes \mathcal{O}\Bigg(-(a-1)\sum_{k\in N}\delta_k - (a-2)\sum_{k,l\in N}\delta_{kl} - \dots - \sum_{J\subset N,\, |J|=a-1}\delta_J\Bigg)$$

for every $a = 1, \dots, n-1$. Let $\hat{\mathbb{G}}$ be the collection of sheaves

$$\hat{\mathbb{G}} = \bigcup_Z (i_Z)_*[\mathbb{G}_{N_1}^\vee \boxtimes \dots \boxtimes \mathbb{G}_{N_t}^\vee]$$

on $\overline{LM}_N$ of the form

$$\mathcal{T} = (i_Z)_*\mathcal{L}, \quad \mathcal{L} = G_{a_1}^\vee \boxtimes \dots \boxtimes G_{a_t}^\vee$$

for all strata $Z = Z_{N_1,\dots,N_t}$ with $N_i \geqslant 2$ for every i and for all $1 \leqslant a_i \leqslant |N_i| - 1$. Here $i_Z\colon Z \hookrightarrow \overline{LM}_N$ is the inclusion map. If $t = 1$, we get line bundles $\mathbb{G}_N$, and for $t \geqslant 2$, these sheaves are torsion sheaves.

THEOREM 1.10. *The set $\hat{\mathbb{G}}$ is a full exceptional collection in $D^b_{\text{cusp}}(\overline{LM}_N)$ and is equivariant under the group $S_2 \times S_N$. The number of objects in $\hat{\mathbb{G}}$ is equal to $!n$, the number of derangements of n objects (permutation without fixed points).*

This is our main theorem; its proof occupies Sections 4 and 5. It gives a new curious formula for the number of derangements:[2]

$$\sum_{\substack{k_1+\dots+k_t=n\\ k_1,\dots,k_t\geqslant 2}} \binom{n}{k_1\ \dots\ k_t}(k_1-1)\cdots(k_t-1) = \,!n, \quad \text{where } \binom{n}{k_1\ \dots\ k_t} = \frac{n!}{k_1!\cdots k_t!}. \tag{1.3}$$

As a corollary, we see that K-theory of $\overline{LM}_N$ is a permutation representation of $S_2 \times S_n$ in a very concrete way, which should be contrasted with description of its equivariant Euler–Poincaré polynomial as an alternating sum in [BM14].

The ordering of $\hat{\mathbb{G}}$ that turns it into an exceptional collection is quite elaborate and discussed in Section 4. The real difficulty though is to prove fullness, which is done in Section 5. Note that fullness would follow at once if phantom subcategories (admissible subcategories with trivial K-group) did not exist on smooth projective toric varieties.

Remark 1.11. The line bundles $G_1, \dots, G_{n-1}$ on $\overline{LM}_n$ may appear ad hoc, but in fact they have a very nice description in terms of the (minimal) wonderful compactification $\overline{\mathrm{PGL}_n}$ of PGL_n (which contains $\overline{LM}_n$ as the closure of the maximal torus). Namely, they are precisely the restrictions of generators of the nef cone of $\overline{\mathrm{PGL}_n}$; see Proposition 4.14 for a more precise statement. It would be interesting to relate derived categories of $\overline{\mathrm{PGL}_n}$ and $\overline{LM}_n$.

[2]We are unaware of a combinatorial "bijective" proof of this identity.

It is worth noting that we do not know any smooth projective toric varieties X with an action of a finite group Γ normalizing the torus action which do not have a Γ-equivariant exceptional collection $\{E_i\}$ of maximal possible length (equal to the topological Euler characteristic of X). Its existence would imply that the K-group $K_0(X)$ is a permutation Γ-module. In the Galois setting (when X is defined over a field which is not algebraically closed and Γ is the absolute Galois group), an analogous statement was conjectured by A. S. Merkurjev and I. A. Panin [MP97]. Of course, one may further wonder whether $\{E_i\}$ is in fact full, which is related to the existence or not of phantom categories on X, another difficult general open question.

We refer to [CT15, CT13, CT12] for background information on the birational geometry of $\overline{M}_{0,n}$, the Losev–Manin space and other related spaces. We refer to [Huy06] for background on semi-orthogonal decompositions.

2. An equivariant version of Orlov's blow-up theorem

Orlov's blow-up theorem Theorem 3.3 is a categorification of the following fact. Let X be a smooth projective variety, and let $Y \subseteq X$ be a smooth subvariety of codimension l. Let $\tilde{X}$ be the blow-up of X along Y. We have a decomposition of cohomology with integral coefficients; see, for example, [Voi07, Theorem 7.31]

$$H^*\big(\tilde{X}\big) \simeq \big[H^*(Y) \otimes H^+\big(\mathbb{P}^{l-1}\big)\big] \oplus H^*(X)\,. \tag{2.1}$$

Now consider the following more general situation. Let $Y_1, \ldots, Y_n \subseteq X$ be smooth transversal subvarieties of codimension l. For any subset $I \subseteq \{1, \ldots, n\}$, we denote the intersection $\cap_{i \in I} Y_i$ by Y_I. In particular, $Y_\emptyset = X$. Let $q\colon \tilde{X} \to X$ be an iterated blow-up of (proper transforms of) $Y_1, \ldots, Y_n$. Since the intersection is transversal, blow-ups can be done in any order. The analogue of (2.1) in this situation was worked out in [BM13, Proposition 6.1]:

$$H^*(\tilde{X}) \simeq \bigoplus_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \emptyset}} \Big[H^*(Y_I) \otimes H^+\big(\mathbb{P}^{l-1}\big)^{\otimes |I|}\Big] \oplus H^*(X)\,, \tag{2.2}$$

which we are going to rewrite as

$$H^*(\tilde{X}) \simeq \bigoplus_{I \subset \{1,\ldots,n\}} \Big[H^*(Y_I) \otimes H^+\big(\mathbb{P}^{l-1}\big)^{\otimes |I|}\Big]\,.$$

The analogue of Theorem 3.3 is also straightforward. We fix the following notation. Let E_i be the exceptional divisor over Y_i for every $i = 1, \ldots, n$. For any subset $I \subseteq \{1, \ldots, n\}$, let

$$E_I = q^{-1}(Y_I) = \cap_{i \in I} E_i\,.$$

In particular, $E_\emptyset = \tilde{X}$. Let $i_I\colon E_I \hookrightarrow \tilde{X}$ be the inclusion.

Lemma 2.1. *Let $\{F_I^\beta\}$ be a (full) exceptional collection in $D^b(Y_I)$ for every subset $I \subseteq \{1, \ldots, n\}$. There exists a (full) exceptional collection in $D^b(\tilde{X})$ with blocks*

$$B_{I,J} = (i_I)_* \left[\big(Lq|_{E_I}\big)^* \big(F_I^\beta\big) \left(\sum_{i=1}^n J_i E_i\right)\right]$$

for every subset $I \subseteq \{1, \ldots, n\}$ (including the empty set) and for every n-tuple of integers J such that $J_i = 0$ if $i \notin I$ and $1 \leqslant J_i \leqslant l-1$ for $i \in I$.

The blocks are ordered in any linear order which respects the following partial order: B_{I^1,J^1} precedes B_{I^2,J^2} if $\sum_{i=1}^n J_i^1 E_i \geqslant \sum_{i=1}^n J_i^2 E_i$ (as effective divisors).

Proof. We argue by induction on n, the case $n=1$ being Orlov's theorem. We decompose $q\colon \tilde X \to X$ as a blow-up $q_0\colon X' \to X$ of Y_n and an iterated blow-up $q'\colon \tilde X \to X'$ of proper transforms $Y'_1,\dots,Y'_{n-1}$ of $Y_1,\dots,Y_{n-1}$. By Orlov's theorem, X' carries a (full) exceptional collection E'^{α}, namely

$$i'_*\bigl[(q_0|_E)^*(F_n^\beta)((l-1)E)\bigr],\dots,i'_*\bigl[(q_0|_E)^*(F_n^\beta)(E)\bigr],Lq_0^*(F_\emptyset^\beta)\,.$$

Here $i'\colon E\hookrightarrow X'$ is the exceptional divisor, and $q_0|_E$ is a projective bundle.

More generally, for every subset $I'\subseteq\{1,\dots,n-1\}$, let $Y'_{I'}=\cap_{i\in I'}Y'_i$ be the proper transform of $Y_{I'}$ isomorphic to the blow-up of $Y_{I'}$ in $Y_{I'\cup\{n\}}$. By Orlov's theorem, $Y'_{I'}$ carries a (full) exceptional collection $F'^{\beta}_{I'}$, namely

$$(i'_{I'})_*\bigl[(q_0|_{E_n^{I'}})^*(F^\beta_{I'\cup\{n\}})((l-1)E)\bigr],\dots,(i'_{I'})_*\bigl[(q_0|_{E_n^{I'}})^*(F^\beta_{I'\cup\{n\}})(E)\bigr],L(q_0|_{Y'_{I'}})^*(F^\beta_{I'})\,.$$

Here $i'_{I'}\colon E_n^{I'}\hookrightarrow Y'_{I'}$ is the exceptional divisor over $Y_{I'\cup\{n\}}$.

Applying the inductive assumption gives an exceptional collection on $\tilde X$ with blocks

$$(i_{I'})_*\left[(Lq'|_{E_{I'}})^*(F'^\beta_{I'})\left(\sum_{i=1}^{n-1}J_iE_i\right)\right]$$

for every subset $I'\subseteq\{1,\dots,n-1\}$ (including the empty set) and for every $(n-1)$-tuple of integers J such that $J_i=0$ if $i\notin I'$ and $1\leqslant J_i\leqslant l-1$ for $i\in I'$.

The blocks are ordered in any linear order which respects the following partial order: $B_{I'^1,J^1}$ precedes $B_{I'^2,J^2}$ if $\sum_{i=1}^{n-1}J_i^1E_i\geqslant\sum_{i=1}^{n-1}J_i^2E_i$ (as effective divisors). We have to check that these blocks are the same as in the statement of the lemma. It is clear that

$$(Lq'|_{E_{I'}})^*\bigl(L(q_0|_{Y'_{I'}})^*(F^\beta_{I'})\bigr)\simeq(Lq|_{E_{I'}})^*(F^\beta_{I'})\,.$$

This takes care of the last element in $F'^\beta_{I'}$. For the rest, we have to show that

$$(i_{I'})_*\left[(Lq'|_{E_{I'}})^*\bigl((i'_{I'})_*\bigl[(q_0|_{E_n^{I'}})^*(F_I^\beta)(J_nE)\bigr]\bigr)\left(\sum_{i=1}^{n-1}J_iE_i\right)\right]\simeq(i_I)_*\left[(Lq|_{E_I})^*(F_I^\beta)\left(\sum_{i=1}^{n}J_iE_i\right)\right],$$

where $I=I'\cup\{n\}$. It suffices to show that

$$(Lq'|_{E_{I'}})^*\bigl((i'_{I'})_*\bigl[(q_0|_{E_n^{I'}})^*(F_I^\beta)(J_nE)\bigr]\bigr)\left(\sum_{i=1}^{n-1}J_iE_i\right)\simeq(\phi)_*\left[(Lq|_{E_I})^*(F_I^\beta)\left(\sum_{i=1}^{n}J_iE_i\right)\right],$$

where $\phi\colon E_I\hookrightarrow E_{I'}$ is the inclusion. Applying the projection formula, we reduce this to

$$(Lq'|_{E_{I'}})^*\bigl((i'_{I'})_*\bigl[(q_0|_{E_n^{I'}})^*(F_I^\beta)\bigr]\bigr)\simeq(\phi)_*\bigl[(Lq|_{E_I})^*(F_I^\beta)\bigr]\,,$$

which follows by flat base change.

The last order of business is to prove the claim about the order of the blocks. We made a choice of blowing up Y_n first; accordingly, the collection has blocks $B_{I',J'}$ for every subset $I'\subseteq\{1,\dots,n-1\}$ (including the empty set) and for every $(n-1)$-tuple of integers J' such that $J'_i=0$ if $i\notin I'$ and $1\leqslant J'_i\leqslant l-1$ for $i\in I'$. The blocks are ordered in any linear order which respects the following partial order: $B_{I'^1,J'^1}\prec B_{I'^2,J'^2}$ if $\sum_{i=1}^{n-1}J_i'^1E_i>\sum_{i=1}^{n-1}J_i'^2E_i$ (as effective divisors). Each block $B_{I',J'}$ is a sequence of blocks $B_{I,J}$ from the statement of the lemma, where $I\cap\{1,\dots,n-1\}=I'$ and $J_i=J'_i$ for $i<n$. They are ordered in the decreasing order by J_n. In particular, if B_{I^1,J^1} precedes B_{I^2,J^2}, then either $\sum_{i=1}^{n}J_i^1E_i-\sum_{i=1}^{n}J_i^2E_i$ is an effective divisor, or $\sum_{i=1}^{n-1}J_i^2E_i-\sum_{i=1}^{n-1}J_i^1E_i$ is not effective. Therefore, it suffices to prove that for any two blocks

B_{I^1,J^1} and B_{I^2,J^2}, if $\sum_{i=1}^n J_i^1 E_i - \sum_{i=1}^n J_i^2 E_i$ is not an effective divisor, then $\{B_{I^1,J^1}, B_{I^2,J^2}\}$ is an exceptional sequence. If $\sum_{i=1}^{n-1} J_i^1 E_i - \sum_{i=1}^{n-1} J_i^2 E_i$ is not effective, then we are done by the above. But if it is effective, then $\sum_{i=2}^n J_i^1 E_i - \sum_{i=2}^n J_i^2 E_i$ is not effective, and we are again done by the above (by changing the order of blow-ups and blowing up Y_1 first). □

Remark 2.2. The same argument shows, more generally, that even in the absence of exceptional collections, there exists a semi-orthogonal decomposition of $D^b(\tilde{X})$ with blocks

$$B_{I,J} = (i_I)_*\left[\left(Lq|_{E_I}\right)^*\left(D^b(Y_I)\right)\left(\sum_{i=1}^n J_i E_i\right)\right]$$

(with the same notation and order as in the lemma). We stated the lemma for exceptional collections with an eye toward its equivariant version.

Continuing with the set-up of Lemma 2.1, let G be a finite group acting on X permuting $Y_1, \ldots, Y_n$. Then it also acts on $\tilde{X}$, and the morphism q is G-equivariant. Let $G_I \subseteq G$ be the normalizer of Y_I for each subset $I \subseteq \{1, \ldots, n\}$ (in particular, $G_\emptyset = G$).

Lemma 2.3. *Let $\{F_I^\beta\}$ be a (full) G_I-equivariant exceptional collection in $D^b(Y_I)$ for every subset $I \subseteq \{1, \ldots, n\}$. We assume that if $Y_I = gY_{I'}$ for some $g \in G$, then $\{F_I^\beta\} = g\{F_{I'}^\beta\}$. There exists a (full) G-equivariant exceptional collection in $D^b(\tilde{X})$ with blocks $B_{I,J}$ (the same as in Lemma 2.1).*

Proof. It suffices to observe that the blocks $B_{I,J}$ are permuted by G. □

Next we recall a few facts and notation from [BM13] in order to prove Theorem 1.2. The subgroup $S_k \times S_{n-k} \subseteq S_n$ preserves the weight $\mathbf{a}$ and therefore acts on $\overline{M}^n_{k,l}$. We have $(S_k \times S_{n-k})$-equivariant reduction morphisms

$$\overline{M}^n_{k,1} \to \overline{M}^n_{k,2} \to \cdots \to \overline{M}^n_{k,r(n,k)}\,, \tag{2.3}$$

where the first map is an isomorphism. Each of the maps in (2.3) is an iterated blow-up of transversal loci of the same codimension permuted by $S_k \times S_{n-k}$. Specifically, for every subset $I \subset \{k+1, \ldots, n\}$ of cardinality $l+1$, let $\overline{M}^n_{k,l+1}(I) \subseteq \overline{M}^n_{k,l+1}$ be the closure of the locus where points marked by I collide. The reduction morphism $\overline{M}^n_{k,l} \to \overline{M}^n_{k,l+1}$ is the blow-up along the transversal union $\cup_I \overline{M}^n_{k,l+1}(I)$ of subvarieties of codimension l, where I runs over all subsets of $\{k+1, \ldots, n\}$ of cardinality $l+1$; see [BM13, Lemma 3.1]. Intersections of these loci are described in [BM13, Section 3.2] as follows. Let $I_1, \ldots, I_m \subset \{k+1, \ldots, n\}$ be subsets of cardinality $l+1$. Then $\cap_{i=1}^m \overline{M}^n_{k,l+1}(I_i) \neq \emptyset$ if and only if the subsets $I_1, \ldots, I_m$ are disjoint. In this case, the intersection is isomorphic to $\overline{M}^{n-lm}_{k+m,l+1}$. Moreover, the stabilizer of this stratum in $S_k \times S_{n-k}$ acts on it through a subquotient contained in $S_{k+m,n-lm-k-m}$. Applying Lemma 2.3 proves Theorem 1.2.

