§1. GEOMETRY OF LINES (JAN 19,21,24,26,28)

Let's start with a familiar example of a moduli space. Recall that the Grassmannian $G(r,n)$ parametrizes r-dimensional linear subspaces of $\mathbb{C}^n$. For example,

$$G(1,n)=\mathbb{P}^{n-1}$$

is a *projective space*[1]. Let's try to understand the next case, $G(2,n)$. The projectivization of a 2-dimensional subspace $U\subset\mathbb{C}^n$ is a line $l\subset\mathbb{P}^{n-1}$, so in essence $G(2,n)$ is a moduli space of lines in the projective space. Most of our discussion remains valid for general $G(r,n)$, but the case of $G(2,n)$ is notationally easier.

§1.1. **Grassmannian as a complex manifold.** Thinking about $G(2,n)$ as just a set is boring: we need to introduce some geometry on it. We care about two flavors of geometry, *Analytic Geometry* and *Algebraic Geometry*. We won't need much of either in the beginning and will develop the latter substantially as we go along. A basic object of analytic geometry is a *complex manifold*, i.e. a *Hausdorff topological space* X covered by *charts* X_i homeomorphic to open subsets of $\mathbb{C}^n$. Coordinate functions on $\mathbb{C}^n$ are called *local coordinates* in the chart. On the overlaps $X_i\cap X_j$ we thus have two competing systems of coordinates, and the main requirement is that *transition functions* between these coordinate systems are *holomorphic*. Maps between complex manifolds are presumed to be *holomorphic maps*, i.e. maps that are holomorphic in charts.

Let's see how this is done for the Grassmannian. Any 2-dimensional subspace $U\subset\mathbb{C}^n$ is a row space of a $2\times n$ matrix A of rank 2. Let A_{ij} denote the 2×2 submatrix of A with columns i and j and let $p_{ij}=\det A_{ij}$ be the corresponding minor. Since $\operatorname{rank} A=2$, we can find some $i<j$ such that $p_{ij}\neq 0$ (*why?*) . Then $(A_{ij})^{-1}A$ has a form

$$\begin{bmatrix}\dots & * & 1 & * & \dots & * & 0 & * & \dots\\ \dots & * & 0 & * & \dots & * & 1 & * & \dots\end{bmatrix}\tag{1.1.1}$$

where the i-th column is $\begin{bmatrix}1\\0\end{bmatrix}$, the j-th column is $\begin{bmatrix}0\\1\end{bmatrix}$, and the remaining $n-2$ columns are arbitrary. Notice that multiplying A by an invertible 2×2 matrix on the left does not change the row space.

We cover $G(2,n)$ by $\binom{n}{2}$ charts

$$X_{ij}=\{U\in G(2,n)\,|\,U \text{ is represented by a matrix } A \text{ like (1.1.1)}\}.$$

Geometrically, this chart parametrizes 2-dimensional subspaces that surject onto the coordinate subspace $\langle e_i,e_j\rangle$ under projection along the complementary coordinate subspace (*why?*) . Each subspace from X_{ij} is a row space of a unique matrix (1.1.1), in particular X_{ij} can be identified with $\mathbb{C}^{2(n-2)}$.

To show that $G(2,n)$ is a complex manifold we have to check that the transition functions between charts X_{ij} and $X_{i'j'}$ are holomorphic. Any

[1] I will often italicize various words in these lecture notes. If you see something italicized, make a pause and ask yourself: do I know what this means? I hope that this will help you to learn the vocabulary faster.

$2\times n$ matrix that represents a subspace $U\in X_{ij}\cap X_{i'j'}$ has $p_{ij}\neq 0$ and $p_{i'j'}\neq 0$. In the chart X_{ij}, the subspace is represented by a matrix A as in (1.1.1). In the chart $X_{i'j'}$, the subspace is represented by a matrix $(A_{i'j'})^{-1}A$. The matrix entries of it depend holomorphically (in fact just rationally) on the matrix entries of A, thus the transition functions are indeed holomorphic.

In this example (and in general) the structure of a topological space on X is introduced simultaneously with constructing charts: the subset is declared open iff its intersection with each chart is open. It is easy to check (*why?*) that $G(2,n)$ is indeed Hausdorff.

