1. Let $R$ be an integrally closed Noetherian domain with field of fractions $K$. Let $L/K$ be a finite extension (not necessarily Galois). Let $T$ be the integral closure of $R$ in $L$. Show that there exists $b_1, \ldots, b_n \in T$ which form a basis of $L$ over $K$.

2. Let $R$ be an integrally closed Noetherian domain with field of fractions $K$. Let $L/K$ be a finite Galois extension with Galois group $G = \{1, \ldots, n\}$. Let $b_1, \ldots, b_n \in T$ be a basis of $L$ over $K$. Consider the determinant $d = \det |\sigma_i(b_j)|$. (a) Show that $d \in T$. (b) Show that $d^2 \in R$.

3. (continuation of the previous problem). $d^2 T \subset Rb_1 + \ldots + Rb_n$.

4. Let $R$ be an integrally closed Noetherian domain with field of fractions $K$. Let $L/K$ be a finite extension (not necessarily Galois). Let $T$ be the integral closure of $R$ in $L$. Show that $T$ is a finitely generated $R$-module.

5. Let $K/Q$ be a finite extension (not necessarily Galois). Show that $O_K$ is a finitely generated free abelian group.

6. Let $k$ be a field. Let $F \subset k(x)$ be a subfield properly containing $k$. Show that $k(x)/F$ is a finite extension.

7. Let $K_1$ and $K_2$ be algebraically closed extensions of $\mathbb{C}$ of transcendence degree over $\mathbb{C}$ of degree 11. Show that any homomorphism $f : K_1 \rightarrow K_2$ is an isomorphism.

8. Let $k \subset K \subset E$ be finitely generated field extensions. Show that

$$\text{tr.deg.} \ E/k = \text{tr.deg.} \ K/k + \text{tr.deg.} \ E/K.$$

9. A matroid is a set $E$ and a non-empty family of finite subsets of $E$ called independent sets such that

- Every subset of an independent set is independent.
- If $A$ and $B$ are two independent sets and $|A| > |B|$ then there exists $a \in A - B$ such that $B \cup \{a\}$ is independent.

Show that the following are matroids: (a) $E$ is a vector space; independent sets are linearly independent sets of vectors. (b) $E$ is a field extension of $k$; independent sets are algebraically independent sets. (c) $E$ is the set of edges of a graph; independent sets are subsets of edges without loops.

10. A basis of a matroid is a maximal (by inclusion) independent set. Show that if a matroid has a finite basis then all its bases have the same number of elements (called dimension of the matroid).

11. Let $k$ be a field and let $A$ be a finitely generated $k$-algebra. Let $B \subset A$ be a $k$-subalgebra such that $A$ is integral over $B$. Show that $A$ is a finitely generated $B$-module and $B$ is a finitely generated $k$-algebra (Hint: consider a $k$-subalgebra $C \subset B$ generated by coefficients of monic equations satisfied by generators of $A$).

12. Prove Noether’s normalization lemma when $k$ is an arbitrary field by using a change of variables $y_i' = y_i - \lambda_i y_i^{\rho_i}$ instead of $y_i' = y_i - \lambda_i y_{r+1}$. 