3. The cuspidal block

Recall that by Definition 1.4, we call an object $E \in D^b(X)$ *cuspidal* with respect to a given collection of morphisms $\pi_i\colon X \to X_i$ (for $i \in I$) between smooth projective varieties if

$$R\pi_{i*}E = 0 \quad \text{for every } i \in I\,.$$

The *cuspidal block* is the full triangulated subcategory of cuspidal objects $D^b_{\text{cusp}}(X) \subset D^b(X)$.

Lemma–Definition 3.1. *In the set-up of Definition 1.4, the* support *of any cuspidal object is a union of irreducible closed subsets $Z \subset X$ such that*

$$\dim \pi_i(Z) < \dim Z \quad \text{for every } i \in S\,.$$

We call any subset Z with this property (independently of whether they are the support of a cuspidal object or not) massive. *Recall that the topological support of an object $E \in D^b(X)$ is the support of its cohomology sheaves.*

Proof. Let Z be the topological support of $E \in D^b(X)$. Suppose that Z contains an irreducible component Z_0 such that $\dim \pi_i(Z_0) = \dim Z_0$. We write $\pi := \pi_i$ and $Y := X_i$ as we will not need other maps and spaces. By passing to an open subset of Y and taking its preimage under π, we can assume that Z is a disjoint union of Z_0 and Z_1 (with Z_1 possibly empty and not necessarily irreducible). We may also assume that $\pi|_{Z_0}$ is finite. It is well known [Orl11, Section 2] that by changing E to an isomorphic object, we may assume that E is a bounded complex of sheaves supported on Z. Thus $E = Ri_*\tilde{E}$, where $i\colon \tilde{Z} \hookrightarrow X$ is an infinitesimal thickening of Z and $\tilde{E} \in D^b\big(\tilde{Z}\big)$. Note that $\tilde{Z}$ is a disjoint union of subschemes $\tilde{Z}_0$ and $\tilde{Z}_1$ (with reduced subschemes Z_0 and Z_1). In particular, $\tilde{E} = \tilde{E}_0 \oplus \tilde{E}_1$, where $\tilde{E}_0$ and $\tilde{E}_1$ are pull-backs of $\tilde{E}$ to Z_0 and Z_1, respectively. It follows that $R\tilde{\pi}_*\big(\tilde{E}_0\big) = 0$, where $\tilde{\pi} = \pi \circ i$. Since $\tilde{E}_0 \neq 0$ and the map $\tilde{\pi}$ is affine, this gives a contradiction. Indeed, if $\pi\colon X \to Y$ is an affine morphism of schemes, then $R\pi_* E = 0$ for some $E \in D_{Q\mathrm{Coh}}(\mathcal{O}_X)$ implies that $E = 0$; see [Sta20, Tag 0AVV]. □

We refer to the survey [Kuz16] for definitions and basic facts concerning semi-orthogonal decompositions in algebraic geometry. The following is well known; see, for example, [Kuz08, Lemma 2.4].

Proposition 3.2. *Let $\pi\colon X \to Y$ be a morphism of smooth projective varieties such that $R\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Then $L\pi^* D^b(Y)$ is an admissible subcategory of $D^b(X)$, and there is a semi-orthogonal decomposition*

$$D^b(X) = \big\langle D^b_{\mathrm{cusp}}(X), L\pi^* D^b(Y)\big\rangle\,.$$

In particular, $D^b_{\mathrm{cusp}}(X)$ is an admissible subcategory.

Classical situations of this sort are provided by Orlov's theorems [Orl93] on derived categories of a projective bundle and of a blow-up, which can be reformulated as follows.

Theorem 3.3. *Let $\pi\colon X \to Y$ be a projective bundle of rank r, with Y a smooth projective variety. Then $D^b_{\mathrm{cusp}}(X)$ is an admissible subcategory of $D^b(X)$ and $D^b_{\mathrm{cusp}}(X)$ has a semi-orthogonal decomposition*

$$\big\langle \pi^* D^b(Y) \otimes \mathcal{O}_\pi(-r),\, \ldots,\, \pi^* D^b(Y) \otimes \mathcal{O}_\pi(-1)\big\rangle\,.$$

Theorem 3.4. *Let $p\colon X \to Y$ be a blow-up of a smooth subvariety Z of codimension $r+1$ of a smooth projective variety Y. Let $i\colon E \to X$ be the exceptional divisor, and let $\pi = p|_Z$. Then $D^b_{\mathrm{cusp}}(X)$ is an admissible subcategory of $D^b(X)$ and has a semi-orthogonal decomposition*

$$\big\langle Ri_*\big[\pi^* D^b(Z) \otimes \mathcal{O}_\pi(-r)\big],\, \ldots,\, Ri_*\big[\pi^* D^b(Z) \otimes \mathcal{O}_\pi(-1)\big]\big\rangle\,.$$

In order to generalize Proposition 3.2 to the set-up of several morphisms, we impose compatibility conditions. In subsequent sections, we will consider several variants of moduli spaces of rational pointed curves, which will all fit into this framework.

THEOREM 3.5. *Let $\mathbb{N}$ be the category of finite subsets of a fixed set with inclusions as morphisms. Let X be a contravariant functor from $\mathbb{N}$ to the category of smooth projective varieties. For every $T \subseteq S$, we refer to the morphism $X_S \to X_T$ as the* forgetful map *and denote it by $\pi_{S\setminus T}$. We impose three assumptions:*

(1) *We have*

$$R\pi_{i*}\mathcal{O}_{X_S} = \mathcal{O}_{X_{S\setminus\{i\}}} \quad \text{for every } i \in S\,. \tag{3.1}$$

(2) *For all $i, k \in S$, $i \neq k$, the morphisms*

$$\pi_i \colon X_{S\setminus\{k\}} \to X_{S\setminus\{i,k\}}\,, \quad \pi_k \colon X_{S\setminus\{i\}} \to X_{S\setminus\{i,k\}} \quad \textit{are } \mathrm{Tor}\textit{-independent} \tag{3.2}$$

(*as defined in* [Sta20, Definition 36.21.2]).

(3) *If we let*

$$Y := X_{S\setminus\{i\}} \times_{X_{S\setminus\{i,k\}}} X_{S\setminus\{k\}}$$

and $\alpha_{i,k} \colon X_S \to Y$ is the map induced by π_i and π_k, we have

$$R\alpha_{i,k_*}\mathcal{O}_{X_S} = \mathcal{O}_Y\,. \tag{3.3}$$

Under these assumptions, we have a semi-orthogonal decomposition (*s.o.d.*)

$$D^b(X_S) = \left\langle D^b_{\mathrm{cusp}}(X_S), \left\{L\pi_K^* D^b_{\mathrm{cusp}}(X_{S\setminus K})\right\}_{K\subset S}, L\pi_S^* D^b(X_\emptyset)\right\rangle,$$

where K runs over proper subsets of S in order of increasing cardinality. In particular, $D^b_{\mathrm{cusp}}(X_S)$ is an admissible subcategory of $D^b(X_S)$.

Following a suggestion of A. Kuznetsov, we start with an abstract "inclusion–exclusion" principle in triangulated categories. Perhaps we should remark that semi-orthogonal decompositions do not intersect well in general, as a simple example of $D^b(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1)\rangle = \langle \mathcal{O}(2), \mathcal{O}(3)\rangle$ shows. However, we have the following.

LEMMA 3.6. *Let $\mathcal{T}$ be a triangulated category with several semi-orthogonal decompositions*

$$\mathcal{T} = \langle A_1, B_1\rangle = \langle A_2, B_2\rangle = \cdots = \langle A_n, B_n\rangle\,.$$

Suppose that the projection functors $\beta_i \colon \mathcal{T} \to B_i$ (in the ith decomposition) have the property that for every j,

$$\beta_i(A_j) \subset A_j\,, \quad \beta_i(B_j) \subset B_j\,.$$

Then we have a s.o.d.

$$\mathcal{T} = \langle \mathcal{T}_K\rangle_K\,, \quad \text{where} \quad \mathcal{T}_K = (\cap_{i\notin K} A_i) \cap (\cap_{i\in K} B_i)$$

and K runs over subsets of $\{1, \dots, n\}$ in the order of increasing cardinality. In particular, $\mathcal{T}_\emptyset = A_1 \cap \cdots \cap A_n$ is an admissible subcategory of $\mathcal{T}$.

Proof. For all subsets $T \subseteq S := \{1, \dots, n\}$, we consider a full triangulated subcategory $A_T = \cap_{i\in T} A_i$. We prove, more generally, that there is a semi-orthogonal decomposition $A_T = \langle \mathcal{T}_K\rangle$, where K runs over subsets of S containing T in order of increasing cardinality. The case $T = \emptyset$ is the statement in the theorem.

We argue by induction on $n = |S|$ and by downwards induction on $|T|$ for a fixed n. If $n = 1$ or $T = S$, then there is nothing to prove. Let $i \in S \setminus T$. Without loss of generality, we assume $i = 1$.

We claim that the semi-orthogonal decomposition $\mathcal{T} = \langle A_1, B_1\rangle$ induces a semi-orthogonal decomposition

$$A_T = \langle A_T \cap A_1, A_T \cap B_1\rangle\,. \tag{3.4}$$

Indeed, the semi-orthogonality is obvious, and, moreover, every object X in A_T fits into a distinguished triangle $\beta_1(X) \to X \to Y \to$ with $Y \in A_1$. Since β_1 preserves A_T by our assumptions, $\beta_1(X) \in A_T \cap B_1$. It follows that $Y \in A_T$ as well.

By the induction assumption, we have semi-orthogonal decompositions

$$A_T \cap A_1 = A_{T\cup\{1\}} = \langle \mathcal{T}_K\rangle \quad \text{and} \quad A_T = \langle \mathcal{T}'_{K'}\rangle\,,$$

where K runs over subsets of S containing $T \cup \{1\}$ and K' over subsets of $S\backslash\{1\}$ containing T and

$$\mathcal{T}'_{K'} = \left(\cap_{i\notin K\cup\{1\}} A_i\right) \cap \left(\cap_{i\in K} B_i\right)\,.$$

We claim that the semi-orthogonal decomposition $A_T = \langle \mathcal{T}'_{K'}\rangle$ induces the semi-orthogonal decomposition

$$A_T \cap B_1 = \left\langle \mathcal{T}'_{K'} \cap B_1\right\rangle = \left\langle \mathcal{T}_{K'\cup\{1\}}\right\rangle\,.$$

Indeed, the semi-orthogonality is clear. By the definition of the semi-orthogonal decomposition, for every object $X \in A_T \cap B_1$, we can write a sequence of morphisms ("filtration")

$$0 \to \cdots \to T_{K'_1} \to T_{K'_2} \to \cdots \to X \to 0$$

such that every morphism is included in the distinguished triangle

$$T_{K'_1} \to T_{K'_2} \to X_{K'_1} \to$$

with $X_{K'_1} \in \mathcal{T}'_{K'_1}$. Applying the functor β_1 to this sequence and using our assumptions gives a filtration of X with subquotients $\beta_1\left(X_{K'_1}\right) \in \mathcal{T}'_{K'_1} \cap B_1$.

Combining these observations with (3.4), we get a semi-orthogonal decomposition

$$A_T = \left\langle \mathcal{T}_K, \mathcal{T}_{K'\cup\{1\}}\right\rangle\,,$$

where K runs over subsets of S containing $T \cup \{1\}$ and K' runs over subsets of $S \setminus \{1\}$ containing T, both in order of increasing cardinality.

Finally, we have to show that we can reorder blocks to put them in the order of increasing cardinality. If $|K_1| < |K_2|$, then choose an index $j \in K_2 \setminus K_1$. Then $\mathcal{T}_{K_1} \subset A_j$ and $\mathcal{T}_{K_2} \subset B_j$. Thus $\mathcal{T}_{K_1} \subset \mathcal{T}_{K_2}^{\perp}$. □

Proof of Theorem 3.5. By Proposition 3.2, we have s.o.d.'s $D^b(X_S) = \langle A_i, B_i\rangle$, where

$$A_i = \left\{E \in D^b(X_S) \,|\, R\pi_{i*}E = 0\right\} \quad \text{and} \quad B_i = L\pi_i^*\left(D^b\left(X_{S\backslash\{i\}}\right)\right)\,.$$

We now apply Lemma 3.6 to $\langle A_i, B_i\rangle$. The projection operators are

$$\beta_i = L\pi_i^* R\pi_{i*}\,.$$

Note that for all $i, k \in S$ with $i \neq k$ and all $E \in D^b(X_{S\backslash\{k\}})$, we have

$$R\pi_{i*}L\pi_k{}^*E \simeq L\pi_k{}^*R\pi_{i*}E \tag{3.5}$$

since, by assumption, π_i and π_k are Tor-independent. This follows from assumption (2) combined with cohomology and base change: if π'_i and π'_k are the projection maps from $Y = X_{S\backslash\{i\}} \times_{X_{S\backslash\{i,k\}}} X_{S\backslash\{k\}}$ and $\alpha\colon X_S \to Y$ is the canonical map, we have

$$R\pi_{i*}L\pi_k^*E = R\pi'_{i*}R\alpha_*L\alpha^*L\pi'_k{}^*E = R\pi'_{i*}L\pi'_k{}^*E = L\pi_k{}^*R\pi_{i*}E\,,$$

where the second equality is by the projection formula and (3.3). It follows that

$$R\pi_{j*}L\pi_i^*R\pi_{i*} = L\pi_i^*R\pi_{j*}R\pi_{i*} = L\pi_i^*R\pi_{i*}R\pi_{j*}\,,$$

and in particular $\beta_i(A_j) \subset A_j$. Also,

$$L\pi_i^*R\pi_{i*}L\pi_j^* = L\pi_i^*L\pi_j^*R\pi_{i*} = L\pi_j^*L\pi_i^*R\pi_{i*}\,,$$

and thus $\beta_i(B_j) \subset B_j$.

It remains to show that, in the notation of Lemma 3.6, we have $D^b(X_S)_K = L\pi_K^* D^b_{\text{cusp}}(X_{S\setminus K})$ for every subset $K \subset T$. Equivalently,

$$\bigcap_{i\in K} B_i = L\pi_K^* D^b(X_{S\setminus K})\,. \tag{3.6}$$

We can assume that $K = \{1, \ldots, k\}$. Then it follows from (3.5) that $\beta_1 \circ \cdots \circ \beta_k = L\pi_K^* R\pi_{K*}$. Thus every object from the left-hand side of (3.6) is isomorphic to an object from the right-hand side, and vice versa. □

Example 3.7. Let $X_S = (\mathbb{P}^1)^S$ with projections as forgetful maps. Conditions (1), (2) and (3) of Theorem 3.5 are clearly satisfied. The subset X_S is the only massive one. Applying Theorem 3.3 repeatedly, it follows that

$$D^b_{\text{cusp}}(X_S) = \langle \mathcal{O}(-1, -1, \ldots, -1)\rangle\,;$$

that is, every object in $D^b_{\text{cusp}}(X_S)$ is isomorphic to $\mathcal{O}(-1, -1, \ldots, -1) \otimes_k K$, where K is a complex of vector spaces. Moreover, the semi-orthogonal decomposition of Theorem 3.5 is induced by a standard exceptional collection of $2^{|S|}$ line bundles $\mathcal{O}(n_1, \ldots, n_{|S|})$, where $n_i = 0$ or -1 for every i.

Note that this collection is obviously equivariant under the action of $\mathrm{Aut}(X_S)$, which is the semidirect product of S_n and $(\mathrm{PGL}_2)^n$ for $n = |S|$. Various moduli spaces considered in this paper can be viewed as "compactified quotients" of this basic example modulo $\mathbb{G}_m$ or PGL_2.

