The Moduli Space of Rank 2 Vector Bundles on Projective Curves

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1 Introduction

Let $C$ be a smooth projective curve over $\mathbb{C}$ and let $L$ be a line bundle on $C$. Consider the moduli functor

$$\mathcal{VB}_C(2, L) : \text{Algebraic Varieties} \rightarrow \text{Sets}$$

defined on objects as follows

$$X \mapsto \left\{ \text{vector bundles } E \text{ of rank } 2 \text{ on } C \times X, \text{ with } \text{Det} \ E_x = L \right\} / \text{isomorphism}$$

Namely, the functor maps every algebraic variety $X$ to the set of families of vector bundles of rank 2 on $C$ parametrized by the points in $X$ and having fixed determinant bundle equal to $L$.

It can be shown that this functor cannot be represented by an algebraic variety. In order to construct a moduli space we therefore need to restrict our attention to a subclass of vector bundles satisfying some additional conditions: as we will see, the stability together with some restriction on the degree of $L$ will be sufficient to construct a coarse moduli space which turns out to be in fact a fine moduli space when the degree of $L$ is odd.

More precisely the following result holds true.

**Theorem 1.1.** Suppose $\deg(L)$ is odd and greater or equal to $4g - 1$. Then the subfunctor $\mathcal{SU}_C(2, L) \subset \mathcal{VB}_C(2, L)$, defined on objects as

$$X \mapsto \left\{ \text{stable vector bundles } E \text{ of rank } 2 \text{ on } C \times X, \text{ with } \text{Det} \ E_x = L \right\} / \text{isomorphism}$$

is representable, and its fine moduli space is $\mathcal{SU}_C(2, L) := \mathcal{SU}_C(2, L)(\text{point})$.

Our goal in this work is to show that

$$\mathcal{SU}_C(2, L) = \left\{ \text{stable vector bundles } E \text{ of rank } 2 \text{ on } C, \text{ with } \text{Det} \ E = L \right\} / \text{isomorphism}$$

can actually be given the structure of an algebraic variety. In order to achieve this goal we are going to prove the following, more precise result.
Theorem 1.2. Suppose that the line bundle $L$ has degree $\geq 4g - 1$.

(i) There exists a Proj quotient

$$\operatorname{Alt}_{N,2}^{ss}(H^0(L))/\operatorname{GL}(N)$$

which is a projective variety of dimension $3g - 3$

(ii) The open set

$$\operatorname{Alt}_{N,2}^{s}(H^0(L))/\operatorname{GL}(N)$$

has an underlying set $\operatorname{SU}_C(2, L)$. Moreover it is non singular and at each point $E \in \operatorname{SU}_C(2, L)$ its tangent space is isomorphic to $H^1(\mathfrak{sl} E)$

(iii) If $\deg(L)$ is odd,

$$\operatorname{SU}_C(2, L) \cong \operatorname{Alt}_{N,2}^{s}(H^0(L))/\operatorname{GL}(N) = \operatorname{Alt}_{N,2}^{ss}(H^0(L))/\operatorname{GL}(N).$$

This is thus a smooth projective variety.

2 Notations and general facts

Let $E$ be a vector bundle on a curve $C$. We recall that $E$ is always associated with its sheaf of sections, which is a locally free sheaf. With an abuse of notation we will refer to this sheaf using the same letter $E$.

Definition 2.1. We say that a vector bundle $E$ is indecomposable if it cannot be written as a direct sum of two non zero vector bundles

Definition 2.2. We say that a vector bundle $E$ is simple if the only sections of its endomorphism bundle are scalars, $H^0(\mathcal{E}nd E) = \mathbb{C}$.

We observe that if a vector bundle is simple, then it is indecomposable. The following result about indecomposable vector bundles is going to be used in the proof of the main theorem.