§1.2. **Moduli space or a parameter space?** All lines in $\mathbb{P}^{n-1}$ are isomorphic to $\mathbb{P}^1$ and to each other. So $G(2,n)$ is not really a *moduli space*, but rather a *parameter space*: it classifies not geometric objects up to isomorphism but rather it classifies geometric sub-objects (lines) in a fixed geometric object (projective space). This distinction is mostly philosophical (depends on how do you decide when two objects are equivalent). Later we will study *Chow varieties* and *Hilbert schemes*: those parametrize all geometric sub-objects (technically called algebraic subvarieties or subschemes) of $\mathbb{P}^n$.

As a rule, parameter spaces are easier to construct than moduli spaces. To construct an honest moduli space $\mathcal{M}$ of geometric objects X, one can

- embed these objects in $\mathbb{P}^{n-1}$ for some n;
- construct a "parameter space" $\mathcal{H}$ for the embedded objects;
- divide $\mathcal{H}$ by the equivalence relation (two embedded objects are equivalent if they are abstractly isomorphic) to get $\mathcal{M}$.

In many cases of interest objects are abstractly isomorphic if and only if they are projectively equivalent in $\mathbb{P}^{n-1}$ (i.e. differ by an element of GL_n). So in effect we will have to construct an *orbit space*

$$\mathcal{M}=\mathcal{H}/\,\mathrm{GL}_n$$

and a *quotient map*

$$\mathcal{H}\to\mathcal{M}$$

that sends each point to its orbit. For example, $G(2,n)/\,\mathrm{GL}_n$ is a point – all lines of $\mathbb{P}^{n-1}$ are abstractly (and projectively!) isomorphic.

Often a more general procedure is necessary:

- first construct a parameter space $\mathcal{H}$ of pairs (X,v), where v is some sort of an extra data on X (often called "marking"). For example, v can be an embedding $X\hookrightarrow\mathbb{P}^n$ but often it's something else.
- then construct a "forgetful" map $\mathcal{H}\to\mathcal{M}$ by "forgetting" marking v. Usually this map is a quotient map for the group action.

In any case, the basic principle is

1.2.1. PRINCIPLE. A good model for a moduli space is provided by an orbit space for a group action.

So we will have to understand how to construct quotients by group actions. Those techniques are provided by *invariant theory* – the second component from the title of this course.

1.2.2. REMARK. For now, the term "quotient map" will have a very crude set-theoretic meaning: we just require that the fibers are exactly the orbits for the group action. We are not going to worry (as we should) about the relationship between geometries on the source and on the target of the quotient map. Later on, when we have more examples to play with, we will give a much more refined definition of the "quotient map". Likewise for now an "orbit space" has simply a set-theoretic meaning: the set of orbits.

§1.3. **Stiefel coordinates.** To illustrate these ideas, let's construct the Grassmannian itself as a quotient! We can mark a subspace by its basis: consider triples (U, v_1, v_2), where $U \subset \mathbb{C}^n$ is a subspace with basis $\{v_1, v_2\}$. Writing down v_1, v_2 in terms of the standard basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$, we see that these triples are parametrized by an open subset

$$\mathrm{Mat}^0_{2,n} \subset \mathrm{Mat}_{2,n}$$

of matrices of rank 2 (with rows v_1 and v_2) . This is a very simple space – in this business an open subset of an affine space is the easiest space you can possibly hope for!

Matrix coordinates on $\mathrm{Mat}^0_{2,n}$ in times long gone were known as *Stiefel coordinates* on the Grassmannian. The "forgetful" map

$$\mathrm{Mat}^0_{2,n} \to G(2, n) \tag{1.3.1}$$

has the following meaning: rank 2 matrices X and X' have the same row space if and only if $X = gX'$ for some matrix $g \in \mathrm{GL}_2$. This (1.3.1) is a quotient map for the action of GL_2 on $\mathrm{Mat}^0_{2,n}$ by left multiplication.

§1.4. **Plücker coordinates.** Since we already have a good grasp of the Grassmannian, we don't really need invariant theory to construct the map (1.3.1). Nevertheless, let me use this example to explain how to use invariants to construct quotient maps (and thus moduli spaces). In fact, this will tell us something new about the Grassmannian.

1.4.1. DEFINITION. We start with a very general situation: let G be a group acting on a set X. A function $f : X \to \mathbb{C}$ is called an *invariant function* if it is constant along G-orbits, i.e. if

$$f(gx) = f(x) \quad \text{for any } x \in X,\ g \in G.$$

Invariant functions $f_1, \ldots, f_r$ form a *complete system of invariants* if they separate orbits. This means that for any two orbits O_1 and O_2, there exists at least one function f_i such that $f_i|_{O_1} \neq f_i|_{O_2}$.