Proof of Proposition 1.6. Recall that we need to prove that $D^b(\overline{M}_N)$ admits a semi-orthogonal decomposition

$$D^b(\overline{M}_N) = \langle D^b_{\text{cusp}}(\overline{M}_N), \{\pi_K^* D^b_{\text{cusp}}(\overline{M}_{N\setminus K})\}_{K\subset N}, \mathcal{O}\rangle\,, \tag{3.7}$$

where K runs over subsets with $1 \leqslant |K| \leqslant n-4$ in order of increasing cardinality $|K|$. We apply Theorem 3.5. All conditions (1), (2) and (3) are satisfied. Recall that a simple criterion for Tor-independence for maps $X \to S$ and $T \to S$ is that one of them is flat. Hence, condition (2) holds as the forgetful maps $\pi_i\colon \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ are flat. Condition (3) holds as the map is birational and the image has rational singularities [Kee92, Introduction, p. 548]. □

The following proof is similar.

Proof of Proposition 1.8. Recall that we need to prove that $D^b(\overline{LM}_N)$ admits the semi-orthogonal decomposition

$$D^b(\overline{LM}_N) = \langle D^b_{\text{cusp}}(\overline{LM}_N), \{\pi_K^* D^b_{\text{cusp}}(\overline{LM}_{N\setminus K})\}_{K\subset N}, \mathcal{O}\rangle\,,$$

where subsets K with $1 \leqslant |K| \leqslant n-2$ are ordered by increasing cardinality. We apply Theorem 3.5 to the forgetful maps

$$\pi_i\colon \overline{LM}_N \to \overline{LM}_{N\setminus\{i\}} \quad (i \in N)\,.$$

All conditions (1), (2) and (3) are satisfied. Note again that the forgetful maps π_i for $i \in N$ are flat (they give the universal family); hence, condition (2) holds. Condition (3) holds because α_{ij} is birational and Y has toroidal, and therefore rational, singularities. $\square$

4. The exceptional collection $\hat{\mathbb{G}}$ on the Losev–Manin space

Proposition 4.1. *An irreducible subset $Z \subset \overline{LM}_N$ is massive if and only if Z is a boundary stratum of the form $Z_{N_1,\dots,N_t}$ with $|N_i| \geqslant 2$ for $i = 1, \dots, t$.*

Proof. Let Z be a boundary stratum. If $N_i = \{a\}$ for some i, then π_a restricted to Z is one-to-one. Hence Z is not a massive subset. On the other hand, if $|N_i| \geqslant 2$ for every i, then Z is a massive subset. It remains to show that if Z is a proper irreducible subset of a boundary stratum which intersects its interior, then Z is not massive. But the interior of any stratum is an algebraic torus $\mathbb{G}_m^r$, and projections onto coordinate subtori are realizable as forgetful maps. Thus Z cannot be massive. $\square$

Proposition 4.2. *The ranks of the K-groups of $D^b(\overline{LM}_n)$ and $D^b_{\text{cusp}}(\overline{LM}_n)$ are equal to $n!$ and $!n$, respectively.*

Proof. Since $\overline{LM}_N$ is a toric variety, its K-group is a free Abelian group and its topological Euler characteristic (and thus the rank of its K-group) is equal to the number of torus fixed points, which are clearly parametrized by permutations of N. The second part of the proposition follows because both the rank of the K-group of $D^b_{\text{cusp}}(\overline{LM}_n)$ (by Proposition 1.8) and $!n$ (by obvious reasons) satisfy the same recursion

$$n! = !n + \sum_{1 \leqslant k \leqslant n-1} \binom{n}{k} !(n-k) + 1\,. \tag{4.1}$$

Hence these numbers agree. $\square$

Proof of formula (1.3). Recall that formula (1.3) states that

$$\sum_{\substack{k_1+\dots+k_t=n \\ k_1,\dots,k_t \geqslant 2}} \binom{n}{k_1 \;\dots\; k_t} (k_1-1)\cdots(k_t-1) = !n\,, \quad \text{where } \binom{n}{k_1 \;\dots\; k_t} = \frac{n!}{k_1! \dots k_t!}\,.$$

We denote the left-hand side by d_n and set $d_0 = 1$ and $d_1 = 0$. Let

$$A = \sum_{n \geqslant 2} (n-1) \frac{x^n}{n!} = x^2 \left(\frac{e^x - 1}{x}\right)' = e^x(x-1) + 1\,.$$

Then we have

$$\sum_{m \geqslant 0} \frac{d_m}{m!} x^m = 1 + A + A^2 + A^3 + \dots = \frac{1}{1-A} = \frac{e^{-x}}{1-x}\,.$$

But (4.1) implies that

$$\frac{1}{1-x} = \left(\sum_{m \geqslant 0} !m \frac{x^m}{m!}\right)\left(\sum_{n \geqslant 0} \frac{x^n}{n!}\right)$$

(where we set $!0 = 0! = 1$). Hence $d_n = !n$, and we are done. $\square$

LEMMA 4.3. (1) *Every G_a is S_N-invariant, and Cremona action takes it to G_{n-a}.*

(2) *We have $G_1 = \psi_0$ and $G_{n-1} = \psi_\infty$.*

(3) *For every boundary divisor $\delta = \delta_{N_1} \simeq \overline{LM}_{N_1} \times \overline{LM}_{N_2}$, we have*

$$G_{a|\delta} = \begin{cases} G_a \boxtimes \mathcal{O} & \text{if } a < |N_1|\,, \\ \mathcal{O} & \text{if } a = |N_1|\,, \\ \mathcal{O} \boxtimes G_{a-|N_1|} & \text{if } a > |N_1|\,. \end{cases}$$

Proof. Direct calculation. □

Notation 4.4. For an object $F \in D^b(X)$, we often use the notation $R\Gamma(F)$ instead of $R\Gamma(X, F)$ when the space X is clear from the context.

LEMMA 4.5. (1) *Every G_a is nef (and hence globally generated), of relative degree 1 with respect to any forgetful map π_i, for $i \in N$.*

(2) *We have $G_a^\vee \in D^b_{\text{cusp}}(\overline{LM}_n)$. In particular, each $G_a^\vee$ is acyclic.*

(3) *We have $R\mathrm{Hom}(G_a, G_b) = 0$ if $a \neq b$.*

(4) *We have $R\Gamma(-\psi_0 + G_a - G_b) = R\Gamma(-\psi_\infty + G_b - G_a) = 0$ if $a < b$.*

In particular, $\mathbb{G}_N$ is an $(S_2 \times S_n)$-equivariant exceptional (in fact pairwise orthogonal) collection of $n-1$ line bundles in $D^b_{\text{cusp}}(\overline{LM}_N)$.

Proof. Since $\overline{LM}_n$ is a toric variety, part (1) will follow if G_a is non-negative on toric boundary curves. This follows from Lemma 4.3(3) by induction on the dimension. Since the restriction of $G_a^\vee$ to each fiber of π_i has vanishing cohomology, part (2) follows by cohomology and base change. Since $R\mathrm{Hom}(G_a, G_b) = R\Gamma(-G_a + G_b)$ and we can assume $a > b$ (by applying Cremona action), parts (3) and (4) both follow from Lemma 4.6. □

LEMMA 4.6. *Consider the divisor*

$$D = -dH + \sum m_I E_I$$

on $\overline{M}_{0,n}$ or $\overline{LM}_N$ written in some Kapranov model. The divisor D is acyclic if

$$1 \leqslant d \leqslant n-3 \quad \text{and} \quad 0 \leqslant m_I \leqslant n - 3 - \#I\,.$$

Proof. By consecutively restricting to boundary divisors E_I starting with those with the largest $\#I$ and continuing to those with smaller $\#I$, we see that all the restrictions are acyclic; hence, D has the same cohomology as $-dH$. Clearly, $-dH$ is acyclic if and only if $1 \leqslant d \leqslant n-3$. □

LEMMA 4.7. *The set $\hat{\mathbb{G}}$ is a collection of $!n$ sheaves in $D^b_{\text{cusp}}(\overline{LM}_N)$.*

Proof. Follows from (1.3) and Lemma 4.5(2). □

It is worth mentioning that if $i\colon Z \hookrightarrow X$ is a closed embedding of smooth projective varieties and $Z \neq X$, then the functor $Ri_*\colon D^b(Z) \to D^b(X)$ is in general not fully faithful. Therefore, even though all sheaves in $\hat{\mathbb{G}}$ are clearly exceptional in the derived category of their respective support (being line bundles on a rational variety), we still have to prove the following.

LEMMA 4.8. *All sheaves in $\hat{\mathbb{G}}$ are exceptional.*

Proof. All sheaves in $\hat{\mathbb{G}}$ are of the form $i_*i^*\mathcal{L} = Ri_*Li^*\mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $\overline{LM}_N$ and i is an embedding of some massive stratum Z. We have

$$\begin{aligned} R\mathrm{Hom}(Ri_*Li^*\mathcal{L}, Ri_*Li^*\mathcal{L}) &= R\mathrm{Hom}\left(\mathcal{L}\overset{L}{\otimes} Ri_*\mathcal{O}_Z, \mathcal{L}\overset{L}{\otimes} Ri_*\mathcal{O}_Z\right) \\ &= R\mathrm{Hom}(Ri_*\mathcal{O}_Z, Ri_*\mathcal{O}_Z)\,. \end{aligned}$$

So it suffices to prove that $Ri_*\mathcal{O}_Z = i_*\mathcal{O}_Z$ is an exceptional object. Let Z be the intersection of boundary divisors $D_1, \dots, D_s$. Resolving $i_*\mathcal{O}_Z$ by the Koszul complex

$$\cdots \to \oplus_{1\leqslant i<j\leqslant s}\mathcal{O}(-D_i - D_j) \to \oplus_{1\leqslant i\leqslant s}\mathcal{O}(-D_i) \to \mathcal{O} \to i_*\mathcal{O}_Z \to 0\,,$$

we see that it suffices to prove that

$$R\Gamma(\mathcal{O}_Z(D_{i_1} + \cdots + D_{i_k})) = 0$$

for all $1 \leqslant i_1 < \cdots < i_k \leqslant s$. Using that $\mathcal{O}_Z(D_i)$ has the form

$$\mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O} \boxtimes \mathcal{O}(-\psi_\infty) \boxtimes \mathcal{O}(-\psi_0) \boxtimes \mathcal{O} \cdots \boxtimes \mathcal{O}\,,$$

we conclude that this is indeed the case. □

Lemma 4.9. *The set $\hat{\mathbb{G}}$ is an exceptional collection with respect to the following order. Let $\mathcal{T}, \mathcal{T}' \in \hat{\mathbb{G}}$. Let $(k_1, \dots, k_t; a_1, \dots, a_t)$ and $(k_1', \dots, k_s'; a_1', \dots, a_s')$ be the corresponding data. Then $\mathcal{T} > \mathcal{T}'$ if the sequence $(a_1, -k_1, a_2, -k_2, \dots)$ is lexicographically (that is, alphabetically) larger than $(a_1', -k_1', a_2', -k_2', \dots)$.*

Proof. Let Z and Z' be massive strata supporting sheaves $\mathcal{T} > \mathcal{T}'$ in $\hat{\mathbb{G}}$. These sheaves have the form $Ri_{Z*}\mathcal{L}$ and $Ri_{Z'*}\mathcal{L}'$, respectively. We have to show that $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Let U be the smallest stratum containing both Z and Z'. Then U is the intersection of boundary divisors $D_1, \dots, D_s$ (these divisors are precisely the divisors containing both Z and Z'). We have

$$R\mathrm{Hom}(Ri_{Z*}\mathcal{L}, Ri_{Z'*}\mathcal{L}') = R\mathrm{Hom}(Lj_{Z'}{}^*Rj_{Z*}\mathcal{L}, \mathcal{L}')\,.$$

By [Huy06, Corollary 11.4(i)], it suffices to prove that

$$R\mathrm{Hom}(Rj_{Z*}\mathcal{L}, Rj_{Z'*}\mathcal{L}'(D)) = 0$$

for every $D = D_{i_1} + \cdots + D_{i_k}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant s$, where j_Z and $j_{Z'}$ denotes the embeddings of Z and Z', respectively, into U. Let $W = Z \cap Z'$. We can assume that W is non-empty as otherwise there is nothing to prove. Let $i_{W,Z}\colon W \hookrightarrow Z$ and $i_{W,Z'}\colon W \hookrightarrow Z'$ be the inclusions. We note that Z and Z' intersect transversally along W in U, and therefore j_Z and $j_{Z'}$ are Tor-independent. Next we apply cohomology and base change:

$$\begin{aligned} R\mathrm{Hom}(Rj_{Z*}\mathcal{L}, Rj_{Z'*}\mathcal{L}'(D)) &= R\mathrm{Hom}(Lj_{Z'}{}^*Rj_{Z*}\mathcal{L}, \mathcal{L}'(D)) = R\mathrm{Hom}(Ri_{W,Z'}{}_*Li^*_{W,Z}\mathcal{L}, \mathcal{L}'(D)) \\ &= R\mathrm{Hom}(Li^*_{W,Z}\mathcal{L}, Li^!_{W,Z'}\mathcal{L}'(D))\,, \end{aligned}$$

where for some morphism $f\colon X \to Y$, we denote the adjoint functor to $Rf_*(-)$ by $Lf^!(-)$. By Grothendieck duality, we have for $E \in D^b(Y)$ that $Lf^!(E) = Lf^*(E) \otimes \omega_f[\dim(f)]$. Here, $\omega_f = \omega_X \otimes f^*\omega_Y^*$ and $\dim(f) = \dim(X) - \dim(Y)$. So it suffices to prove that

$$R\mathrm{Hom}\left(Li^*_{W,Z}\mathcal{L}, Li^*_{W,Z'}\mathcal{L}' \otimes (D + c_1(\mathcal{N}))\right) = 0\,,$$

where $c_1(\mathcal{N})$ is the first Chern class of the normal bundle $\mathcal{N} := \mathcal{N}_{W,Z'}$, that is, the sum of all boundary divisors that cut out W inside Z' or, alternatively, the sum of boundary divisors that cut out Z but do not contain Z'.

Now we proceed case by case. We write

$$W = \overline{LM}_{K_1} \times \overline{LM}_{K_2} \times \cdots ,$$
$$R\mathrm{Hom}\left(Li_{W,Z}{}^{*}\mathcal{L}\,, Li_{W,Z'}{}^{*}\mathcal{L}'(D+N)\right) = C_1 \boxtimes C_2 \boxtimes \cdots ,$$

where C_1 is computed on $\overline{LM}_{K_1}$, etc. Note that if $N = N_1 \sqcup \cdots \sqcup N_t$ and $N = N_1' \sqcup \cdots \sqcup N_{t'}'$ are the two partitions corresponding to $\mathcal{T}$ and $\mathcal{T}'$, respectively (hence, $|N_i| = k_i$ and $|N_i'| = k_i'$ for all i), then $W \neq \emptyset$ implies that either $N_1 \subseteq N_1'$ or $N_1' \subseteq N_1$. In particular, we have $|K_1| = \min(k_1, k_1')$, and if $k_1 = k_1'$, then we have $N_1 = N_1'$.

Case 1. Suppose $a_1 > a_1'$. We would like to show that $C_1 = 0$.

If $k_1 < k_1'$, then $C_1 = R\mathrm{Hom}(-G_{a_1}, -G_{a_1'} - \psi_\infty)$, where $-\psi_\infty$ is a contribution from N (there is no contribution to C_1 from D). Hence, $C_1 = 0$ by Lemma 4.5(4).

If $k_1 = k_1'$, then there is no contribution from $c_1(\mathcal{N})$ to C_1, and we have that either $C_1 = R\mathrm{Hom}(-G_{a_1}, -G_{a_1'}) = 0$ (if D does not include D_{K_1}) or $C_1 = R\mathrm{Hom}(-G_{a_1}, -G_{a_1'} - \psi_\infty) = 0$ (if D includes D_{K_1}).

Finally, if $k_1 > k_1'$, then there are no contributions from $c_1(\mathcal{N})$ or D to C_1 and $C_1 = R\mathrm{Hom}(L, -G_{a_1'}) = 0$, where $L = -G_{a_1}$ if $a_1 < k_1'$ and $L = \mathcal{O}$ otherwise. In both cases, $C_1 = 0$ by Lemma 4.5.