Lemma 2.3. If $E$ is an indecomposable vector bundle. Then

$$h^1(\mathfrak{sl} E) = (r^2 - 1)(g - 1),$$

where $\mathfrak{sl} E$ is the kernel of the sheaf morphism

$$\operatorname{tr} : \mathcal{E}nd E \to \mathcal{O}_C.$$ 

Now let us introduce the notion of stability for vector bundles: in order to do that we are going to need some preliminary definitions.
Definition 2.4. Let $r$ be the rank of the locally free sheaf $E$. We define the determinant bundle as
\[
\text{Det } E = \Lambda^r E,
\]
where $\Lambda^r E$ is defined as the sheafification of the presheaf
\[
U \mapsto \Lambda^i_{\mathcal{O}_C(U)} E(U).
\]

Since $r$ is the rank of the locally free sheaf, then $\text{Det} E$ is an invertible sheaf, and it can always be associated to a divisor $D$ on the curve $C$.

Definition 2.5. Let $E$ be a vector bundle on a curve $C$, we define the degree of $E$ as the degree of the divisor associated to $\text{Det} E$.

The following definition of a (semi)stable vector bundle is due to Mumford:

Definition 2.6. Let $E$ be a vector bundle, then:

(i) $E$ is (slope-)semistable if and only if for any subbundle $M \subset E$ it holds:
\[
\frac{\deg(M)}{\text{rank}(M)} \leq \frac{\deg(E)}{\text{rank}(E)}. \tag{1}
\]

(ii) $E$ is (slope-)stable if and only if for any subbundle $M \subset E$ it holds:
\[
\frac{\deg(M)}{\text{rank}(M)} < \frac{\deg(E)}{\text{rank}(E)}. \tag{2}
\]

Observe that if $\deg(E)$ and $\text{rank}(E)$ are coprime, then stability and semistability are the equivalent. This is, in particular, the case if $\text{rank}(E) = 2$ and $\deg(E)$ is odd.

Remark 2.7. If $E$ is a vector bundle of rank 2 then (1) is equivalent to
\[
\deg(M) \leq \frac{1}{2} \deg(E)
\]
for any subbundle $M \subset E$, while (2) is equivalent to
\[
\deg(M) < \frac{1}{2} \deg(E)
\]
for any subbundle $M \subset E$.

Proposition 2.8. Every stable vector bundle is simple.
3 Gieseker points

We note first that from our assumptions on $E$, it follows that $E$ is generated by
global sections and $H^1(E) = 0$. We call $S = \{s_1, \cdots, s_N\}$ a marking if $s_1, \cdots, s_N$
are linearly independent $H^0(E)$-elements. Here and in the following through this
paper, we denote $N := \deg(E) + 2 - 2g$. With the above remark, we see that any
marking $S$ forms a basis of $H^0(E)$ in our case.

For a bundle $E$ and a marking $S = \{s_1, \cdots, s_N\}$, we define the Gieseker point
$T_{E,S}$ as

$$T_{E,S} := \begin{pmatrix}
s_1 \\
\vdots \\
s_N
\end{pmatrix} \wedge (s_1, \cdots, s_N) \in \text{Alt}_N(H^0(L)),$$

where $\text{Alt}_N(H^0(L))$ denotes the skew-symmetric $N \times N$-matrix with coefficients
in $H^0(L)$ and $L = \text{Det}(E)$ is the determinant bundle of $E$.

$T_{E,S}$ can be understood as a matrix, so that it defines a sheaf homomorphism
from $\mathcal{O}_{\mathbb{P}^N}^{\oplus N}$ to $L^{\oplus N}$, namely

$$\mathcal{O}_{\mathbb{P}^N}^{\oplus N} \xrightarrow{(s_1, \cdots, s_N)} E^{(s_1, \cdots, s_N)^T} L^{\oplus N}.$$

Since the second map is an injection, the image $\text{Im}(T_{E,S})$ is isomorphic to $E$. In
particular, the rank of the image is two. Thus we get, $T_{E,S} \in \text{Alt}_{N,2}(H^0(L))$,
where $\text{Alt}_{N,2}(H^0(L))$ denotes the set of rank $\leq 2$ elements in $\text{Alt}_N(H^0(L))$.