In this case then the map

$$F : X \to \mathbb{C}^r, \quad F(x) = (f_1(x), \ldots, f_r(x))$$

is obviously a quotient map (onto its image): its fibers are exactly the orbits!

Often we want to have a quotient map with target $\mathbb{P}^r$ rather than $\mathbb{C}^r$. Thus we need the following generalization:

1.4.2. DEFINITION. Fix a homomorphism

$$\chi : G \to \mathbb{C}^*.$$

A function $f : X \to \mathbb{C}$ is called a *semi-invariant of weight* χ if

$$f(gx) = \chi(g)f(x) \quad \text{for any } x \in X,\ g \in G$$

(notice that an invariant function is a special case of a semi-invariant of weight $\chi = 1$). Suppose $f_0, \ldots, f_r$ are semi-invariants of *the same weight* χ. We will call them a *complete system of semi-invariants of weight* χ if

- for any $x \in X$, there exists a function f_i such that $f_i(x) \neq 0$;
- for any two points $x, x' \in X$ not in the same orbit, we have

$$[f_0(x) : \ldots : f_r(x)] \neq [f_0(x') : \ldots : f_r(x')].$$

The first condition means that we have a map

$$F : X \to \mathbb{P}^r, \quad F(x) = [f_0(x) : \ldots : f_r(x)],$$

which is clearly constant along G-orbits:

$$[f_0(gx) : \ldots : f_r(gx)] = [\chi(g)f_0(x) : \ldots : \chi(g)f_r(x)] = [f_0(x) : \ldots : f_r(x)].$$

The second condition means that F is a quotient map onto its image.

1.4.3. EXAMPLE. $G = \mathrm{GL}_2$ acts by left multiplication on $\mathrm{Mat}^0_{2,n}$. Consider the 2×2 minors p_{ij} as functions on $\mathrm{Mat}^0_{2,n}$. It is convenient to set $p_{ji} := -p_{ij}$ for $j > i$. Consider the homomorphism

$$\det : \mathrm{GL}_2 \to \mathbb{C}^*.$$

1.4.4. PROPOSITION. *The minors p_{ij} form a complete system of semi-invariants on $\mathrm{Mat}^0_{2,n}$ of weight* $\det$.

Proof. We have

$$p_{ij}(gA) = \det(g)p_{ij}(A) \quad \text{for any} \quad g \in \mathrm{GL}_2,\ A \in \mathrm{Mat}^0_{2,n}.$$

It follows that p_{ij}'s are semi-invariants of weight $\det$. For any $A \in \mathrm{Mat}^0_{2,n}$, at least one of the p_{ij}'s does not vanish. So we have a map

$$F : \mathrm{Mat}^0_{2,n} \to \mathbb{P}^{\binom{n}{2}-1}$$

given by the minors p_{ij}.

Now take $A, A' \in \mathrm{Mat}^0_{2,n}$ such that $F(A) = F(A')$. We have to show that A and A' are in the same G-orbit. Suppose $p_{ij}(A) \neq 0$, then certainly $p_{ij}(A') \neq 0$. By twisting A and A' by some elements of GL_2, we can assume that both A and A' have a form (1.1.1). In particular,

$$p_{ij}(A) = p_{ij}(A') = 1.$$

Since $F(A) = F(A')$, we now have

$$p_{i'j'}(A) = p_{i'j'}(A') \quad \text{for any } i', j'.$$

Now it's really easy to see that $A = A'$: a key point is a trivial observation that an element in the first (resp. second) row and the k-th column of A (1.1.1) can be computed as p_{kj} (resp. p_{ik}). Thus $A = A'$. □

Since we already know that

$$\mathrm{Mat}^0_{2,n}/GL_2 = G(2, n),$$

this gives an inclusion

$$i : G(2, n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$$

called the Plücker embedding. The minors p_{ij} are in this context called *Plücker coordinates* on $G(2, n)$.

1.4.5. REMARK. A little warning: for now we have only proved that i is the inclusion of sets. It is also clear that i is a holomorphic map of manifolds: in each chart it is given by 2×2 minors of a matrix representing a 2-dimensional subspace in this chart, these minors are obviously holomorphic. To show that this inclusion is an *embedding* of complex manifolds, a little extra work is required, see below.