Case 2. Suppose $a_1 = a_1'$ and $k_1 < k_1'$. As in case 1, we have that

$$C_1 = R\mathrm{Hom}(-G_{a_1}, -G_{a_1} - \psi_\infty) = 0\,.$$

Case 3. Suppose $a_1 = a_1'$, $k_1 = k_1'$ and that D includes D_{K_1}.

In this case also, $C_1 = R\mathrm{Hom}(-G_{a_1}, -G_{a_1} - \psi_\infty) = 0$.

Case 4. Suppose $a_1 = a_1'$, $k_1 = k_1'$ and that D does not include D_{K_1}.

In this case, $C_1 = R\mathrm{Hom}(-G_{a_1}, -G_{a_1}) = \mathbb{C}$ is useless. However, we can now proceed exactly as above, restricting to the next Losev–Manin factor $\overline{LM}_{K_2}$ in W. Note that, in general, the factors $\overline{LM}_{K_i}$ appearing in W need not be positive-dimensional, but in this case, since $K_1 = K_1'$, we must have that $|K_2| \geqslant 2$, and we can proceed by induction. The lemma follows. ☐

The Cremona action gives another possible linear order.

COROLLARY 4.10. *The set $\hat{\mathbb{G}}$ is an exceptional collection with respect to the order $<'$:*

$$(k_1, \ldots, k_t;\, a_1, \ldots, a_t) >' (k_1', \ldots, k_t';\, a_1', \ldots, a_t')$$

if the sequence $(k_t - a_t, -k_t, k_{t-1} - a_{t-1}, -k_{t-1}, \ldots)$ is lexicographically larger than $(k_s' - a_s', -k_s', k_{s-1}' - a_{s-1}', -k_{s-1}', \ldots)$.

Remark 4.11. The linear orders $<$ and $<'$ are clearly not $(S_2 \times S_N)$-equivariant. The lemma shows that both orders refine the $(S_2 \times S_N)$-equivariant relation $\prec$ given by paths in the quiver with arrows

$$\mathcal{T} \to \mathcal{T}' \iff R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') \neq 0\,.$$

In other words, this quiver has no cycles. It would be nice to describe it combinatorially. It would be even better to explicitly describe the algebra

$$\bigoplus_{\mathcal{T}\prec\mathcal{T}'} R\mathrm{Hom}(\mathcal{T},\mathcal{T}')\,.$$

Here are some easy observations about the quiver:

(1) If there is an arrow between $\mathcal{T}$ and $\mathcal{T}'$, then the corresponding strata have a non-empty intersection.

(2) The line bundles can be arranged to be at the right of torsion sheaves in the collection: for any torsion sheaf $\mathcal{T}'$ in $\hat{\mathbb{G}}$ and any line bundle $\mathcal{T} = G_a^\vee$, we have (in the notation of the proof of Lemma 4.9)

$$R\mathrm{Hom}(\mathcal{T},\mathcal{T}') = R\Gamma\left(G_{a|Z'}\otimes\mathcal{T}'\right) = C_1\boxtimes C_2\boxtimes\cdots$$

and $C_1 = R\mathrm{Hom}\left(L, G^\vee_{a_1'}\right)$, where $L = G_{a_1}$ if $a_1 < k_1'$ and $L=\mathcal{O}$ otherwise. In both cases, $C_1 = 0$ by Lemma 4.5.

(3) It is not true in general that sheaves can be pre-ordered by codimension of support. For example, on $\overline{LM}_8$, the sheaf $\mathcal{T}'$ with data $(3,2,3;2,1,1)$ and support Z' has to be to the right of the sheaf $\mathcal{T}$ with data $(3,5;1,3)$ and support Z with $Z'\subseteq Z$, as an easy computation as above shows that $R\mathrm{Hom}(\mathcal{T},\mathcal{T}')\neq 0$.

Let us give more information about the quiver. We first introduce some terminology.

Definition 4.12. Let $\mathcal{T}\in\hat{\mathbb{G}}$ with support Z:

$$Z = \overline{LM}_{K_1}\times\overline{LM}_{K_2}\times\cdots\times\overline{LM}_{K_t}\,,$$
$$\mathcal{T} = G^\vee_{a_1}\boxtimes\cdots\boxtimes G^\vee_{a_t}\,.$$

(1) We call $\overline{LM}_{K_1}$ the *first component* of Z, $\overline{LM}_{K_2}$ the *second component* of Z, etc. and $\overline{LM}_{K_t}$ the *last component* of Z.

(2) We say that we *remove the component* $\overline{LM}_{K_i}$ from $\mathcal{T}$ if we consider the sheaf $\tilde{\mathcal{T}}$ given by

$$\tilde{Z} = \overline{LM}_{K_1}\times\cdots\times\overline{LM}_{K_{i-1}}\times\overline{LM}_{K_{i+1}}\times\cdots\times\overline{LM}_{K_t}\,,$$
$$\tilde{\mathcal{T}} = G^\vee_{a_1}\boxtimes\cdots\boxtimes G^\vee_{a_{i-1}}\boxtimes G^\vee_{a_{i+1}}\boxtimes\cdots\boxtimes G^\vee_{a_t}\,.$$

(3) We say that the *end data* of $\mathcal{T}$ is $(k_1,k_t;\,k_1-a_1,a_t)$. Clearly, different objects in $\hat{\mathbb{G}}$ could have the same end data.

(4) Recall from the proof of Lemma 4.9 that to show that $R\mathrm{Hom}(\mathcal{T},\mathcal{T}')=0$, it suffices to show that

$$R\mathrm{Hom}\left(Li^*_{W,Z}\mathcal{T}, Li^*_{W,Z'}\mathcal{T}'\otimes(N+D)\right)=0\,,$$

where $W = Z\cap Z'$, N is the first Chern class of the normal bundle $\mathcal{N}_{W,Z'}$, that is, the sum of boundary divisors that cut out Z, and $D = D_{i_1}+\cdots+D_{i_r}$ is a (possibly empty) sum of boundary divisors containing both Z and Z'. We let

$$W = \overline{LM}_{S_1}\times\overline{LM}_{S_2}\times\cdots\,,$$
$$R\mathrm{Hom}\left({i_{W,Z}}^*\mathcal{T}, {i_{W,Z'}}^*\mathcal{T}'(N+D)\right) = C_{S_1}\boxtimes C_{S_2}\boxtimes\cdots\,.$$

In what follows, we will refer to the C_{S_i} as the *components* of $R\mathrm{Hom}(\mathcal{T},\mathcal{T}')$.

Lemmas 4.9 and 4.10 have the following corollary, which can be used as an algorithm to determine, given a pair of torsion objects $\mathcal{T}$ and $\mathcal{T}'$ in $\hat{\mathbb{G}}$, whether $R\mathrm{Hom}(\mathcal{T},\mathcal{T}')=0$ or $R\mathrm{Hom}(\mathcal{T}',\mathcal{T})=0$.

Corollary 4.13. *Let $\mathcal{T}$ and $\mathcal{T}'$ be torsion sheaves in $\hat{\mathbb{G}}$ with supports Z and Z' and end data $(k_1, k_t;\, b_1, b_t)$ and $(k_1', k_s';\, b_1', b_s')$. If the inequalities*

$$b_1 + b_t \leqslant b_1' + b_s' , \quad k_1 + k_t - b_1 - b_t \geqslant k_1' + k_s' - b_1' - b_s'$$

both hold and one of them is a strict inequality, then $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Moreover, if both inequalities are equalities, then $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') \neq 0$ is only possible when

$$b_1 = b_1' , \quad b_t = b_s' , \quad k_1 = k_1' , \quad k_t = k_s'$$

and the first and last components are the same, that is, $K_1 = K_1'$ and $K_t = K_s'$. Whenever all these conditions hold, we have that

$$R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0 \quad \text{if} \quad R\mathrm{Hom}(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}') = 0 ,$$

where $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}}'$ are the sheaves obtained from $\mathcal{T}$ and $\mathcal{T}'$, respectively, after removing the first and last components $\overline{LM}_{K_1}$ and $\overline{LM}_{K_t}$.

Proof. Recall that we have

$$a_1 = k_1 - b_1 , \quad a_1' = k_1' - b_1' , \quad a_t = b_t , \quad a_s' = b_s' .$$

By Lemma 4.9, if $k_1 - b_1 > k_1' - b_1'$, then $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Similarly, by Lemma 4.10, if $k_t - b_t > k_s' - b_s'$, then $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Since we assume

$$(k_1 - b_1) + (k_t - b_t) \geqslant (k_1' - b_1') + (k_s' - b_s') ,$$

it follows that we must have $k_1 - b_1 = k_1' - b_1'$, $k_t - b_t = k_s' - b_s'$. Now if $-k_1 > -k_1'$, again by Lemma 4.9, we have $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Similarly, if $-k_t > -k_s'$, by Lemma 4.10, we have $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Hence, we may assume $k_1 \geqslant k_1'$ and $k_t \geqslant k_s'$. But then $(k_1 + k_1') - (k_t + k_s') \geqslant 0$, while

$$(k_1 + k_1') - (k_t + k_s') = (b_1 + b_t) - (b_1' + b_s') \geqslant 0 .$$

Hence, we must have $k_1 = k_1'$ and $k_t = k_s'$ and hence $b_1 = b_1'$ and $b_t = b_s'$.

If these equalities hold, for the intersection $Z \cap Z'$ to be non-empty, we must have that the first and last components are the same, that is, $K_1 = K_1'$ and $K_t = K_s'$. As in the proof of Lemma 4.9 (case 4), we can remove the first and last components, $\overline{LM}_{K_1}$ and $\overline{LM}_{K_t}$, from Z and Z' and proceed with the rest. □

We finish this section by relating line bundles $G_1, \ldots, G_{n-1}$ on $\overline{LM}_n$ to the wonderful compactification of PGL_n. Following [Bri07], we identify $\mathrm{Pic}\,\overline{\mathrm{PGL}_n}$ with the weight lattice of PGL_n. Let $\alpha_1, \ldots, \alpha_{n-1}$ be simple roots, and let $\omega_1, \ldots, \omega_{n-1}$ be fundamental co-weights. It is shown in [Bri07] that $\alpha_1, \ldots, \alpha_{n-1}$ and $\omega_1, \ldots, \omega_{n-1}$ span the effective cone and the nef cone, respectively, of $\overline{\mathrm{PGL}_n}$. We identify $\overline{LM}_n$ with the closure of the maximal torus in PGL_n.

Proposition 4.14. *Divisors on $\overline{\mathrm{PGL}_n}$ corresponding to $\omega_1, \ldots, \omega_{n-1}$ restrict to divisors $G_1, \ldots, G_{n-1}$ on the Losev–Manin space $\overline{LM}_n$.*

Proof. First we consider divisors $D_1, \ldots, D_n$ on $\overline{\mathrm{PGL}_n}$ which correspond to simple roots $\alpha_1, \ldots, \alpha_{n-1}$. We will show that they restrict to total boundary divisors

$$\Delta_1 = \sum \delta_{0i} , \quad \ldots , \quad \Delta_{n-1} = \sum \delta_{0 i_1 \ldots i_{n-1}}$$

on the Losev–Manin space $\overline{LM}_n$. Indeed, it is known (see, for example, [Bri07]) that $D_1, \ldots, D_{n-1}$ are the boundary divisors of $\overline{\mathrm{PGL}_n}$, that is,

$$\overline{\mathrm{PGL}_n} \setminus \mathrm{PGL}_n = D_1 \cup \cdots \cup D_{n-1} .$$

In particular, every D_i restricts to a linear combination of boundary divisors of $\overline{LM}_n$. Since each of these divisors is PGL_n-invariant (acting by conjugation), the restriction is invariant under S_n (= Weyl group); that is, it is a linear combination of total boundary divisors.

The compactification $\overline{\mathrm{PGL}_n}$ is a spherical homogeneous space for the group $\mathrm{PGL}_n \times \mathrm{PGL}_n$, extending its action on PGL_n by left and right translations. The group of semi-invariant functions $\Lambda = k(\mathrm{PGL}_n)^{(\mathcal{B})}/k^*$ (where $\mathcal{B} = B^- \times B$ is a Borel subgroup of $\mathrm{PGL}_n \times \mathrm{PGL}_n$) is identified with the root lattice of PGL_n, which in turn is identified with the lattice of characters $M = k(T)^{(T)}/k^*$ of the maximal torus in PGL_n via the restriction of the $\mathcal{B}$-semi-invariant functions. Every boundary divisor D_i determines a functional $\rho(D_i)\colon \Lambda \to \mathbb{Z}$ (and so an element of the dual weight lattice $\Lambda^* = \mathbb{Z}^n/\langle 1, \ldots, 1\rangle$) by taking a divisorial valuation of a function in Λ along D_i. In fact, we have $\rho(D_i) = \omega_i = e_1 + \cdots + e_i \mod \langle 1, \ldots, 1\rangle$, the fundamental coweight (see, for example, [Bri07]). These vectors span the Weyl chamber, and the fan of $\overline{LM}_n$ (as a toric variety) is precisely the fan of its Weyl group translates. Moreover, vectors ω_i are primitive vectors along the rays which give boundary divisors $\delta_{0,1,\ldots,i}$; see [LM00]. So we are done by [BK05, Lemma 6.1.6].

By pulling back a coordinate hyperplane in $\mathbb{P}^{n-1}$ and symmetrizing, we get the formula

$$\psi_0 = \frac{n-1}{n}\Delta_1 + \cdots + \frac{1}{n}\Delta_{n-1}\,.$$

Combining it with Definition 1.9 yields the following formula for the G_i:

$$G_i = \sum_{j=1}^{n-1} B_{ij}\Delta_j\,,$$

where $B_{ij} = i(n-j)/n$ if $i \leqslant j$ and $B_{ij} = B_{ji}$ if $i > j$. It is well known and easy to prove that the inverse of the matrix B is the Cartan matrix of the root system A_{n-1}, and therefore $\psi_i = \sum_{j=1}^{n-1} B_{ij}\alpha_j$, which finishes the proof. □

5. Fullness of the exceptional collection $\hat{\mathbb{G}}$

We will need the following more general set-up.

Definition 5.1. For every integer $r \geqslant -1$, define a contravariant functor X^r from $\mathbb{N}$ to the category of smooth projective varieties as follows. Let X^r_N be an iterated blow-up of $\mathbb{P}^{n+r}$ in n general points (marked by N) followed by the blow-up of the $\binom{n}{2}$ proper transforms of the lines passing through two points, the $\binom{n}{3}$ proper transforms of the planes passing through three points, etc. For example,

$$X^{-1}_N = \overline{LM}_N\,, \quad X^r_\emptyset = \mathbb{P}^r\,.$$

For every $M \subseteq N$, the forgetful morphism $\pi_{N\setminus M}\colon X^r_N \to X^r_M$ is induced by a linear projection from points in $N \setminus M$.

For every subset $S \subseteq N$ of cardinality at most $n + r - 1$, we denote by $E_S \subseteq X^r_N$ the exceptional divisor over a subspace spanned by points in S.

Proposition 5.2. *All conditions of Theorem 3.5 are satisfied. Thus we have a semi-orthogonal decomposition*

$$D^b(X^r_S) = \left\langle D^b_{\mathrm{cusp}}(X^r_S), \left\{L\pi^*_K D^b_{\mathrm{cusp}}(X^r_{S\setminus K})\right\}_{K\subset S}, L\pi^*_N D^b(\mathbb{P}^r)\right\rangle,$$

where K runs over proper subsets of S in order of increasing cardinality.

Notation 5.3. For every $i \in N$, we have a birational morphism

$$f_i \colon X_N^r \to X_{N\setminus i}^{r+1},$$

obtained by blowing down exceptional divisors E_S, for $i \in S$, in the order of decreasing cardinality.

Definition 5.4 (Strata in $\boldsymbol{X_N^r}$). Consider partitions

$$N = N_1 \sqcup \cdots \sqcup N_k, \quad |N_u| > 0 \quad (u = 1, \ldots, k-1).$$

Set

$$Z_{N_1,\ldots,N_k} = E_{N_1} \cap E_{N_1 \cup N_2} \cap \cdots \cap E_{N_1 \cup \cdots \cup N_{k-1}}.$$

We call $Z_{N_1,\ldots,N_k}$ a *stratum* in X_N^r. We call a stratum in X_N^r *massive* if it is the image of a massive stratum in $\overline{LM}_{n+r+1}$ via the birational map $\overline{LM}_{n+r+1} \to X_N^r$ which is the composition of the maps f_i for those $i \notin N$.

