Homework 5

1. Show that a finite field extension $F/K$ is solvable if and only if $\text{Gal}(L/K)$ is solvable, where $L$ is the Galois closure of $F$ in $\bar{K}$.

2. Let $p$ be a prime number and let $\zeta_p \in \mathbb{C}$ be a primitive $p$-th root of unity. Show that $\text{Gal}(\mathbb{Q}(\zeta_p, \sqrt[p]{2})/\mathbb{Q})$ is a semidirect product of $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_p^\times$.

3. Suppose $D_4$ acts on $F = \mathbb{C}(x_1, \ldots, x_4)$ by permutations of variables (here we identify variables with vertices of the square). Show that $F^{D_4}$ is generated over $\mathbb{C}$ by 4 rational functions.

4. Let $M$ be a module over a ring $R$. A sequence of submodules $M = M_1 \supset M_2 \supset \ldots \supset M_r = 0$ is called a filtration of $M$ (of length $r$). A module $M$ is called simple if it does not contain any proper submodules other than $0$ and itself. A filtration is called simple if each $M_i/M_{i+1}$ is simple. A module $M$ is said to be of finite length if it admits a simple finite filtration. Two filtrations of $M$ are called equivalent if they have the same length and the same collection of subquotients $\{M_1/M_2, M_2/M_3, \ldots, M_{r-1}/M_r\}$ (up to isomorphism and renumbering). Prove that if $M$ has finite length then any two simple filtrations of $M$ are equivalent and any filtration of $M$ can be refined to a simple filtration.

5. (a) Let $f(x) \in K[x]$ be an irreducible separable polynomial with roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n \in \bar{K}$. Suppose that there exist rational functions $\theta_1(x), \ldots, \theta_n(x) \in K(x)$ such that $\alpha_i = \theta_i(\alpha)$ for any $i$. Suppose also that $\theta_i(\theta_j(\alpha)) = \theta_j(\theta_i(\alpha))$ for any $i, j$. Show that the Galois group of the splitting field of $f(x)$ is Abelian. (b) Give an example of the situation as in part (a) with $K = \mathbb{Q}$ and such that the Galois group of $f(x)$ is not cyclic. Give a specific polynomial $f(x)$, and compute its roots and functions $\theta_i$.

6. (a) Let $\bar{K}$ be an algebraic closure of $K$. Show that there exists the unique maximal (by inclusion) subfield $K_{ab} \subset \bar{K}$ such that $K_{ab}/K$ is Galois and the Galois group $\text{Gal}(K_{ab}/K)$ is Abelian. (b) Deduce from the Kronecker–Weber Theorem that $\mathbb{Q}_{ab} = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_n)$.

7. Let $K = \mathbb{C}[z^{-1}, z]$ be the field of Laurent series (series in $z$, polynomials in $z^{-1}$). Let $K_m = \mathbb{C}[z^{\frac{1}{m}}, z^{-\frac{1}{m}}] \supset K$. (a) Show that $K_m/K$ is Galois with a Galois group $\mathbb{Z}/m\mathbb{Z}$. (b) Show that any Galois extension $F/K$ with a Galois group $\mathbb{Z}/m\mathbb{Z}$ is isomorphic to $K_m$. (c) Show that $K_{ab} = \bigcup_{m \geq 1} K_m$ the field of so called Puiseux series.

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4This was proved by Abel himself.

5Sir Isaac Newton proved that the field of Puiseux series in in fact algebraically closed.