Furthermore, the change of basis $\tilde{S} = S \cdot g$ by a $\text{GL}(N)$-element $g$ implies the
$\text{GL}(N)$-action on $\text{Alt}_{N,2}(H^0(L))$ given by $T_{E,\tilde{S}} = g^T T_{E,S} g$. Therefore we get the map

$$\text{SU}_C(2, L) \longrightarrow \text{Alt}_{N,2}(H^0(L))/\text{GL}(N)$$

$$E \quad \mapsto \quad [T_{E,S}].$$

Conversely, because for any $T_{E,S}$ as a linear map $\mathcal{O}_{\mathbb{P}^N}^{\oplus N} \rightarrow L^{\oplus N}$, the image $T_{E,S}$
defines the vector bundle isomorphic to $E$, the map above is injective.

4 Stability

We introduce another sort of stability conditions for rank 2 bundles:

**Definition 4.1.** Suppose, $E$ is a rank 2 vector bundle.

(i) $E$ is called $H^0$-semistable if for any line subbundle $M \subset E$, it holds:

$$h^0(M) \leq \frac{1}{2} h^0(E).$$

(ii) $E$ is called $H^0$-stable if for any line subbundle $M \subset E$, it holds:

$$h^0(M) < \frac{1}{2} h^0(E).$$
The two stability conditions are related as in the following proposition.

**Proposition 4.2.** If $H^1(E) = 0$ and $\deg(E) \geq 4g - 2$,

\begin{align*}
E \text{ is } H^0\text{-semistable} & \iff E \text{ is semistable}, \\
E \text{ is } H^0\text{-stable} & \implies E \text{ is stable}.
\end{align*}

**Proof.** By the Riemann-Roch theorem, we have

\begin{align*}
    h^0(E) - h^1(E) &= \deg(E) + 2(1 - g), \\
    h^0(M) - h^1(M) &= \deg(M) + 1 - g
\end{align*}

for any line subbundle $M$, thus

\begin{align*}
\frac{1}{2} h^0(E) - h^0(M) \\
&\leq \frac{1}{2} h^0(E) - h^0(M) + h^1(M) \\
&= \frac{1}{2} \deg(E) - \deg(M).
\end{align*}

From this, it follows

\[ \frac{1}{2} h^0(E) - h^0(M) > (\text{resp.} \geq) 0 \implies \frac{1}{2} \deg(E) - \deg(M) > (\text{resp.} \geq) 0. \]

We still need to prove the converse direction for the semistability. Assume, $\frac{1}{2} h^0(E) - h^0(M) < 0$, then $h^0(M) > \frac{1}{2} h^0(E) \geq g$. The second inequality follows from the Riemann-Roch theorem

\[ h^0(E) = \deg(E) + 2(1 - g) \geq 2g. \]

This implies $h^1(M) = 0$, for otherwise we can find $n \geq 0$ so that $h^1(M(np)) = 1$ for a point $p \in C$, then it holds $\deg(M) + n \leq 2g - 2$. Also by the Riemann-Roch theorem,

\[ h^0(M(np)) - 1 = \deg(M) + n + 1 - g, \]

thus

\[ h^0(M) \leq h^0(M(np)) = \deg(M) + n + 2 - g \leq g. \]

This is a contradiction. Therefore, in this case, we get

\[ (0 >) \frac{1}{2} h^0(E) - h^0(M) = \frac{1}{2} \deg(E) - \deg(M). \]

We have proved that $E$ is $H^0$-semistable if $E$ is semistable. \hfill $\Box$
For the space $\text{Alt}_N(H^0(L))$, we also introduce the notion of stability.

**Definition 4.3.** Consider $\text{Alt}_N(H^0(L))$ as an affine space on which $\text{SL}(N)$ acts.

(i) $T \in \text{Alt}_N(H^0(L))$ is **stable** in the $\text{SL}(N)$-action if and only if

(a) The orbit $\text{SL}(N) \cdot T$ is closed, and

(b) The stabilizer $\text{Stab}(T)$ is finite.