For $r \geqslant 0$, each stratum $Z_{N_1,\ldots,N_k}$ is the image of a stratum in $\overline{LM}_{n+r+1}$. For all $r \geqslant -1$, we have an identification

$$Z_{N_1,\ldots,N_k} \simeq \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_{k-1}} \times X_{N_k}^r,$$

where $X_{N_k}^r$ is the blow-up of $\mathbb{P}^{|N_k|+r}$ at the linear subspaces spanned by the points in N_k. If $r \geqslant 0$, a stratum $Z_{N_1,\ldots,N_k}$ is massive if and only if $|N_u| \geqslant 2$ for all $u = 1, \ldots, k-1$ and $|N_k| + r > 0$.

Definition 5.5. We let $\hat{\mathbb{G}}_N^r$ be a collection of objects in $D^b(X_N^r)$ defined inductively as follows:

$$\hat{\mathbb{G}}_N^{-1} := \hat{\mathbb{G}}_N, \quad \hat{\mathbb{G}}_N^{r+1} = Rf_{i*}(\hat{\mathbb{G}}_N^r).$$

Definition 5.6. Consider the following line bundles on X_N^r (for $r \geqslant -1$):

$$G_a^{r\vee} = -aH + (a-1)\sum_{i \in N} E_i + (a-2)\sum_{i,j \in N} E_{ij} + \cdots \quad (a = 1, \ldots, n+r)$$

(the coefficient in front of the exceptional divisor must be positive).

Lemma 5.7. *For every $E \in D^b_{\mathrm{cusp}}(X_N^r)$, we have $Rf_{i*}(E) \in D^b_{\mathrm{cusp}}\big(X_{N\setminus i}^{r+1}\big)$. In particular, the collection $\hat{\mathbb{G}}_N^r$ is contained in $D^b_{\mathrm{cusp}}(X_N^r)$. Moreover, we have*

$$Rf_{i*}G_a^{r\vee} = G_a^{r+1\vee} \quad (a = 1, \ldots, n+r).$$

In particular, $\hat{\mathbb{G}}_N^r$ contains the line bundles $G_a^{r\vee}$ (for $1 \leqslant a \leqslant n+r$) and the following torsion objects:

$$\mathcal{T} = (i_Z)_*\mathcal{L}, \quad \mathcal{L} = G_{a_1}^\vee \boxtimes \cdots \boxtimes G_{a_{k-1}}^\vee \boxtimes G_{a_k}^{r\ \vee}$$

for all massive strata $Z = Z_{N_1,\ldots,N_k}$ in X_N^r, all $1 \leqslant a_u \leqslant |N_u| - 1$ for $u = 1, \ldots, k-1$ and all $1 \leqslant a_k \leqslant |N_k| + r$.

Proof. The first statement follows from the commutative diagram

$$\begin{array}{ccc} X_N^r & \xrightarrow{f_i} & X_{N\setminus i}^{r+1} \\ \Big\downarrow{\scriptstyle \pi_j} & & \Big\downarrow{\scriptstyle \pi_j} \\ X_{N\setminus j}^r & \xrightarrow{f_i} & X_{N\setminus i,j}^{r+1}. \end{array} \tag{5.1}$$

To prove the rest of the lemma, it suffices to prove $Rf_{i*}G_a^{r\vee}=G_a^{r+1\vee}$. Set $t=\min\{a-1,n\}$. Note that

$$Lf_i^*G_a^{r+1\vee} = f_i^*G_a^{r+1\vee} = -aH + (a-1)\sum_{j\in N\setminus\{i\}} E_j + (a-2)\sum_{j,k\in N\setminus\{i\}} E_{jk} + \cdots + (a-t)\sum_{J\subseteq N\setminus\{i\},\,|J|=t} E_J\,, \tag{5.2}$$

as the pull-backs $f_i^*E_J$ are simply the proper transforms of the divisors E_J under the blow-up map f_i. In particular, $f_i^*G_a^{r+1\vee} = G_a^{r+1\vee}+F$, where

$$F=(a-1)E_i+(a-2)\sum_{j\in N\setminus\{i\}} E_{ij}+\cdots+(a-t)\sum_{J\subseteq N, i\in J,\,|J|=t} E_J\,.$$

Note that the coefficient $a-|J|$ of any E_J appearing in F satisfies

$$1\leqslant a-|J|\leqslant n+r-j<n+r-j+1=\operatorname{codim}_{X_N^r}E_J\,.$$

This implies, after repeatedly applying Lemma 5.8 to the morphisms that successively blow down the divisors E_J with $i\in J$, for a fixed $|J|$ (starting from the larger $|J|$ to the smaller), that $Rf_{i*}\mathcal{O}(F)=\mathcal{O}$. It follows that

$$Rf_{i*}G_a^{r\vee}=Rf_{i*}\big({f_i}^*G_a^{r+1\vee}\otimes\mathcal{O}(F)\big)=G_a^{r+1\vee}\otimes Rf_{i*}\mathcal{O}(F)=G_a^{r+1\vee}\,. \qquad\square$$

The following lemma is well known.

Lemma 5.8. *Let $p\colon X\to Y$ be a blow-up of a smooth subvariety Z of codimension $r+1$ of a smooth projective variety Y. Let E be the exceptional divisor. Then for all $1\leqslant i\leqslant r$, we have*

$$Rp_*\mathcal{O}_X(iE)=\mathcal{O}_Y\,.$$

Lemma 5.9. *All sheaves in $\hat{\mathbb{G}}_N^r$ are exceptional.*

The proof is identical to that of Lemma 4.8, and we omit it. The same direct computation as in Lemma 4.3 shows that the line bundles G_a^r satisfy the same restriction properties as the line bundles G_a.

Lemma 5.10. *For $S\subseteq N$, identifying the exceptional divisor $E_S\subseteq X_N^r$ with the product $\overline{LM}_S\times X_{N\setminus S}^r$, we have*

$$G^r_{a|E_S}=\begin{cases} G_a\boxtimes\mathcal{O} & \text{if } a<|S|\,,\\ \mathcal{O} & \text{if } a=|S|\,,\\ \mathcal{O}\boxtimes G^r_{a-|S|} & \text{if } a>|S|\,.\end{cases}$$

The analogue of Lemma 4.5 is the following.

Lemma 5.11. (1) *Every G_a^r is nef (hence, globally generated), of relative degree 1 with respect to any forgetful map π_i, for $i\in N$.*

(2) *Each $(G_a^r)^\vee$ is acyclic.*

(3) *We have $R\mathrm{Hom}(G_a,G_b)=0$ if $a>b$.*

(4) *We have $R\Gamma(-\psi_0+G_a-G_b)=0$ if $a<b$.*

The proof is identical to that of Lemma 4.5, and we omit it. Note that unlike in the case $r=-1$, when $r\geqslant 0$, there is no more S_2-symmetry, and it is generally false that $R\mathrm{Hom}(G_a,G_b)=0$

if $a < b$. For example, if $r \geqslant 0$, then $R\mathrm{Hom}(G_1, G_2) = R\Gamma\big(H - \sum_{i\in N} E_i\big) \neq 0$. As a result, the order $<$ of Lemma 4.9 does not generally descend to an order on $\hat{\mathbb{G}}^r_N$ which makes $\hat{\mathbb{G}}^r_N$ an exceptional collection (for example, if $N = \emptyset$). However, the order $<'$ from Corollary 4.10 descends to an order on $\hat{\mathbb{G}}^r_N$ which makes $\hat{\mathbb{G}}^r_N$ an exceptional collection.

LEMMA 5.12. *For all $r \geqslant -1$, the set $\hat{\mathbb{G}}^r_N$ is an exceptional collection with respect to the following order. Let $\mathcal{T}, \mathcal{T}' \in \hat{\mathbb{G}}^r_N$. Let $(k_1, \dots, k_t; a_1, \dots, a_t)$ and $(k'_1, \dots, k'_s; a'_1, \dots, a'_s)$ be the corresponding data. Then $\mathcal{T} >' \mathcal{T}'$ if the sequence*

$$(k_t - a_t, -k_t, k_{t-1} - a_{t-1}, -k_{t-1}, \dots)$$

is lexicographically (that is, alphabetically) larger than

$$(k'_s - a'_s, -k'_s, k'_{s-1} - a'_{s-1}, -k'_{s-1}, \dots)\,.$$

Proof. The proof is similar to that of Lemma 4.9. We sketch the proof for completeness. Let Z and Z' be massive strata supporting sheaves $\mathcal{T} > \mathcal{T}'$ in $\hat{\mathbb{G}}$. These sheaves have the form $Ri_{Z*}\mathcal{L}$ and $Ri_{Z'*}\mathcal{L}'$, respectively. We have to show that $R\mathrm{Hom}(\mathcal{T}, \mathcal{T}') = 0$. Let U be the smallest stratum containing both Z and Z'. Then U is the intersection of codimension 1 strata (exceptional divisors) $D_1, \dots, D_s$ containing both Z and Z'. Let $W = Z \cap Z'$. We can assume that W is non-empty as otherwise there is nothing to prove. Let $i_{W,Z} \colon W \hookrightarrow Z$ and $i_{W,Z'} \colon W \hookrightarrow Z'$ be the inclusions. As in the proof of Lemma 4.9, it suffices to prove that

$$R\mathrm{Hom}(Li^*_{W,Z}\mathcal{L}, Li^*_{W,Z'}\mathcal{L}'(D + c_1(\mathcal{N}))) = 0\,,$$

where $c_1(\mathcal{N})$ is the first Chern class of the normal bundle $\mathcal{N} := \mathcal{N}_{W,Z'}$, that is, the sum of all the exceptional that cut out Z but do not contain Z'. We write

$$W = \overline{LM}_{K_1} \times \overline{LM}_{K_2} \times \dots \times \overline{LM}_{K_{s-1}} \times X^r_{K_t}\,,$$
$$R\mathrm{Hom}(Li_{W,Z}{}^*\mathcal{L}, Li_{W,Z'}{}^*\mathcal{L}'(D + c_1(\mathcal{N}))) = C_1 \otimes C_2 \otimes \dots \otimes C_t\,,$$

where C_1 is $R\mathrm{Hom}$ between components of line bundles $Li_{W,Z}{}^*\mathcal{L}$ and $Li_{W,Z'}{}^*\mathcal{L}'(D + c_1(\mathcal{N}))$ corresponding to $\overline{LM}_{K_1}$, etc.

Case 1. Suppose $k_t - a_t > k'_t - a'_t$. We prove that $C_t = 0$. If $k_t < k'_t$, then $a'_t > a_t + (k'_t - k_t) > (k'_t - k_t)$. Hence,

$$C_t = R\mathrm{Hom}(-G_{a_t}, -G_{a'_t - (k'_t - k_t)} - \psi_\infty)\,,$$

where $-\psi_0$ is a contribution from $c_1(\mathcal{N})$ (there is no contribution from D). As $a'_t - (k'_t - k_t) > a_t$, it follows that $C_t = 0$ by Lemma 5.11(4).

If $k_t = k'_t$, then $a'_t > a_t$. As there is no contribution from $c_1(\mathcal{N})$ to C_t, we have that either $C_t = R\mathrm{Hom}(-G_{a_t}, -G_{a'_t}) = 0$ (if D does not include D_{K_1}), or $C_t = R\mathrm{Hom}(-G_{a_t}, -G_{a'_t} - \psi_0) = 0$ (if D includes D_{K_1}).

If $k_t > k'_t$, then there are no contributions from $c_1(\mathcal{N})$ or D to C_t and

$$C_t = R\mathrm{Hom}(L, -G_{a'_t}) = 0\,,$$

where $L = -G_{a_t - (k_t - k'_t)}$ if $a_t > k_t - k'_t$ and $L = \mathcal{O}$ otherwise. As, by assumption, $a_t - (k_t - k'_t) < a'_t$, it follows that in both cases, $C_t = 0$ by Lemma 4.5.

Case 2. Suppose $k_t - a_t = k'_t - a'_t$ and $k_t < k'_t$. Then $a'_t = (k'_t - k_t) + a_t$ and hence $a'_t > (k'_t - k_t)$. As in case 1, we have

$$C_t = R\mathrm{Hom}(-G_{a_t}, -G_{a'_t - (k'_t - k_t)} - \psi_0) = R\mathrm{Hom}(-G_{a_t}, -G_{a_t} - \psi_0) = 0\,.$$

Case 3. Suppose $a_t = a'_t$, $k_t = k'_t$ and that D includes D_{K_t}. In this case, we have

$$C_t = R\mathrm{Hom}(-G_{a_t}, -G_{a_t} - \psi_0) = 0\,.$$

Case 4. Suppose $a_t = a'_t$, $k_t = k'_t$ and that D does not include D_{K_t}. In this case,

$$C_t = R\mathrm{Hom}(-G_{a_t}, -G_{a_t}) = \mathbb{C}$$

is useless. However, we can now use Corollary 4.10 as the remaining factors are Losev–Manin spaces (or, alternatively, proceed exactly as above, by restricting to the next Losev–Manin factor $\overline{LM}_{K_{t-1}}$ in W). The lemma follows. □

Lemma 5.13. *Let $r \geqslant -1$. For every $\mathcal{T} \in \hat{\mathbb{G}}_N^{r+1}$ and every $j \in N$, we have*

$$R\pi_{j_*} L{f_i}^* \mathcal{T} = 0\,.$$

Proof. We use the commutative diagram (5.1). Since π_j is flat and $X^r_{N\setminus j} \times_{X^{r+1}_{N\setminus ij}} X^{r+1}_{N\setminus i}$ has toroidal, and hence rational, singularities, the claim follows by cohomology and base change. □

To finish the proof of Theorem 1.10, we prove the following crucial result.

Lemma 5.14. *Let $r \geqslant -1$. For every $\mathcal{T} \in \hat{\mathbb{G}}_N^{r+1}$,*

$$\mathrm{Cone}\,[L{\pi_i}^* R\pi_{i*} L{f_i}^* \mathcal{T} \to L{f_i}^* \mathcal{T}]$$

belongs to the subcategory generated by $\hat{\mathbb{G}}_N^r$.

We postpone the proof of Lemma 5.14 to the end of this section. We use Lemma 5.14 to prove the following result (that implies Theorem 1.10).

Proposition 5.15. *If $N \neq \emptyset$, the subcategory $D^b_{\mathrm{cusp}}(X^r_N)$ is generated by $\hat{\mathbb{G}}^r_N$.*

This proves the following theorem.

Theorem 5.16. *For all $r \geqslant -1$, the set $\hat{\mathbb{G}}^r_N$ is a full exceptional collection in $D^b_{\mathrm{cusp}}(X^r_N)$.*

In particular, when $r = -1$, this gives that the set $\hat{\mathbb{G}}_N$ is a full exceptional collection in $D^b_{\mathrm{cusp}}\big(\overline{LM}_N\big)$ (Theorem 1.10).

Proof of Proposition 5.15. We argue by induction on the dimension $n+r$ and for a fixed $n+r$, by induction on n. The base of induction is X^{r-1}_1. Note that we have a $\mathbb{P}^1$-bundle $\pi_1 \colon X^{r-1}_1 \to \mathbb{P}^{r-1}$. By Orlov's Theorem 3.3, the category $D^b_{\mathrm{cusp}}\big(X^{r-1}_1\big)$ is generated by

$$\pi_1^* D^b\big(\mathbb{P}^{r-1}\big) \otimes \mathcal{O}(-E_1) = \langle \mathcal{O}(-rH + (r-1)E_1), \ldots, \mathcal{O}(-2H + E_1), \mathcal{O}(-H)\rangle\,,$$

which is precisely our claim in this case.

