Homework 11

1. Describe explicitly (i.e. how each element of the group acts) all one-dimensional complex representations of (a) \((\mathbb{Z}/2\mathbb{Z})^r\), (b) \(D_n\).

2. Let \(V\) be an irreducible complex representation of a finite group \(G\). Show that there exists a unique (up to a positive scalar) \(G\)-invariant Hermitian inner product on \(V\).

3. Let \(k = \mathbb{F}_q\) be a finite field with \(q = p^n\) elements. Let \(G\) be a \(p\)-group. Show that any irreducible representation of \(G\) over \(k\) is trivial.

4. Let \(G\) be a group (not necessarily finite) and let \(V = \mathbb{C}[G]\) be its regular representation. Let \(U \subset V\) be a vector subspace spanned by vectors \([gh] - [hg]\) for any \(g, h \in G\). Suppose that \(\dim V/U < \infty\). Show that \(G\) has only finitely many conjugacy classes and that their number is equal to \(\dim V/U\).

5. Let \(G\) be a group (not necessarily finite) and let \(V = \mathbb{C}[G]\) be its regular representation. Let \(U \subset V\) be a vector subspace spanned by vectors \([gh] - [hg]\) for any \(g, h \in G\). Suppose that \(\dim V/U < 1\). Show that \(G\) has only finitely many conjugacy classes and that their number is equal to \(\dim V/U\).

6. Let \(G\) be a group and let \(V\) be a \(G\)-module over a field \(k\). Let \(K/k\) be a field extension. Let \(V_K := V \otimes_k K\). Show that \(V_K\) has a natural \(G\)-module structure over \(K\) compatible with extension of scalars: \(V_K \cong V \otimes_{k[G]} K[G]\).

7. Let \(V\) be an irreducible representation of a finite group \(G\).
   (a) Let \(x \in V, x \neq 0\). Prove that \(\dim V \leq [G : G_x]\).
   (b) Let \(H \subset G\) be an Abelian subgroup. Prove that \(\dim V \leq [G : H]\).

8. Let \(V\) denote the two-dimensional real representation of \(D_n\) given by the natural embedding \(D_n \subset O_2(\mathbb{R})\).
   (a) Choose a system of generators of \(D_n\) and write down matrices of these elements in some basis of \(V\).
   (b) Show that \(V_C\) is irreducible.

9. Let \(G\) be the group of all rotations of \(\mathbb{R}^3\) which preserve a regular tetrahedron centered at the origin. Let \(V\) denote the three-dimensional representation of \(G\) over \(\mathbb{R}\) given by the natural embedding \(G \subset O_3(\mathbb{R})\). Choose generators of \(G\) and write down their matrices in some basis of \(V\).

10. Let \(G\) be the group of all rotations of \(\mathbb{R}^3\) which preserve a regular icosahedron centered at the origin. Let \(V\) denote the three-dimensional representation of \(G\) over \(\mathbb{R}\) given by the natural embedding \(G \subset O_3(\mathbb{R})\). Show that \(V_C\) is irreducible.

11. Let \(G\) be a finite Abelian group and let \(\hat{G}\) be its Pontryagin dual group (over \(\mathbb{C}\)). For any function \(f : G \to \mathbb{C}\), its Fourier transform \(\hat{f} : \hat{G} \to \mathbb{C}\) is by definition the following function:
    \[
    \hat{f}(\rho) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g) \rho(g).
    \]
    Compute \(\hat{f}\) (here we identify \(\hat{G}\) with \(G\)).

12. Describe explicitly all irreducible representations and build the character table for (a) \(A_4\), (b) \(S_4\).