1. For any \( k \geq 0 \), let \( p_k = \sum_{i=1}^{n} \alpha_i^k \). Show that
\[
\begin{vmatrix}
1 & \cdots & 1 \\
\alpha_1 & \cdots & \alpha_n \\
\vdots & \ddots & \vdots \\
\alpha_1^{n-1} & \cdots & \alpha_n^{n-1}
\end{vmatrix} = \prod_{i>j} (\alpha_i - \alpha_j);
\]
\[
\begin{vmatrix}
p_0 & p_1 & \cdots & p_{n-1} \\
p_1 & p_2 & \cdots & p_n \\
p_2 & p_3 & \cdots & p_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_n & \cdots & p_{2n-2}
\end{vmatrix} = \prod_{i>j} (\alpha_i - \alpha_j)^2.
\]

2. (continuation of the previous problem). (a) Let \( p \) be an odd prime number. Show that the discriminant of the cyclotomic polynomial \( \Phi_p(x) \) is equal to \((-1)^{\frac{p-1}{2}} p^{p-2}\). (b) Use (a) to give a different proof of the Kronecker–Weber theorem for quadratic extensions.

3. Let \( q \) be an odd prime and let \( a \) be an integer coprime to \( q \). Then
\[
\left( \frac{a}{q} \right) \equiv a^{\frac{q-1}{2}} \mod q.
\]

4. Let \( p \) be a on odd prime. Let \( \alpha \) be a primitive 8-th root of unity in \( \mathbb{F}_p \) and let \( y = \alpha + \alpha^{-1} \). (a) Show that \( y^p = (-1)^{\frac{p-1}{2}} y \). (b) Show that \( y^2 = 2 \).
(c) Show that \( \left( \frac{2}{p} \right) = (-1)^{\frac{p^2-1}{8}} \).

5. Compute \( \left( \frac{2013}{1597} \right) \).

6. Let \( R \) be a domain with the field of fractions \( K \). Let \( F/K \) be an algebraic extension and let \( S \) be the integral closure of \( R \) in \( F \). For any \( \alpha \in F \), show that there exists \( r \in R \) such that \( r \alpha \in S \).

7. Suppose \( n, m \geq 2 \) are coprime positive integers. Show that \( \mathbb{C}[x, y]/(x^n - y^m) \) is a domain and find its normalization.

8. Let \( A \) be an integrally closed domain with the field of fractions \( K \). Let \( F/K \) be an algebraic extension of fields. Let \( \alpha \in F \). Show that \( \alpha \) is integral over \( A \) if and only if its minimal polynomial has coefficients in \( A \).

9. Prove that the Gauss Lemma holds not only in a UFD but in any integrally closed domain \( R \) in the following form: suppose \( f(x) \in R[x] \) is a monic polynomial and \( f(x) = g(x)h(x) \), where \( g(x), h(x) \in K[x] \) are monic polynomials (here \( K \) is the field of fractions of \( R \)). Then \( g(x), h(x) \in R[x] \).

10. Let \( A \subset B \) be domains and suppose that \( B \) is integral over \( A \). Let \( I \subset B \) be an ideal. Show that \( B/I \) is integral over \( A/A \cap I \).

11. (a) Let \( A \subset B \) be rings and suppose that \( B \) is integral over \( A \). Show that \( A \) is a field if and only if \( B \) is a field. (b) Let \( A \subset B \) be rings and suppose that \( B \) is integral over \( A \). Let \( \mathfrak{p} \subset B \) be a prime ideal. Show that \( \mathfrak{p} \) is a maximal ideal of \( B \) if and only if \( \mathfrak{p} \cap A \) is a maximal ideal of \( A \).

12. Let \( A \) be an integrally closed domain with the field of fractions \( K \). Let \( F/K \) be a Galois extension with the Galois group \( G \). Let \( B \) be the integral closure of \( A \) in \( F \). Show that \( G \) preserves \( B \) and that \( B^G = A \).