Assume $n \geqslant 2$. Choose an object $E \in D^b_{\mathrm{cusp}}(X^r_N)$ such that $R\mathrm{Hom}(\mathcal{T}, E) = 0$ for every $\mathcal{T} \in \hat{\mathbb{G}}^r_N$. We need to show that $E = 0$. We first show that $Rf_{i*}E = 0$ for all $i \in N$. Let $i \in N$. By Lemma 5.7, we have $Rf_{i*}E \in D^b_{\mathrm{cusp}}(X^{r+1}_{N\setminus\{i\}})$. By the inductive assumption, to prove that $Rf_{i*}E = 0$, it is sufficient to prove that $R\mathrm{Hom}(\mathcal{T}, Rf_{i*}E) = 0$ for every $\mathcal{T} \in \hat{\mathbb{G}}^{r+1}_{N\setminus\{i\}}$. Note that

$$R\mathrm{Hom}(\mathcal{T}, Rf_{i*}E) = R\mathrm{Hom}(Lf_i^* \mathcal{T}, E)\,.$$

If we let $C = \mathrm{Cone}\,[L{\pi_i}^* R\pi_{i*} L{f_i}^* \mathcal{T} \to L{f_i}^* \mathcal{T}]$, it follows by Lemma 5.14 that $R\mathrm{Hom}(C, E) = 0$. Using the distinguished triangle

$$L{\pi_i}^* R\pi_{i*} L{f_i}^* \mathcal{T} \to L{f_i}^* \mathcal{T} \to C \to L{\pi_i}^* R\pi_{i*} L{f_i}^* \mathcal{T}[1]$$

and the fact that for all $F \in D^b(X^r_{N\setminus\{i\}})$, we have (since $E \in D^b_{\mathrm{cusp}}(X^r_N)$)

$$R\mathrm{Hom}(L\pi_i{}^*F, E) = R\mathrm{Hom}(F, R\pi_{i*}E) = 0\,,$$

it follows that $R\mathrm{Hom}(Lf_i^*\mathcal{T}, E) = 0$. This proves that $Rf_{i*}E = 0$ for all $i \in N$. In particular, by Lemma 3.1, the support $\mathrm{Supp}\,E$ of E is contracted by all birational maps f_i, for $i \in N$:

$$\mathrm{Supp}\,E \subseteq \mathrm{Exc}(f_1) \cap \cdots \cap \mathrm{Exc}(f_n)\,.$$

Since $\mathrm{Exc}(f_i) = \bigcup_{i\in S} E_S$ and $E_S \cap E_T \neq \emptyset$ if and only if $S \subseteq T$ or $T \subseteq S$, this implies that $\mathrm{Exc}(f_1)\cap\cdots\cap\mathrm{Exc}(f_n)$ can be non-empty only if $r \geqslant 1$, in which case this intersection is contained in E_N, the exceptional divisor corresponding to blowing up the proper transform $\Delta_N \cong \overline{LM}_N$ of the subspace spanned by the points in N (the last blow-up). It follows that

$$\mathrm{Supp}\,E \subseteq E_N \cong \Delta_N \times \mathbb{P}^r\,.$$

For every $i \in N$, we can decompose $f_i = f_i' \circ p$, where $p\colon X^r_N \to Y$ blows down E_N (with image Δ_N) and $f_i'\colon Y \to X^{r+1}_{N\setminus\{i\}}$ is the composition of blow-downs of E_S for $S \subsetneq N$ containing i. If we write $E_S' = p(E_S)$, it is still the case that $E_S' \cap E_T' \neq \emptyset$ if and only if $S \subseteq T$ or $T \subseteq S$. It follows that

$$\mathrm{Exc}(f_1') \cap \cdots \cap \mathrm{Exc}(f_n') = \emptyset\,. \tag{5.3}$$

Since $Rf'_{i*}Rp_*E = Rf_{i*}E = 0$ for all i, equation(5.3) and Lemma 3.1 imply that $Rp_*E = 0$. Let $\alpha\colon E_N \hookrightarrow X^r_N$ be the inclusion map. By Orlov's Theorem 3.4, the object E belongs to the subcategory in $D^b(X^r_N)$ with semi-orthogonal decomposition

$$\big\langle R\alpha_*\big[D^b\big(\overline{LM}_N\big) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-r)\big], \ldots, R\alpha_*\big[D^b\big(\overline{LM}_N\big) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-1)\big]\big\rangle\,.$$

In particular, there exist morphisms

$$0 = E_0 \to E_1 \to \cdots \to E_r = E$$

that fit into exact triangles

$$E_{i-1} \to E_i \to F_i \to E_{i-1}[1] \quad \text{with} \quad F_i \in R\alpha_*\big[D^b\big(\overline{LM}_N\big) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i)\big]\,.$$

CLAIM 5.17. *We have* $F_i \in R\alpha_*\left[D^b_{\mathrm{cusp}}\big(\overline{LM}_N\big) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i)\right]$ *for all* $1 \leqslant i \leqslant r$.

The proposition now follows immediately from Claim 5.17: by the inductive hypothesis, the subcategory $R\alpha_*\left[D^b_{\mathrm{cusp}}\big(\overline{LM}_N\big) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i)\right]$ is generated by sheaves that belong to $\hat{\mathbb{G}}^r_N$, but the latter have no non-zero morphisms into E. Thus $E = 0$. □

Proof of Claim 5.17. Let $F_i = R\alpha_*(H_i \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i))$ for some $H_i \in D^b\big(\overline{LM}_N\big)$. We have to show that $H_i \in D^b_{\mathrm{cusp}}\big(\overline{LM}_N\big)$ for all i.

Let $j \in N$, and let $\alpha_j\colon \overline{LM}_{N\setminus\{j\}} \times \mathbb{P}^r \hookrightarrow X^r_{N\setminus\{j\}}$ be the inclusion. Then

$$R\pi_{j*}F_i = R\pi_{j*}R\alpha_*(H_i \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i)) = R\alpha_{j*}(R\pi_{j*}H_i \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i))\,,$$

where $R\pi_{j*}H_i \in D^b\big(\overline{LM}_{N\setminus\{j\}}\big)$. In particular, $R\pi_{j*}H_i = 0$ if and only if $R\pi_{j*}F_i = 0$. Note that $R\pi_{j*}F_i$ belongs to the subcategory

$$R\alpha_{j*}\big[D^b\big(\overline{LM}_{N\setminus\{j\}}\big) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i)\big]\,.$$

Suppose $R\pi_{j*}F_p \neq 0$ for some $j \in N$, and choose the maximal p with this property. Applying $R\pi_{j*}$ to the filtration gives morphisms

$$0 = R\pi_{j*}E_0 \to R\pi_{j*}E_1 \to \cdots \to R\pi_{j*}E_r = R\pi_{j*}E = 0$$

that fit into exact triangles

$$R\pi_{j*}E_{i-1} \to R\pi_{j*}E_i \to R\pi_{j*}F_i \to R\pi_{j*}E_{i-1}[1]\,.$$

In particular, $R\pi_{j*}E_p = R\pi_{j*}E_{p+1} = \cdots = R\pi_{j*}E_r = R\pi_{j*}E = 0$ and $R\pi_{j*}F_p \simeq R\pi_{j*}E_{p-1}[1]$. However, $R\pi_{j*}E_{p-1}[1]$ belongs to the subcategory generated by

$$R\alpha_j * \left[D^b\left(\overline{LM}_{N'}\right) \boxtimes \mathcal{O}_{\mathbb{P}^r}(-i)\right]$$

for $i < p$ and thus cannot have a non-zero morphism to $R\pi_{j*}F_p$. □

We now prove Lemma 5.14. The proof occupies the rest of this section. We first prove the case when $\mathcal{T}' = G_a^{r+1\vee}$ on $X^{r+1}_{N\setminus\{i\}}$ in Lemma 5.18.

Lemma 5.18. *Let $r \geqslant -1$. For all $1 \leqslant a \leqslant n+r$ and $i \in N$, we have*

$$\pi_{i*}\left(f_i^* G_a^{r+1\vee}\right) = 0\,,$$

$$R^1\pi_{i*}\left(f_i^* G_a^{r+1\vee}\right) = \begin{cases} G^r_{a-1}{}^\vee \oplus G^r_{a-2}{}^\vee \oplus \cdots \oplus G_1^{r\vee} & \text{if } a \geqslant 2\,,\\ 0 & \text{if } a = 1\,,\end{cases}$$

$$\mathrm{Cone}\left[L\pi_i{}^* R\pi_{i*} Lf_i{}^* G_a^{r+1\vee} \to Lf_i{}^* G_a^{r+1\vee}\right] = G_a^{r\vee} \oplus G^r_{a-1}{}^\vee \oplus \cdots \oplus G_1^{r\vee}\,.$$

Proof. If $a = 1$, then $f_i^* G_1^{r+1\vee} = -H = G_1^{r\vee}$ and the statements follow at once as $G_1^{r\vee}$ is a cuspidal object. Now, assume $a \geqslant 2$. For clarity, first consider the situation when $a \leqslant n$. In this case, we have

$$(G_a^r)^\vee = -aH + (a-1)\sum_{j\in N\setminus i} E_j + \cdots + 1\cdot \sum_{J\subseteq N\setminus i,\,|J|=a-1} E_J\,.$$

Define divisors on X_N^r as follows:

$$E^s = \sum_{i\in I\subseteq N,\,|I|=s} E_I \quad (1 \leqslant s \leqslant a-1)\,,$$

$$F_s = E^1 + E^2 + \cdots + E^s = E_i + \sum_j E_{ij} + \sum_{j,k} E_{ijk} + \cdots + \sum_{i\in I\subseteq N,\,|I|=s} E_I\,,$$

$$H_1 = f_i^* G_a^{r+1}\,,\quad H_2 = H_1 + F_1\,,\quad H_3 = H_2 + F_2\,,\quad \ldots\quad,\quad H_a = H_{a-1} + F_{a-1} = G_a^{r\vee}\,.$$

There are two sets of exact sequences that we will use, identifying as usual divisors with the corresponding line bundles:

(A)

$$0 \to H_1 \to H_2 \to H_{2|F_1} \to 0\,,$$
$$0 \to H_2 \to H_3 \to H_{3|F_2} \to 0\,,$$
$$\vdots$$
$$0 \to H_{a-1} \to H_a \to H_{a|F_{a-1}} \to 0\,.$$

(B) For $2 \leqslant k \leqslant a-1$ and $1 \leqslant s \leqslant k-1$, letting

$$Q^k_{s+1} = \left(H_{k+1} - F_s\right)_{|E^{s+1}} = \bigoplus_{i\in I\subseteq N,\,|I|=s+1} \left(H_{k+1} - F_s\right)_{|E_I}\,,$$

we have exact sequences

$$0 \to Q_2^k \to H_{k+1|F_2} \to H_{k+1|F_1} \to 0\,,$$
$$0 \to Q_3^k \to H_{k+1|F_3} \to H_{k+1|F_2} \to 0\,,$$
$$\vdots$$
$$0 \to Q_k^k \to H_{k+1|F_k} \to H_{k+1|F_{k-1}} \to 0\,.$$

Note that $F_{s+1} = F_s + E^{s+1}$ and that the sequences of type (B) are obtained by tensoring with H_{k+1} the canonical sequence

$$0 \to \mathcal{O}(-F_s)_{|E^{s+1}} \to \mathcal{O}_{F_{s+1}} \to \mathcal{O}_{F_s} \to 0\,. \tag{5.4}$$

Lemma 5.18 follows at once from taking $k = 1$ in parts (A1) and (A2) in Claim 5.19. Parts (B1)–(B3) in the claim refer to the exact sequences of type (B), while parts (A1)–(A2) in the claim refer to the exact sequences of type (A). Parts (B1)–(B3) will be used to prove parts (A1)–(A2) (this is why they appear first).

We now discuss the case when $a > n$. In this case, we have

$$(G_a^r)^\vee = -aH + (a-1)\sum_{j\in N\setminus i} E_j + \cdots + (a-(n-1))E_{N\setminus i}\,.$$

We define F_s as above in the range $1 \leqslant s \leqslant n$ and let

$$F_{a-1} = \cdots = F_{n+1} = F_n = E^1 + E^2 + \cdots + E^n\,.$$

We define H_k as above, for all $1 \leqslant k \leqslant a$. As before, $H_a = (G_a^r)^\vee$. We use the exact sequences of type (A). In order to analyze the sheaves $H_{k+1|F_k}$, there are two cases to consider: (1) $1 \leqslant k \leqslant n < a$ and (2) $n < k \leqslant a-1$. For a fixed k, we consider the sequences (5.4) of type (B), where for $1 \leqslant s \leqslant k-1$, the quotient Q_{s+1}^k is defined as before if $s \leqslant n-1$, while if $k-1 \geqslant s \geqslant n$, we let

$$Q_{n+1}^k = \cdots = Q_k^k = 0\,.$$

Hence, the exact sequences of type (B) that we consider are

$$0 \to Q_2^k \to H_{k+1|F_2} \to H_{k+1|F_1} \to 0\,,$$
$$0 \to Q_3^k \to H_{k+1|F_3} \to H_{k+1|F_2} \to 0\,,$$
$$\vdots$$
$$0 \to Q_n^k \to H_{k+1|F_n} \to H_{k+1|F_{n-1}} \to 0\,.$$

The rest of the proof is identical, as Claim 5.19 still holds. □

Claim 5.19. (B1) *For $2 \leqslant k \leqslant a-1$ and $1 \leqslant s \leqslant \min\{k-1, n-1\}$, we have*

$$R\pi_{i*}Q_{s+1}^k = 0\,.$$

(B2) *For all $1 \leqslant s \leqslant k \leqslant a-1$, we have*

$$R^l\pi_{i*}\big(H_{k+1|F_s}\big) = 0 \quad \textit{for all } l > 0 \quad \textit{and} \quad \pi_{i*}\big(H_{k+1|F_s}\big) = \big(G_k^r\big)^\vee\,.$$

(B3) *For all $1 \leqslant s \leqslant k \leqslant a-1$, the canonical map*

$$\pi_i{}^*\pi_{i*}\big(H_{k+1|F_s}\big) \to \big(H_{k+1|F_s}\big)$$

is surjective with kernel $\pi_i^*(G_k^r)^\vee \otimes \mathcal{O}(-F_s)$*. Moreover,*

$$\operatorname{Cone}\big[L\pi_i{}^* R\pi_{i*}(H_{k+1|F_s}) \to (H_{k+1|F_s})\big] = \big(\pi_i^*(G_k^r)^\vee \otimes \mathcal{O}(-F_s)\big)[1]\,.$$

In particular,

$$\operatorname{Cone}\big[L\pi_i{}^* R\pi_{i*}(H_{k+1|F_k}) \to (H_{k+1|F_k})\big] = (G_k^r)^\vee[1]\,.$$

(A1) *For all* $1 \leqslant k \leqslant a-1$*, we have*

$$\pi_{i*}(H_k) = 0 \quad \textit{and} \quad R^1\pi_{i*}(H_k) = (G_{a-1}^r)^\vee \oplus (G_{a-2}^r)^\vee \oplus \cdots \oplus (G_k^r)^\vee\,.$$

(A2) *For all* $1 \leqslant k \leqslant a$*, we have*

$$\operatorname{Cone}\big[L\pi_i{}^* R\pi_{i*}(H_k) \to (H_k)\big] = (G_a^r)^\vee \oplus (G_{a-1}^r)^\vee \oplus \cdots \oplus (G_k^r)^\vee\,.$$

Proof. We prove parts (B1)–(B3). From the commutative diagram

$$\begin{array}{ccc} \overline{LM}_I \times X^r_{N\setminus I} = E_I & \longrightarrow & X^r_N \\ \Big\downarrow{\scriptstyle (\pi_i,\mathrm{Id})} & & \Big\downarrow{\scriptstyle \pi_i} \\ \overline{LM}_J \times X^r_{N\setminus I} = E_J & \longrightarrow & X^r_{N\setminus i}\,, \end{array} \tag{5.5}$$

it follows that

$$R\pi_{i*}\big(-\psi_x \boxtimes (G_{k-s}^r)^\vee\big) = R\pi_{i*}(-\psi_x) \boxtimes (G_{k-s}^r)^\vee = 0$$

as $R\pi_{i*}(-\psi_x) = 0$. Hence, (5.20) implies that $R\pi_{i*}Q^k_{s+1} = 0$, thus proving part (B1). Note that it suffices to prove parts (B2) and (B3) for $1 \leqslant s \leqslant \min\{k, n\}$, as $F_n = F_{n+1} = \cdots = F_{a-1}$. Clearly, part (B2) follows immediately from part (B1), the exact sequences of type (B) and the diagram (5.10). We now prove part (B3) by induction on s (for a fixed k). Set

$$h_s \colon \pi_i{}^*\pi_{i*}(H_{k+1|F_s}) \to (H_{k+1|F_s})\,, \quad \mathcal{K}_s = \operatorname{Ker}(h_s)\,.$$

We use the following two observations: (1) for any sheaf $\mathcal{T}$, the canonical map $\pi_i{}^*\pi_{i*}(\mathcal{T}) \to \mathcal{T}$ is non-zero whenever $\pi_{i*}(\mathcal{T})$ is non-zero, and (2) if $F \subset X$ is an effective divisor and $\mathcal{L}$ is a line bundle on X, the only non-zero morphism $\mathcal{L} \to \mathcal{L}_{|F}$ is the restriction map (with kernel $\mathcal{L}(-F)$).