What is a rationale for considering minors p_{ij} and not something else as a complete system of semi-invariants? Well, let's consider *all* possible semi-invariants on $\mathrm{Mat}^0_{2,n}$ which are polynomials in $2n$ matrix entries. In fact, by continuity, this is the same thing as polynomial semi-invariants on $\mathrm{Mat}_{2,n}$. Let

$$\mathcal{O}(\mathrm{Mat}_{2,n}) = \mathbb{C}[a_{1i}, a_{2i}]_{1\le i\le n}$$

denote the algebra of polynomial functions on $\mathrm{Mat}_{2,n}$. It is easy to see (*why?*) that the only holomorphic homomorphisms $\mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}^*$ are powers of the determinant. Let

$$R_i = \mathcal{O}(\mathrm{Mat}_{2,n})^{\mathrm{GL}_2}_{\det^i}$$

be a subset of polynomial semi-invariants of weight $\det^i$. Notice that the scalar matrix $\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ acts on R_i by multiplying it on $\det^i \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = t^{2i}$. It follows that all polynomials in R_i have degree $2i$, in particular

$$R_i = 0 \quad \text{for } i < 0, \quad R_0 = \mathbb{C}.$$

We assemble all semi-invariants in one package (*algebra of semi-invariants*):

$$R = \bigoplus_{i\ge 0} R_i \subset \mathcal{O}(\mathrm{Mat}_{2,n}).$$

Since the product of semi-invariants of weights χ and χ' is a semi-invariant of weight $\chi \cdot \chi'$, R is a *graded subalgebra* of $\mathcal{O}(\mathrm{Mat}_{2,n})$.

The following theorem was classically known as *the First Fundamental Theorem of invariant theory*.

1.4.6. THEOREM. *The algebra R is generated by the minors p_{ij} for $1 \le i < j \le n$.*

Thus considering only p_{ij}'s makes sense: all semi-invariants can not separate orbits any more effectively than the generators. We are not going to use this theorem and the proof. But this raises some general questions:

- is the algebra of polynomial invariants (or semi-invariants) always finitely generated?
- do these basic invariants separate orbits?
- how to compute these basic invariants?

We will see that the answer to the first question is positive under very general assumptions (*Hilbert's finite generation theorem*). The answer to the second question is "not quite" but a detailed analysis of what's going on is available (Hilbert–Mumford's *stability* and the *numerical criterion* for it). As far as the last question is concerned, the generators of the algebra of invariants can be computed explicitly only in a handful of cases – from this perspective we are lucky that we have Plücker generators.

§1.5. **Grassmannian as a projective variety.** What is the image of a Plücker embedding $G(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$? Let U be a row space of a matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \end{bmatrix}$$

and consider a bivector

$$b = (a_{11}e_1 + \ldots + a_{1n}e_n) \wedge (a_{21}e_1 + \ldots + a_{2n}e_n) = \sum_{i<j} p_{ij} e_i \wedge e_j.$$

Thus if we identify $\mathbb{P}^{\binom{n}{2}-1}$ with the projectivization of $\Lambda^2\mathbb{C}^n$, the map i simply sends a subspace U generated by vectors $u, u' \in \mathbb{C}^n$ to $u \wedge u'$. Therefore, the image of i is a subset of *decomposable* bivectors. Let's show that this subset is a projective algebraic variety

1.5.1. DEFINITION. Let $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomials. The vanishing set

$$X = V(f_1, \ldots, f_r) = \{x \in \mathbb{P}^n \mid f_1(x) = \ldots = f_r(x) = 0\}$$

is called a *projective algebraic variety*. For each chart $U_i \subset \mathbb{P}^n$ (points where $x_i \neq 0$), $X \cap U_i \subset U_i \simeq \mathbb{A}^n$ is an *affine algebraic variety* given by vanishing of $f_1, \ldots, f_r$ *dehomogenized* with respect to x_i.

What kind of polynomials vanish along the image of i? Notice that we have

$$0 = b \wedge b = \sum_{i<j<k<l} (p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}) e_i \wedge e_j \wedge e_k \wedge e_l.$$

Thus, quadratic polynomals

$$x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}$$

vanish along the image of i. These polynomials are called *Plücker relations*

1.5.2. PROPOSITION. *$i(G(2,n))$ is a projective variety given by vanishing of Plücker relations. The map i is an immersion of complex manifolds.*

In particular, we can use this fact to *redefine* $G(2,n)$ purely algebraically as a projective algebraic variety in the Plücker projective space defined by Plücker relations.