(ii) $T \in \text{Alt}_N(H^0(L))$ is **semistable** in the $\text{SL}(N)$-action if and only if there exists an semi-invariant polynomial $F$ of positive weight such that $F(T) \neq 0$, equivalently if and only if $0 \not\in \text{SL}(N) \cdot T$.

The stability conditions for a bundle $E$ and its Gieseker point $T_{E,S}$ are consistent.

**Proposition 4.4.** If $H^1(E) = 0$, then

$T_{E,S}$ is $\text{SL}(N)$-semistable $\implies$ $E$ is $H^0$-semistable,

$T_{E,S}$ is $\text{SL}(N)$-stable $\implies$ $E$ is $H^0$-stable.

**Remark 4.5.** By the previous proposition, this implies the (semi-)stability of $E$.

**Proof.** Assume, there exists a line subbundle $M$ of $E$ such that $a := h^0(M) \geq \frac{N}{2}$. Take a basis $S$ of $H^0(E)$ by extending a basis of $H^0(M)$, we have the Gieseker point

$$T_{E,S} = \begin{pmatrix} 0 \\ -B^T \\ C \end{pmatrix}.$$ 

Set $b := N - a$. Consider a one-parameter subgroup

$$\mathbb{C}^\times \ni t \mapsto g(t) := \begin{pmatrix} t^{-b}I_a & 0 \\ 0 & t^aI_b \end{pmatrix} \in \text{SL}(N).$$

This acts on $T_{E,S}$ as $g(t)^T T_{E,S} g(t) = \begin{pmatrix} 0 \\ -t^{a-b}B^T \\ t^aC \end{pmatrix}.$

If $E$ is not semistable, i.e. $a > b$, then $g(t)^T T_{E,S} g(t) \to 0$ as $t \to 0$. Thus $T_{E,S}$ is not semistable.

If $E$ is not stable but semistable, i.e. $a = b$, then $g(t)^T T_{E,S} g(t) \to \left( -\frac{B}{B^T}B \right)$ as $t \to 0$. Since the stablizer of $\left( -\frac{B}{B^T}B \right)$ contains all $g(t)$ and thus $\left( -\frac{B}{B^T}B \right)$ is not stable, $T_{E,S}$ cannot be stable.

The harder part to prove is the converse of Proposition 4.5, which is:
Proposition 4.6. If $H^1(E) = 0$, then

$E$ is $H^0$-semistable $\implies T_{E,S}$ is $GL(N)$-semistable.

We prepare two lemmas to prove the proposition.

Lemma 4.7. If $E$ is $H^0$-semistable and $h^0(E) \geq 2$, then $E$ is generated by global sections at a generic point $p \in C$. In particular, $h^0(E(-p)) = h^0(E) - 2$ at a generic point $p \in C$.

Lemma 4.8. If $E$ is $H^0$-semistable and $h^0(E) \geq 4$ then there exists a point $p \in C$ for which the bundle $E(-p)$ is $H^0$-semistable.

Proof of the Proposition. We construct semi-invariants who do not vanish at $T_{E,S}$.

Case 1: $N$ is even.

Using the lemma recursively, we find $\frac{N}{2}$ points $p_1, \ldots, p_{\frac{N}{2}} \in C$ such that $H^0(E(-p_1 - \cdots - p_{\frac{N}{2}})) = 0$. This implies the linear map

$$g = (ev_1, \ldots, ev_{\frac{N}{2}}) : H^0(E) \to \bigoplus_{i=1}^{\frac{N}{2}} E/E(-p_i)$$

is injective, but since both spaces have the same dimension $N$, this is an isomorphism. So we can choose, the basis $S = (s_1, \ldots, s_N)$ so that $s_i$ corresponds to $e_i$, where $e_i, e_{\frac{N}{2}+i}$ are basis of $E/E(-p_i)$. Then, defining $f := ev_1 + \cdots + ev_{\frac{N}{2}}$, we get

$$f(T_{E,S}) = \begin{pmatrix} 0 & I_{\frac{N}{2}} \\ -I_{\frac{N}{2}} & 0 \end{pmatrix}.$$ 

Here we identify $E/E(-p_i) \cong \mathbb{C}$ through some global section of $E$. Thus Pfaff$(f(T_{E,S})) = 1$. Because the pfaffian is a semi-invariant of weight 1, this shows that $T_{E,S}$ is a semistable point.