When $s = 1$, we have from part (B2) and (5.10) that

$$\pi_{i*}(H_{k+1|F_1}) = (G_k^r)^\vee\,, \quad \pi_i{}^*\pi_{i*}(H_{k+1|F_1}) = \pi_i{}^*(G_k^r)^\vee\,,$$
$$H_{k+1|F_1} = (G_k^r)^\vee = \big(\pi_i{}^*(G_k^r)^\vee\big)_{|F_1} \quad \text{on } F_1 = E_i\,.$$

Hence, it follows from observations (1) and (2) that h_1 is surjective and $\mathcal{K}_1 = \pi_i^*(G_k^r)^\vee \otimes \mathcal{O}(-F_1)$. Now assume that h_s is surjective and $\mathcal{K}_s = \pi_i^*(G_k^r)^\vee \otimes \mathcal{O}(-F_s)$. By applying $\pi_i^*\pi_{i*}(-)$ to the exact sequence

$$0 \to Q^k_{s+1} \to (H_{k+1})_{|F_{s+1}} \to (H_{k+1})_{|F_s} \to 0\,, \tag{5.6}$$

it follows from part (B1) that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \pi_i{}^*\pi_{i*}(H_{k+1|F_{s+1}}) & \longrightarrow & \pi_i{}^*\pi_{i*}(H_{k+1|F_s}) & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle h_{s+1}} & & \Big\downarrow{\scriptstyle h_s} & & \\ 0 & \longrightarrow & Q^k_{s+1} & \longrightarrow & (H_{k+1|F_{s+1}}) & \longrightarrow & (H_{k+1|F_s}) & \longrightarrow & 0\,. \end{array}$$

By our inductive assumption, h_s is surjective. By the snake lemma, there is an exact sequence

$$0 \to \mathcal{K}_{s+1} \to \mathcal{K}_s \to Q^k_{s+1} \to \operatorname{Coker}(h_{s+1}) \to 0\,.$$

The induced map $\mathcal{K}_s \to Q^k_{s+1}$ is non-zero. Otherwise, $Q^k_{s+1} \cong \operatorname{Coker}(h_{s+1})$, which implies that the exact sequence (5.6) is split since there is a retract $\big(H_{k+1|F_{s+1}}\big) \to Q^k_{s+1}$. But the sequence (5.6) is obtained by tensoring the canonical sequence (5.4) with a line bundle, and (5.4) is not split, as there are no non-zero morphisms $\mathcal{O}_{F_{s+1}} \to \mathcal{O}_{E^{s+1}}(-F_s)$:

$$\operatorname{Hom}\big(\mathcal{O}_{F_{s+1}}, \mathcal{O}_{E^{s+1}}(-F_s)\big) = \mathrm{H}^0\big(\mathcal{O}_{E^{s+1}}(-F_s)\big) = 0$$

by (5.7), and we have a contradiction. We have $\mathcal{K}_s = \pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_s)$ by the induction assumption. By (5.20), we have $Q^k_{s+1} = \big(\pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_s)\big)_{|E^{s+1}}$. Hence, $Q^k_{s+1} = (\mathcal{K}_s)_{|E^{s+1}}$. By observation (2), the map $\mathcal{K}_s \to Q^k_{s+1}$ is surjective, that is, $\operatorname{Coker}(h_{s+1}) = 0$, and furthermore

$$\mathcal{K}_{s+1} = \mathcal{K}_s\big(-E^{s+1}\big) = \pi_i^*(G^r_k)^\vee \otimes \mathcal{O}\big(-F_s - E^{s+1}\big) = \pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_{s+1})\,.$$

This proves the first statement in part (B3). In particular, this gives

$$\operatorname{Cone}\big[L\pi_i{}^* R\pi_{i*}\big(H_{k+1|F_k}\big) \to \big(H_{k+1|F_k}\big)\big] = \big(\pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_k)\big)[1]\,,$$

and now the last statement in part (B3) follows from

$$G^{r\vee}_k = \pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_k)\,.$$

We now prove parts (A1) and (A2). Apply $\pi_{i*}(-)$ to the exact sequences of type (A). Then part (A1) follows from part (B2) and downward induction, using the fact that there are no non-trivial extensions between $(G^r_k)^\vee$ and $(G^r_{k'})^\vee$ for $k \neq k'$. Similarly, to prove part (A2), we use downward induction on $1 \leqslant k \leqslant a$ and the exact sequences of type (A). As $H_a = (G^r_a)^\vee$, we have

$$\operatorname{Cone}\big[L\pi_i{}^* R\pi_{i*}\big(H_a\big) \to \big(H_a\big)\big] = (G^r_a)^\vee\,.$$

Note that if $\pi\colon X \to Y$ is a morphism between smooth projective varieties and $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence of sheaves on X, there is a distinguished triangle relating the cones $C_i = \operatorname{Cone}[L\pi^* R\pi_* A_i \to A_i]$:

$$C_1 \to C_2 \to C_3 \to C_1[1]\,.$$

Then part (A2) follows from part (B3) by using the fact that there are no non-trivial extensions between $G^{r\vee}_k$ and $G^{r\vee}_{k'}$ for $k \neq k'$. □

Claim 5.20. *For all subsets $I \subseteq N$ with $i \in I$, where $|I| = s+1$ with $1 \leqslant s \leqslant n-1$, on $E_I \cong \overline{LM}_I \times X^r_{N\setminus I}$, we have*

$$F_{s|E_I} = \psi_x \boxtimes \mathcal{O} \tag{5.7}$$

(here x is the attaching point). Now, assume $1 \leqslant k \leqslant a-1$ and $1 \leqslant s \leqslant \min\{k-1, n-1\}$. Then

$$H_{k+1|E_I} = \mathcal{O} \boxtimes \big(G^r_{k-s}\big)^\vee = \big(\pi_i^*(G^r_k)^\vee\big)_{|E_I}\,. \tag{5.8}$$

Hence, we have

$$\big(H_{k+1} - F_s\big)_{|E_I} = (-\psi_x) \boxtimes \big(G^r_{k-s}\big)^\vee = \big(\pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_s)\big)_{|E_I}\,,$$

$$Q^k_{s+1} = \bigoplus_{i\in I\subseteq N,\, |I|=s+1} (-\psi_x) \boxtimes \big(G^r_{k-s}\big)^\vee = \big(\pi_i^*(G^r_k)^\vee \otimes \mathcal{O}(-F_s)\big)_{|E^{s+1}}\,. \tag{5.9}$$

Moreover, on $E_i \cong X^r_{N\setminus\{i\}}$, we have

$$H_{k+1|E_i} = (G^r_k)^\vee\,. \tag{5.10}$$

Proof. To prove (5.7), we let $I = J \cup \{i\}$. Then $|J| = s$. We have

$$(F_s)_{|E_I} = \left(E_i + \sum_{j \in N \setminus i} E_{ij} + \cdots + \sum_{K \subseteq N \setminus i,\, |K| = s-1} E_K \right)_{|E_I}$$
$$= \delta_{J \cup \{x\}} + \sum_{j \in J} \delta_{(J \setminus \{j\}) \cup \{x\}} + \cdots + \sum_{j \in J} \delta_{jx}$$

(as divisors on $\overline{LM}_I$). Using the ψ_x Kapranov model of $\overline{LM}_I$, we have

$$(F_s)_{|E_I} = \Lambda_J + \sum_{j \in J} E_{J \setminus \{j\}} + \cdots + \sum_{j \in J} E_j = H\,.$$

Here Λ_J denotes the class of the proper transform in $\overline{LM}_I$ of the hyperplane in $\mathbb{P}^s$ spanned by the points in J. This proves (5.7).

To see (5.8) and (5.10), recall that if $1 \leqslant k \leqslant a-1$, then

$$H_{k+1} = H_1 + F_1 + \cdots + F_k\,,$$
$$H_1 = -aH + (a-1) \sum_{j \in N \setminus \{i\}} E_j + (a-2) \sum_{j,k \in N \setminus \{i\}} E_{jk} + \cdots + (a-t) \sum_{K \subseteq N \setminus \{i\},\, |K| = t} E_K\,,$$

where $t = \min\{a-1, n-1\}$. There are two cases to consider:

(1) $k \leqslant n$ (with either $n \leqslant a-1$ or $a-1 \leqslant n$)

(2) $k > n$, in which case we must have $a-1 > n$ and $t = n-1$. Note that we must have $r \geqslant 2$ as $n + r - 1 \geqslant a - 1 > n$.

In case (1), we have

$$F_1 + \cdots + F_k = kE_i + (k-1) \sum_{j \in N \setminus \{i\}} E_{ij} + \cdots + 1 \cdot \sum_{i \in K \subseteq N,\, |K| = k} E_K\,.$$

In case (2), we have

$$F_1 + \cdots + F_k = kE_i + (k-1) \sum_{j \in N \setminus \{i\}} E_{ij} + \cdots$$
$$\cdots + (k-n+2) \sum_{i \in K \subseteq N,\, |K| = n-1} E_K + (k-n+1) E_N\,.$$

Now let $1 \leqslant s \leqslant \min\{k-1, n-1\}$, and let $I \subseteq N$, with $i \in I$ and $|I| = s+1$ for some $0 \leqslant s \leqslant k-1$. Let

$$\mathcal{O}(H_{k+1})_{|E_I} = \mathcal{O}(H') \boxtimes \mathcal{O}(H'')\,,$$

where H' is the component on $\overline{LM}_I$ and H'' is the component on $X^r_{N \setminus I}$. We now compute H' and H''. Note that only the divisors E_K with $I \subseteq K \subseteq N$ contribute to H''. For example, H_1 does not; that is, we have

$$H'' = (F_1 + \cdots + F_k)_{|E_I}\,.$$

In case (1), we have

$$H'' = (k-s)(-\psi_x) + (k-s-1)\sum_{k\in N\setminus I}\delta_{k,x} + (k-s-2)\sum_{k,l\in N\setminus I}\delta_{k,l,x} + \dots + 1\cdot \sum_{K\subseteq N\setminus I,\,|K|=k-s-1}\delta_{K\cup\{x\}}\,.$$

(as a divisor on $\overline{M}_{0,N\setminus I}$), while in case (2), we have

$$H'' = (k-s)(-\psi_x) + (k-s-1)\sum_{k\in N\setminus I}\delta_{k,x} + (k-s-2)\sum_{k,l\in N\setminus I}\delta_{k,l,x} + \dots + (k+1-n)\delta_{(N\setminus I)\cup\{x\}}\,.$$

In both cases, $H'' = \left(G^r_{k-s}\right)^\vee$ as by the definition of G^r_{k-s} on $X^r_{N\setminus I}$,

$$\left(G^r_{k-s}\right)^\vee = (k-s)H - (k-s-1)\sum_{j\in N\setminus I}E_j - \dots - (k-s-t')\sum_{K\subseteq N\setminus I,\,|K|=t'}E_K$$

(in the Kapranov model given by ψ_x), where $t' = \min\{k-s-1, n-s-1\}$; that is, $t' = k-s-1$ in case (1), and $t' = n-s-1$ in case (2).

We now calculate H'. Let $I = J\cup\{i\}$. Since $|J| = s \leqslant \min\{k-1, n-1\}$, we have $s\leqslant t = \min\{a-1, n-1\}$. Using the ψ_0 Kapranov model of $\overline{LM}_I$, we obtain that the contribution from H_1 to H' comes from

$$-aH + (a-1)\sum_{j\in J}E_j + \dots + (a-(s-1))\sum_{K\subseteq J,\,|K|=s-1}E_K + (a-s)E_J$$

and equals

$$-aH + (a-1)\sum_{j\in J}E_j + \dots + (a-(s-1))\sum_{K\subseteq J,\,|K|=s-1}E_K + (a-s)\Lambda_J\,,$$

while the contribution from $F_1+\dots+F_k$ to H' comes from

$$kE_i + (k-1)\sum_{j\in J}E_{ij} + \dots + (k-(s-2))\sum_{K\subseteq J,\,|K|=s-2}E_{K\cup\{i\}} + (k-(s-1))\sum_{K\subseteq J,\,|K|=s-1}E_{K\cup\{i\}} + (k-s)E_J\,,$$

and equals

$$kE_i + (k-1)\sum_{j\in J}E_{ij} + \dots + (k-(s-2))\sum_{K\subseteq J,\,|K|=s-2}E_{K\cup\{i\}} + (k-(s-1))\sum_{K\subseteq J,\,|K|=s-1}\Lambda_{K\cup\{i\}} + (k-s)(-\psi_x)\,.$$

Here Λ_S (for $S\subseteq I$ with $|S| = s$) denotes the class of the proper transform in $\overline{LM}_I$ of the hyperplane in $\mathbb{P}^s$ spanned by the points in S, that is,

$$\Lambda_S = H - \sum_{K\subseteq S,\,1\leqslant|K|\leqslant s-1}E_K\,.$$

We now sum up these two terms and compute the coefficient of H to be

$$-a + (a - s) + (k - s + 1)s - (k - s)s = 0\,.$$

Here we use that on $\overline{LM}_I$, the class of ψ_x in the ψ_0 Kapranov model is

$$\psi_x = sH - (s-1)\sum_{j\in I} - (s-2)\sum_{j,k\in I} - \cdots\,.$$

Similarly, the coefficient of E_K for $K \subseteq J$ with $|K| = l$ is

$$(a - l) - (a - s) - (k - s + 1)(s - l) + (k - s)(s - l) = 0\,,$$

while the coefficient of $E_{K\cup\{i\}}$ for $K \subseteq J$ with $|K| = l$ is

$$(k - l) - (k - s + 1)(s - l) + (k - s)(s - l - 1) = 0\,.$$

Hence, $H' = 0$ and $\mathcal{O}(H_{k+1})_{|E_I} = \mathcal{O} \boxtimes \left(G^r_{k-s}\right)^\vee$.

To see that $\left(\pi_i^*(G^r_k)^\vee\right)_{|E_I} = \mathcal{O} \boxtimes \left(G^r_{k-s}\right)^\vee$, we use the commutative diagram (5.5). Note the equality of line bundles $(G^r_k)^\vee{}_{|E_J} = \mathcal{O} \boxtimes \left(G^r_{k-s}\right)^\vee$ (Remark 5.10). This finishes the proof of (5.8). The case when $E_I = E_i$ corresponds to the case $s = 0$, and the above computation shows (5.10). Clearly, (5.20) follows from (5.8) and (5.7). □

To prove the general case of Lemma 5.14, we need the following.

LEMMA 5.21. *If $\pi_i\colon X^r_N \to X^r_{N\setminus i}$ is the forgetful map, then for all $1 \leqslant a \leqslant n + r - 1$, the line bundle $\pi_i^*\left(G^r_a\right)^\vee \otimes \mathcal{O}(-E_i)$ belongs to the subcategory generated by $\hat{\mathbb{G}}^r_N$.*

Proof. Let $t = \min\{a-1, n-1\}$. Keeping the notation of the proof of Lemma 5.18, consider the divisors F_s on X^r_N, for $1 \leqslant s \leqslant t+1$, defined by

$$F_s = E^1 + E^2 + \cdots + E^s = E_i + \sum_j E_{ij} + \sum_{j,k} E_{ijk} + \cdots + \sum_{i\in I\subseteq N,\,|I|=s} E_I\,,$$

and let

$$L^1 = \pi_i^*\left(G^r_a\right)^\vee - F_1\,,\quad L^2 = \pi_i^*\left(G^r_a\right)^\vee - F_2\,,\quad \ldots\quad,\quad L^{t+1} = \pi_i^*\left(G^r_a\right)^\vee - F_{t+1}\,.$$

We claim that $\pi_i^*\left(G^r_a\right)^\vee - F_{t+1} = (G^r_a)^\vee$. This is clear if one considers separately the two cases $a \leqslant n$ and $a > n$. For example, if $a > n$, then

$$(G^r_a)^\vee = -aH + (a-1)\sum_{j\in N} + \cdots + (a-n)E_N = \pi_i^*(G^r_a)^\vee - F_n\,.$$

We have to prove that L^1 belongs to the subcategory generated by $\hat{\mathbb{G}}^r_N$. We use the exact sequences

$$0 \to L^2 \to L^1 \to \bigoplus_{j\in N\setminus i} \left(L^1\right)_{|E_{ij}} \to 0\,,$$

$$0 \to L^3 \to L^2 \to \bigoplus_{j,k\in N\setminus i} \left(L^2\right)_{|E_{ijk}} \to 0\,,$$

$$\vdots$$

$$0 \to L^{t+1} \to L^t \to \bigoplus_{\substack{J\subseteq N\setminus\{i\},\\ |J|=t}} \left(L^t\right)_{|E_{J\cup\{i\}}} \to 0\,.$$

Clearly, it is enough to prove that for all $1 \leqslant s \leqslant t$ and $J \subseteq N \setminus \{i\}$ with $|J| = s$, the sheaves $(L^s)_{|E_{J\cup\{i\}}}$ are in the subcategory generated by $\hat{\mathbb{G}}^r_N$. Note that $E_{J\cup\{i\}}$ is a massive stratum in X^r_N as

$$|J \cup \{i\}| = s+1 \geqslant 2\,, \quad |N \setminus J| + r = n - s + r > 0$$

since $s \leqslant t \leqslant n-1$ and when $r=-1$, we have $t = a-1$ and $a \leqslant n-1$.

