Homework 2

We fix a finite field extension $K \subset F$. We also assume that $F \subset \overline{K}$.

1. Let $\alpha \in F$ and let $f(x)$ be its minimal polynomial over $K$. Show that there exists $k \geq 0$ such that all roots of $f(x)$ in $\overline{K}$ have multiplicity $p^k$ and $\alpha^{p^k}$ is separable over $K$.

2. (a) Show that elements of $F$ separable over $K$ form a field $L$ (called a separable closure of $K$ in $F$). We define $[F : K]_s := [L : K]$.

(b) Show that the separable closure of $L$ in $F$ is equal to $L$. (b) Prove that the number of different homomorphisms $F \to \overline{K}$ over $K$ is equal to $[F : K]_s$.

3. An extension $F/K$ is called purely inseparable if $[F : K]_s = 1$. Show that $F/K$ is purely inseparable if and only if $\text{char } K = p$ and $F$ is generated over $K$ by elements $\alpha_1, \ldots, \alpha_r$ such that the minimal polynomial of each $\alpha_i$ has the form $x^{p^{k_i}} - a_i$ for some $a_i \in K$ and a positive integer $k_i$.

4. Let $L = \overline{F_p}(x, y)$ be the field of rational functions in two variables and let $K = \overline{F_p}(x^p, y^p)$ be its subfield. (a) Show that $L/K$ is an algebraic extension and compute its degree. (b) Show that there exist infinitely many pairwise different intermediate subfields between $K$ and $L$. (c) Show that $L$ cannot be expressed as $K(\alpha)$ for some $\alpha \in L$.

5. A field $k$ is called perfect if either $\text{char } k = 0$ or $\text{char } k = p$ and the Frobenius homomorphism $F : k \to k$ is an isomorphism. Show that if $k$ is perfect then any algebraic extension of $k$ is separable over $k$ and perfect.

6. Let $F$ be a splitting field of the polynomial $f \in K[x]$ of degree $n$. Show that $[F : K]$ divides $n!$ (do not assume that $F$ is separable over $K$).

7. Let $F \subset \overline{K}$ be a finite Galois extension of $K$ and let $L \subset \overline{K}$ be any finite extension of $K$. Consider the natural $K$-linear map $L \otimes_K F \to \overline{K}$. (a) Show that its image is a field, in fact a composite field $LF$. (b) Show that $LF$ is Galois over $L$. (c) Show that $\text{Gal}(LF/L)$ is isomorphic to $\text{Gal}(F/L \cap F)$.

8. (a) Find the minimal polynomial over $\mathbb{Q}$ of $\sqrt[4]{3} + \sqrt[4]{3}$. (b) Compute the Galois group of its splitting field.

9. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of prime degree $p$. Suppose that $f(x)$ has exactly $p - 2$ real roots. Show that the Galois group of the splitting field of $f(x)$ is $S_p$.

10. For any $d \geq 2$, prove existence of an irreducible polynomial in $\mathbb{Q}[x]$ of degree $d$ with exactly $d - 2$ real roots (Hint: take some obvious reducible polynomial with exactly $d - 2$ real roots and perturb it a little bit to make it irreducible).

11. Let $G$ be any finite group. Show that there exist finite extensions $\mathbb{Q} \subset K \subset F$ such that $F/K$ is a Galois extension with a Galois group $G$.

12. Let $F$ be a splitting field of the polynomial $f(x) \in K[x]$. Show that $\text{Gal } F/K$ acts transitively on roots of $f(x)$ if and only if $f(x)$ is irreducible (do not assume that $f(x)$ is separable).

13. Let $F$ be a splitting field of a biquadratic polynomial $x^4 + ax^2 + b \in K[x]$. Show that $\text{Gal}(F/K)$ is isomorphic to a subgroup of $D_4$. 