Proof. We already know that Plücker relations vanish along the image of i, so we just have to work out the vanishing set of Plücker relations

$$X = V(x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}) \subset \mathbb{P}^{\binom{n}{2}-1}.$$

We can do it in charts. For simplicity, let's only consider the chart U_{12}, where we have $x_{12} = 1$. What are the equations of $X \cap U_{12}$? Some of them are

$$x_{kl} = x_{1k}x_{2l} - x_{1l}x_{2k}, \quad 2 < k < l \le n, \tag{1.5.3}$$

i.e. any x_{kl} whatsoever is just a minor of the matrix

$$\begin{bmatrix} 1 & 0 & -x_{23} & -x_{24} & \dots & -x_{2n} \\ 0 & 1 & x_{13} & x_{14} & \dots & x_{1n} \end{bmatrix}$$

It follows that this point of the Plücker vector space is a row space of a matrix above, and all other Plücker relations in this chart are just formal consequences of (1.5.3), i.e. $X = G(2,n)$ set-theoretically.

But of course more is true: $X_{12} = X \cap U_{12}$ is defined by equations (1.5.3), which can be interpreted as follows: X_{12} is embedded in $\mathbb{A}^{\binom{n}{2}-1}$ as a graph of the map

$$\mathbb{A}^{2(n-2)} \to \mathbb{A}^{\binom{n}{2}-1-2(n-2)} = \mathbb{A}^{\binom{n-2}{2}},$$

$$A = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} & \dots & a_{1n} \\ 0 & 1 & a_{23} & a_{24} & \dots & a_{2n} \end{bmatrix} \mapsto \{p_{kl}(A)\}_{3 \le k < l \le n}.$$

In particular,

$$X \cap U_{12} \simeq \mathbb{A}^{2(n-2)}$$

and the transition functions between various affine charts of X are exactly the same as transition functions between charts of $G(2,n)$ (*why?*) . It follows that X is a complex manifold isomorphic to $G(2,n)$ via the map i. □

Notice that of course not any projective (or affine) algebraic variety X is a complex manifold: they are often *singular*. How can we check if X is a smooth manifold? For simplicity (by considering charts), we can assume that $X = \{f_1 = \ldots = f_r = 0\} \subset \mathbb{A}^n$ is an affine variety. Let $p \in X$. Suppose that we can choose l equations (after reordering, let's assume that the first l equations work) such that

- in some complex neighborhood of p, $X = \{f_1 = \ldots = f_l = 0\}$,
- The rank of the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_l}{\partial x_1} & \dots & \frac{\partial f_l}{\partial x_n} \end{bmatrix}$$

  is equal to l.

Then, by the *Implicit Function Theorem*, X is (locally near p) a complex manifold of dimension $n - l$ with tangent space $\operatorname{Ker} J$. Notice that different points $p \in X$ may (and usually will) require different subcollections of l equations.

Here is the Algebraic Geometry's approach. The same variety X can be defined by different sets of equations, and sometimes they will fail to detect smoothness of X. For example, if we define the y-axis of $\mathbb{A}^2$ by the equation $x^2 = 0$ (rather than simply $x = 0$), the Jacobian matrix will have rank 0 (rather than 1), which reflects the fact that a double line should be thought as singular at all points.

By *Hilbert's Nullstellensatz*, the ideal of all polynomials that vanish along X is the *radical* $\sqrt{I}$. Let's suppose that I is already a *radical ideal*, i.e. $I = \sqrt{I}$. If I is not a *prime ideal* then X is *reducible*, i.e. is a union of two algebraic varieties (if $fg \in I$ but $f, g \notin I$ then $V(I) = V(I,f) \cup V(I,g)$). If we want X to be a smooth manifold then these components of X better be smooth individually (and don't intersect). So let's suppose that I is prime. In this case X is called an *irreducible affine variety*.

Let $f_1, \ldots, f_r$ be generators of I (recall that I has finitely many generators by *Hilbert's basis theorem*) and consider the Jacobian matrix J. For any point

$p \in X$, the kernel of J is called a tangent space T_pX (it is easy to see (*why?*) that it does not depend on the choice of generators). Let s be the maximal possible dimension of T_pX. A point p is called *non-singular* if $\dim T_pX = s$, otherwise it is called *singular*. The set of non-singular (resp. singular) points is called a *smooth locus* X_{sm} (resp. *singular locus* X_{sing}).