As in (5.8), we have

$$\left(\pi_i^* G_a^{r\vee}\right)_{|E_{J\cup\{i\}}} = \mathcal{O} \boxtimes (G^r_{a-s})^\vee\,,$$

while by (5.7), we have

$$\mathcal{O}(-F_s)_{|E_{J\cup\{i\}}} = (-\psi_x) \boxtimes \mathcal{O}\,.$$

It follows that $(L^s)_{|E_{J\cup\{i\}}}$ is one of the objects in $\hat{\mathbb{G}}^r_N$, as it equals $(-\psi_x) \boxtimes (G^r_{a-s})^\vee$. □

Proof of Lemma 5.14. Consider the case when $\mathcal{T}$ is a torsion sheaf. Let

$$\mathcal{T} = (i_Z)_* \mathcal{L}, \quad \mathcal{L} = G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes G^{r+1\vee}_{a_l}\,,$$

where $Z = Z_{N_1,N_2,\ldots,N_l}$ is the massive stratum in $X^{r+1}_{N\setminus\{i\}}$ corresponding to a partition $N_1 \sqcup \cdots \sqcup N_l$ of $N \setminus \{i\}$. Since Z is massive, we have $|N_t| \geqslant 2$ for every $1 \leqslant t \leqslant l-1$ and $|N_l| + r + 1 > 0$. The preimage $Z' = f_i^{-1}(Z)$ is a massive stratum in X^r_N, and there is a commutative diagram

$$\begin{array}{ccc} Z' = f_i^{-1}(Z') & \xrightarrow{\ i_{Z'}\ } & X^r_N \\ {\scriptstyle \mathrm{Id}\times f_i^{N_l}}\big\downarrow & & \big\downarrow {\scriptstyle f_i} \\ Z & \xrightarrow{\ i_Z\ } & X^{r+1}_{N\setminus i}\,, \end{array}$$

where $i_{Z'}$ and i_Z are the canonical inclusions, we identify

$$Z = \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_{l-1}} \times X^{r+1}_{N_l}\,, \quad Z' = \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_{l-1}} \times X^r_{N_l\cup\{i\}}$$

and $f_i^{N_l}$ denotes the blow-up map $X^r_{N_l\cup\{i\}} \to X^{r+1}_{N_l}$ (we write f_i whenever there is no risk of confusion). Let $\mathcal{T}' = Lf_i^*\mathcal{T}'$. Then

$$\mathcal{T}' = (i_{Z'})_*\mathcal{L}'\,, \quad \mathcal{L}' = (\mathrm{Id}\times f_i)^*\mathcal{L}' = G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes f_i^* G^{r+1\vee}_{a_l}\,.$$

We compute $\mathrm{Cone}\,[L\pi_i{}^* R\pi_{i*}\mathcal{T}' \to \mathcal{T}']$ by the exact same argument as in the proof of Lemma 5.18. We define divisors $H_1, H_2, \ldots, H_{a_l}$ on $X^r_{N_l\cup\{i\}}$ exactly as before, so that we have

$$H_1 = f_i^*\left(G^{r+1}_{a_l}\right)^\vee\,, \quad H_{a_l} = \left(G^r_{a_l}\right)^\vee.$$

On $X^r_{N_l\cup\{i\}}$, consider the exact sequences of type (A) in the proof of Lemma 5.18. After taking the box product with $G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}}$, one obtains exact sequences on Z'. It is enough to prove that for all $1 \leqslant k \leqslant a_l - 1$, the object $\mathrm{Cone}\,[L\pi_i{}^* R\pi_{i*}\mathcal{T}_k \to \mathcal{T}_k]$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$, where

$$\mathcal{T}_k = (i_{Z'})_*\left(G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes \left(H_{k+1|F_k}\right)\right).$$

On $X^r_{N_l\cup\{i\}}$, we consider the exact sequences of type (B) in the proof of Lemma 5.18 after taking the box product with $G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}}$. Let

$$\mathcal{T}_{k,s} = (i_{Z'})_*\left(G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes Q^k_{s+1}\right),$$
$$\tilde{\mathcal{T}}_k = (i_{Z'})_*\left(G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes \left(H_{k+1|F_1}\right)\right).$$

Then Cone $\left[L\pi_i{}^* R\pi_{i*}\mathcal{T}_k \to \mathcal{T}_k\right]$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$ if and only if

$$\text{Cone}\left[L\pi_i{}^* R\pi_{i*}\mathcal{T}_{k,s} \to \mathcal{T}_{k,s}\right] \quad \text{and} \quad \text{Cone}\left[L\pi_i{}^* R\pi_{i*}\tilde{\mathcal{T}}_k \to \tilde{\mathcal{T}}_k\right]$$

are in the subcategory generated by $\hat{\mathbb{G}}^r_N$. By (5.20), the sheaf Q^k_{s+1} is a direct sum of objects in $\hat{\mathbb{G}}^r_{N_l\cup\{i\}}$. Hence, $\mathcal{T}_{k,s}$ is a direct sum of objects in $\hat{\mathbb{G}}^r_N$. In particular, Cone $\left[L\pi_i{}^* R\pi_{i*}\mathcal{T}_{k,s} \to \mathcal{T}_{k,s}\right] = \mathcal{T}_{k,s}$. We are left to prove that Cone $\left[L\pi_i{}^* R\pi_{i*}\tilde{\mathcal{T}}_k \to \tilde{\mathcal{T}}_k\right]$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$.

For simplicity, set $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_k$. Let $\overline{Z} := \pi_i(Z')$. We make the identification

$$\overline{Z} = \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_{l-1}} \times X^r_{N_l}\,.$$

Then $\pi_i^{-1}(\overline{Z}) = Z^1 \cup \cdots \cup Z^l$, where

$$Z^l = Z' = \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_{l-1}} \times X^r_{N_l\cup\{i\}}\,,$$
$$Z^t = \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_t\cup\{i\}} \times \cdots \times \overline{LM}_{N_{l-1}} \times X^r_{N_l}\,, \quad 1 \leqslant t \leqslant l-1\,.$$

As the divisor E_i in $X^r_{N_l\cup\{i\}}$ can be identified with $\overline{LM}_{\{i\}} \times X^r_{N_l}$, the sheaf $\tilde{\mathcal{T}}$ is supported on the non-massive stratum

$$Z^l \cap Z^{l-1} = \overline{LM}_{N_1} \times \cdots \times \overline{LM}_{N_{l-1}} \times \overline{LM}_{\{i\}} \times X^r_{N_l}\,,$$
$$\tilde{\mathcal{T}} = \left(i_{Z^l\cap Z^{l-1}}\right)_* \mathcal{M}\,, \quad \mathcal{M} = G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes \mathcal{O} \boxtimes G^{r\,\vee}_k\,,$$

where $i_{Z^l\cap Z^{l-1}}\colon Z^l \cap Z^{l-1} \to X^r_N$ is the canonical inclusion. Denote by

$$v\colon Z^l \cap Z^{l-1} \to \pi_i{}^{-1}(\overline{Z})\,, \quad u\colon \pi_i{}^{-1}(\overline{Z}) \to X^r_N$$

the canonical inclusions. Then $i_{Z^l\cap Z^{l-1}} = u \circ v$. Let $\rho = \pi_{i|\pi_i{}^{-1}(\overline{Z})}$. There is a commutative diagram

$$\begin{array}{ccc} \pi_i{}^{-1}(\overline{Z}) = Z^1 \cup \cdots \cup Z^l & \xrightarrow{u} & X^r_N \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi_i} \\ \overline{Z} & \xrightarrow{i_{\overline{Z}}} & X^r_{N\setminus i}\,. \end{array}$$

The restriction maps $\rho_{|Z^t}\colon Z^t \to \overline{Z}$ are induced by the forgetful maps $\overline{LM}_{N_t\cup\{i\}} \to \overline{LM}_{N_t}$ if $t < l$ and $X^r_{N_l\cup\{i\}} \to X^r_{N_l}$ for $t = l$. Note that the restriction map $\rho_{|Z^l\cap Z^{l-1}}\colon Z^l \cap Z^{l-1} \to \overline{Z}$ is an isomorphism. Let

$$\overline{\mathcal{M}} = R\rho_*(Rv_*\mathcal{M}) = G^\vee_{a_1} \boxtimes \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes G^{r\,\vee}_k\,.$$

For all $1 \leqslant t \leqslant l-1$, we have

$$\left(\rho^*\overline{\mathcal{M}}\right)_{|Z^t} = G^\vee_{a_1} \boxtimes \cdots \boxtimes \pi_i^*\left(G^{r\,\vee}_{a_t}\right) \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes G^{r\,\vee}_k\,, \tag{5.11}$$

while

$$\left(\rho^*\overline{\mathcal{M}}\right)_{|Z^l} = G^\vee_{a_1} \boxtimes \cdots \boxtimes G^{r\,\vee}_{a_t} \cdots \boxtimes G^\vee_{a_{l-1}} \boxtimes \pi_i^*\left(G^{r\,\vee}_k\right)\,. \tag{5.12}$$

The strata $Z^1, \ldots, Z^l$ intersect: if $t < s$, then $Z_t \cap Z_s \neq \emptyset$ if and only if $s = t+1$. There are exact sequences

$$0 \to \mathcal{O}_{Z^1\cup\cdots\cup Z^{l-1}}\big(-Z^l\big) \to \mathcal{O}_{Z^1\cup\cdots\cup Z^l} \to \mathcal{O}_{Z^l} \to 0\,,$$
$$0 \to \mathcal{O}_{Z^1\cup\cdots\cup Z^{l-2}}\big(-Z^{l-1}\big) \to \mathcal{O}_{Z^1\cup\cdots\cup Z^{l-1}}\big(-Z^l\big) \to \mathcal{O}_{Z^{l-1}}\big(-Z^l\big) \to 0\,,$$
$$\vdots$$
$$0 \to \mathcal{O}_{Z^1}\big(-Z^2\big) \to \mathcal{O}_{Z^1\cup Z^2}\big(-Z^3\big) \to \mathcal{O}_{Z^2}\big(-Z^3\big) \to 0\,.$$

We also consider the exact sequence

$$0 \to \mathcal{O}_{Z^l}\big(-Z^{l-1}\big) \to \mathcal{O}_{Z^l} \to \mathcal{O}_{Z^l\cap Z^{l-1}} \to 0\,.$$

We tensor all the above sequences with $\rho^*\overline{\mathcal{M}}$. If we write

$$\mathcal{N}^t = \rho^*\overline{\mathcal{M}} \otimes \mathcal{O}_{Z^t}\big(-Z^{t+1}\big) \quad (1 \leqslant t \leqslant l-1)\,, \quad \mathcal{N}^0 = \rho^*\overline{\mathcal{M}} \otimes \mathcal{O}_{Z^l}\big(-Z^{l-1}\big)\,,$$
$$\mathcal{F}^t = \rho^*\overline{\mathcal{M}} \otimes \mathcal{O}_{Z^1\cup\cdots\cup Z^t}\big(-Z^{t+1}\big) \quad (1 \leqslant t \leqslant l-1)\,,$$

then we have exact sequences on $Z^1 \cup \cdots \cup Z^l$

$$0 \to \mathcal{F}^{l-2} \to \rho^*\overline{\mathcal{M}} \to \big(\rho^*\overline{\mathcal{M}}\big)_{|Z^l} \to 0\,,$$
$$0 \to \mathcal{F}^{l-3} \to \mathcal{F}^{l-2} \to \mathcal{N}^{l-1} \to 0\,,$$
$$0 \to \mathcal{F}^{l-4} \to \mathcal{F}^{l-3} \to \mathcal{N}^{l-2} \to 0\,,$$
$$\vdots$$
$$0 \to \mathcal{F}^1 = \mathcal{N}^1 \to \mathcal{F}^2 \to \mathcal{N}^2 \to 0$$

and, furthermore,

$$0 \to \mathcal{N}^0 \to \big(\rho^*\overline{\mathcal{M}}\big)_{|Z^l} \to v_*\mathcal{M} \to 0\,.$$

Consider the push-forwards via $u_*(-)$ to X^r_N of all of the above exact sequences. Recall that $\tilde{\mathcal{T}} = u_*(v_*\mathcal{M})$. To prove that $\operatorname{Cone}\big[L{\pi_i}^*R\pi_{i*}\tilde{\mathcal{T}} \to \tilde{\mathcal{T}}\big]$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$, it suffices to prove that for $\mathcal{N}$ among

$$\rho^*\overline{\mathcal{M}}\,, \quad \mathcal{N}^1\,, \quad \ldots\,, \quad \mathcal{N}^{l-1}\,, \quad \mathcal{N}^0\,,$$

we have that $\operatorname{Cone}\big[L{\pi_i}^*R\pi_{i*}(u_*\mathcal{N}) \to (u_*\mathcal{N})\big]$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$. This is clear for $\rho^*\overline{\mathcal{M}}$, as $u_*\rho^*\overline{\mathcal{M}} = \pi_i^* i_{\overline{Z}*}\overline{\mathcal{M}}$ (since π_i is flat), and we have $\operatorname{Cone}\big[L{\pi_i}^*R\pi_{i*}L\pi_i^*A \to L\pi_i^*A\big] = 0$ for any A. As

$$\mathcal{O}_{Z^t}\big(-Z^{t+1}\big) = \mathcal{O}_{\overline{LM}_{N_1}} \boxtimes \cdots \boxtimes \mathcal{O}_{\overline{LM}_{N_t}}(-\delta_{i,y}) \boxtimes \cdots \boxtimes \mathcal{O}_{X^r_{N_l\cup\{i\}}}\,,$$

where y is one of the attaching points of $\overline{LM}_{N_t}$, using (5.11) and Lemma 5.21, it follows that $u_*\mathcal{N}^t$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$. In particular,

$$R\pi_{i*}\big(u_*\mathcal{N}^t\big) = 0 \quad \text{and} \quad \operatorname{Cone}\big[L{\pi_i}^*R\pi_{i*}\big(u_*\mathcal{N}^t\big) \to \big(u_*\mathcal{N}^t\big)\big] = u_*\mathcal{N}^t\,.$$

Similarly, $u_*\mathcal{N}^0$ is in the subcategory generated by $\hat{\mathbb{G}}^r_N$ since

$$\mathcal{O}_{Z^l}\big(-Z^{l-1}\big) = \mathcal{O}_{\overline{LM}_{N_1}} \boxtimes \cdots \boxtimes \mathcal{O}_{X^r_{N_l\cup\{i\}}}(-E_i)$$

and we may use (5.12) and Lemma 5.21. □

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Ana-Maria Castravet ana-maria.castravet@uvsq.fr

Université Paris-Saclay, UVSQ, CNRS, Laboratoire de Mathématiques de Versailles, 78000, Versailles, France

Jenia Tevelev tevelev@math.umass.edu

Department of Mathematics and Statistics, University of Massachusetts Amherst, 710 North Pleasant Street, Amherst, MA 01003, USA

Laboratory of Algebraic Geometry and its Applications, HSE, Moscow, Russia