Case 2: $N$ is odd.

Using the lemmas, we can find $\frac{N-1}{2} + 1$ points $p_1, \ldots, p_{\frac{N-1}{2}+1} \in C$ so that $h^0(E(-p_1 - \cdots - p_{\frac{N-1}{2}+1})) = 3$ and $E(-p_1 - \cdots - p_{\frac{N-1}{2}+1})$ is $H^0$-semistable. We set $n := \frac{N-1}{2}$.

Choose $p_n, p_{n+1} \in C$ and $\varphi, \psi \in H^0(E(-p_1 - \cdots - p_{n-1}))$ so that $\varphi(p_n) = \psi(p_{n+1}) = 0$. Then there is $t \in H^0(E(-p_1 - \cdots - p_{n-1}))$ with $t(p_n), t(p_{n+1}) \neq 0$. Furthermore, we can choose $p_0 \in C$ so that $\varphi$ and $\psi$ spans the fiber $E_{p_0}$ and $t(p_0) = 0$.

With $f := ev_1 + \cdots + ev_{n-1} + ev_0$, $f' := ev_1 + \cdots + ev_{n-1} + ev_{n+1}$, $h := ev_n$, we will show

$$(\text{rad}(f(T_{E,S})))^T h(T_{E,S}) \text{rad}(f'(T_{E,S})) \neq 0.$$
Here \( \text{rad}(T) \) is the radical vector defined by
\[
\text{rad}(T) = \begin{pmatrix}
Pfaff(T_{11}) \\ -Pfaff(T_{22}) \\ \vdots \\ (-1)^N Pfaff(T_{NN})
\end{pmatrix},
\]
where \( T_{ii} \) denotes the submatrix of \( T \) obtained by eliminating the \( i \)-th row and the \( i \)-th column of \( T \).

Using the fact \( \text{rad}(gTg^t) = \tilde{g}^T \text{rad}(T) \) with the cofactor matrix \( \tilde{g} \), we see that this is, in fact, an semi-invariant of weight 2. Since the rank of \( h(T_{E,S}) \) is two, we can use the formula
\[
(\text{rad}(f(T_{E,S}))^T h(T_{E,S}) \text{rad}(f'(T_{E,S}))) = Pfaff\left( \begin{pmatrix} f(T_{E,S}) \\ -h(T_{E,S}) \\ h(T_{E,S}) \\ -f'(T_{E,S}) \end{pmatrix} \right),
\]
in order to show that the semi-invariant does not vanish at \( T_{E,S} \). As in case 1, choosing a basis, the matrix in the right hand side can be given as
\[
\begin{pmatrix}
0 & -I_n & 0 & 0 & 0 & 0 \\
I_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 & B \\
0 & 0 & 0 & 0 & -I_n & 0 \\
0 & 0 & 0 & I_n & 0 & 0 \\
0 & 0 & -B^T & 0 & 0 & C
\end{pmatrix}.
\]
A, B, C are respectively \( A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \) in the basis \( (\varphi, \psi, t) \). Since \( \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} \) has non-zero determinant, \( Pfaff\left( \begin{pmatrix} f(T_{E,S}) \\ -h(T_{E,S}) \\ h(T_{E,S}) \\ -f'(T_{E,S}) \end{pmatrix} \right) \) is invertible, thus its pfaffian is non-zero. Thus our semi-invariant does not vanish at \( T_{E,S} \). So \( T_{E,S} \) is semi-invariant in \( SL(N) \)-action.

Finally, we show \( T_{E,S} \) is a stable point if \( E \) is stable. For this, we use the lemma:

**Lemma 4.9.** If \( H^1(E) = 0 \), \( E \) is generated by global sections, and \( E \) is simple, then \( \text{Stab}(T_{E,S}) = \pm I_N \).

So, we know that for a stable bundle \( E \) with \( \deg(E) \geq 4g - 1 \), the Gieseker point \( T_{E,S} \) is semistable and the stabilizer is finite, thus it follows that \( T_{E,S} \) is stable. We have proved the map is given by
\[
SU_C(2, L) \rightarrow Alt^s_{N,2}(H^0(L))/GL(N) \\
E \mapsto [T_{E,S}],
\]
where \( Alt^s_{N,2}(H^0(L)) \) is the set of \( SL(N) \)-stable points in \( Alt_{N,2}(H^0(L)) \). We have already mentioned that this is an injection. The map is furthermore a surjection, for any \( T \in Alt^s_{N,2}(H^0(L)) \) defines a bundle \( E := \text{Im}(T) \subset L^\oplus N \) with \([T_{E,S}] = T\) and Proposition 4.4. shows that \( E \) is stable.
5 Smoothness of $\text{Alt}^s_{N,2}(H^0(L))$

Because for any $T \in \text{Alt}^s_{N,2}(H^0(L))$ with $\deg(L) \geq 4g - 2$, the rank 2 bundle $E = \text{Im}(T) \subset L^\oplus N$ satisfies $H^1(E) = 0$, $\det(E) = L$, and $E$ is semistable, the following proposition implies that $\text{Alt}^s_{N,2}(H^0(L))$ is smooth.

**Proposition 5.1.** If $H^1(E) = 0$, then

(i) $\text{Alt}_{N,2}(H^0(L))$ is smooth at $T_{E,S}$.

(ii) If $E$ is simple, $T_{T_{E,S}} \text{Alt}^s_{N,2}(H^0(L))/\mathfrak{gl}(N) \cong H^1(\mathfrak{sl}(E))$.

We only give the proof of (ii).

**Proof of (ii).** Remember that the Gieseker point $T_{E,S}$ defines a rank 2 sheaf homomorphism $O^\oplus N \to L^\oplus N$ and its image is isomorphic to $E$ on generic fibers. Choosing a local basis $\varphi$ of $L$, we can identify $L^\oplus N$ with $(O^\oplus N)^*$ through $L^\oplus N \cong (O^\oplus N)^*$.

Thus on each fiber, we can apply the following linear algebra considerations.

Suppose $V$ is an $N$-dimensional vector space over $\mathbb{C}$ and $f : V \to V^\vee$ is a skew-symmetric linear map of rank $r$ in the sense that $f = -f^\vee$, where $f^\vee : V^{\vee\vee} \cong V \to V^\vee$ is the dual map of $f$. We identify $f$ with a skew-symmetric bilinear form $<, > : V \times V \to \mathbb{C}$. We assume $<, >$ is in the form

$$j_r := \begin{pmatrix} J_r & 0 \\ 0 & 0 \end{pmatrix},$$

where $J_r = \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix}$ is the standard symplectic form matrix. Then we get $\text{Ker}(f) \cong \{0\}^r \oplus \mathbb{C}^{N-r}$ and $\text{Im}(f) \cong \mathbb{C}^r \oplus \{0\}^{N-r}$. $\text{GL}(N)$ transitively acts on the space $\text{Alt}_N(\mathbb{C})$ of skew-symmetric forms, and further on the subvariety $\text{Alt}_{N,r}(\mathbb{C})$ of rank $r$ skew-symmetric forms. Thus $\text{Alt}_{N,r}(\mathbb{C}) \cong \text{Stab}(j_r) \backslash \text{GL}(N)$. Since the stabilizer $\text{Stab}(j_r)$ is given by

$$\text{Stab}(j_r) = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : A \in \text{Sp}(r), D \in \text{GL}(N-r), C \in M(N-r, r; \mathbb{C}) \right\},$$

we get the tangent space to be isomorphic to

$$\mathfrak{gl}(N)/T_{I_N}(\text{Stab}(j_r)) \cong \mathfrak{gl}(r)/\mathfrak{sp}(r) \oplus \mathbb{C}^r \times \mathbb{C}^{N-r} \cong \left\{ \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} : A \in \mathfrak{gl}(r)/\mathfrak{sp}(r), B \in M(r, N-r; \mathbb{C}) \right\}.$$
In particular, for \( r = 2 \) case, because of \( \mathfrak{sp}(2) = \mathfrak{sl}(2) \), we can calculate

\[
T_{jr}(\text{Alt}_{N,2}(\mathbb{C})) = \left\{ \begin{pmatrix} aJ_2 & J_2B \\ B^tJ_2 & 0 \end{pmatrix} : a \in \mathbb{C}, B \in M(2, N - 2; \mathbb{C}) \right\}
\subset \text{Alt}_N(\mathbb{C}) .
\]

We get the short exact sequence

\[
0 \to \mathfrak{sl}(2) \to \mathfrak{gl}(2) \oplus \mathbb{C}^{2 \times (N-2)} \to T_{jr}(\text{Alt}_{N,2}(\mathbb{C})) \to 0 .
\]

Identifying the above matrices as linear maps from \( V \) to \( V^\vee \), the short exact sequence turns to be

\[
0 \to \mathfrak{sl}(\text{Im}(f)) \to \text{Hom}(V, \text{Im}(f)) \to T_f(\text{Alt}_{N,2}(V)) \to 0 .
\]

Applying this for the Gieseker point \( f = T_{E,S} \), we get the short exact sequence of sheaves on generic points, thus that of vector bundles on \( C \):

\[
0 \to \mathfrak{sl}(E) \to \text{Hom}(O^{\oplus N}_C, E) \to T_{T_{E,S}}(\text{Alt}_{N,2}(L)) \to 0 ,
\]

which implies the long exact sequence

\[
0 \to H^0(\mathfrak{sl}(E)) \to H^0(\text{Hom}(O^{\oplus N}_C, E)) \to H^0(T_{T_{E,S}}(\text{Alt}_{N,2}(L))) \to H^1(\mathfrak{sl}(E)) .
\]

Because of \( H^0(\mathfrak{sl}(E)) = 0 \), \( H^0(\text{Hom}(O^{\oplus N}_C, E)) = \text{Hom}(H^0(O^{\oplus N}_C), H^0(E)) \cong \mathfrak{gl}(N) \), and \( H^0(T_{T_{E,S}}(\text{Alt}_{N,2}(L))) = T_{T_{E,S}}(\text{Alt}_{N,2}(H^0(L))) \), we get

\[
T_{T_{E,S}}(\text{Alt}_{N,2}(H^0(L))) / \mathfrak{gl}(N) \cong H^1(\mathfrak{sl}(E)) .
\]

\[\square\]

### 6 Proof of theorem 1.2

Let us use the results from the previous sections. First, thanks to proposition 5.1 the set \( \text{Alt}_{N,2}^*(H^0(L)) \) is smooth. The smoothness of this set is enough to guarantee the existence of the Proj quotient (see [Mukai], Remark 6.14).

Moreover we have seen in section 4 that there is a bijection

\[
\text{SU}_C(2, L) \cong \text{Alt}_{N,2}^*(H^0(L))/\text{GL}(N) .
\]

Since the stabilizer of the action of \( \text{GL}(N)/\{\pm 1\} \) on \( \text{Alt}_{N,2}^*(H^0(L)) \) is trivial and, again by proposition 5.1, \( \text{Alt}_{N,2}^*(H^0(L)) \) is smooth, it follows that \( \text{SU}_C(2, L) \) is nonsingular (see [Mukai], 9.52).

Finally, due to Proposition 5.1,

\[
\dim(\text{Alt}_{N,2}^*(H^0(L)))/\text{GL}(N) = \dim(H^1(\mathfrak{sl}(E))) = 3g - 3 .
\]

This concludes the proof of (i) and (ii). The proof of (iii) follows because the stability and the semistability of \( E \) are equivalent when \( \deg(L) \) is odd.
References