If X is irreducible then the *coordinate algebra*

$$\mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n]/I$$

is a domain and its quotient field is called the *field of rational functions* on X, denoted by $\mathbb{C}(X)$. The *dimension* of X is defined as follows:

$$\dim X = tr.deg._{\mathbb{C}}\mathbb{C}(X)$$

(the *transcendence degree*). For a proof of the following, see [Ш1, II.1.1].

1.5.4. THEOREM.

- $s = \dim X$.
- $X_{sm} \subset X$ *is Zariski-open.*

If X is a non-singular algebraic variety then X is also a complex manifold of the same dimension. To distinguish X the variety from X the complex manifold, the latter is denoted by X^{an}.

§1.6. **Second fundamental theorem – relations.** We proved that $G(2, n)$ is defined by Plücker relations in the Plücker projective space. We can ask for more: is it possible to describe *all* polynomials in $\binom{n}{2}$ variables x_{ij} that vanish along $G(2, n)$? Algebraically, we consider the homomorphism of polynomial algebras

$$\psi : \mathbb{C}[x_{ij}]_{1 \le i < j \le n} \to \mathbb{C}[a_{1i}, a_{2i}]_{1 \le i \le n}, \quad x_{ij} \mapsto p_{ij}(A)$$

and we ask: what is the kernel of ψ? (Notice that the image of ψ is equal to the algebra of GL_2-semi-invariants by the First fundamental theorem of invariant theory, but we are not going to use this). Let $I = \operatorname{Ker}\psi$. A good way of thinking about I is that its elements are relations between 2×2 minors of a general $2 \times n$ matrix.

1.6.1. THEOREM (Second fundamental theorem of invariant theory). *The kernel of ψ is generated (as an ideal) by* Plücker relations

$$x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}$$

for all fourtuples $i < j < k < l$.

Proof. The proof consists of several steps.

Step 1. Plücker relations are in I. We already know this, see Proposition 1.5.2. Let $I' \subset I$ be an ideal generated by the Plücker relations. The goal is to show that $I = I'$.

Step 2. This is called the *straightening law* – it was introduced by Alfred Young (who has Young diagrams named after him). We encode each monomial $x_{i_1j_1} \ldots x_{i_kj_k}$ in a *Young tableaux*

$$\begin{bmatrix} i_1 & i_2 & \ldots & i_k \\ j_1 & j_2 & \ldots & j_k \end{bmatrix} \tag{1.6.2}$$

The tableaux is called *standard* if it has increasing rows:

$$i_1 \leq i_2 \leq \ldots \leq i_k \quad \text{and} \quad j_1 \leq j_2 \leq \ldots \leq j_k.$$

In this case we also call the corresponding monomial a *standard monomial*. We claim that any monomial in x_{ij}'s is equivalent modulo I' (i.e. modulo Plücker relations) to a linear combination of standard monomials. Indeed, suppose that $\mathbf{x} = x_{i_1 j_1} \ldots x_{i_k j_k}$ is not standard. By reordering the variables in $\mathbf{x}$, we can assume that

$$i_1 \leq i_2 \leq \ldots \leq i_k$$

and if $i_l = i_{l+1}$ for some l then $j_l \leq j_{l+1}$. Let l be the largest index such that $j_l > j_{l+1}$. We argue by induction on l that $\mathbf{x}$ is a linear combination of standard monomials. We have

$$i_l < i_{l+1} < j_{l+1} < j_l.$$

Consider the Plücker relation

$$x_{i_l j_l} x_{i_{l+1} j_{l+1}} = -x_{i_l i_{l+1}} x_{j_{l+1} j_l} + x_{i_l j_{l+1}} x_{i_{l+1} j_l} \mod I'$$

Since the tableaux

$$\begin{bmatrix} i_l & j_{l+1} \\ i_{l+1} & j_l \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i_l & i_{l+1} \\ j_{l+1} & j_l \end{bmatrix}$$

are standard, once we substitute $-x_{i_l i_{l+1}} x_{j_{l+1} j_l} + x_{i_l j_{l+1}} x_{i_{l+1} j_l}$ for $x_{i_l j_l} x_{i_{l+1} j_{l+1}}$ in $\mathbf{x}$ we will get a linear combination of two monomials which can be both written as linear combinations of standard monomials by inductive assumption.

Step 3. Finally, we claim that standard monomials are linearly independent modulo I, i.e.

$$\{\psi(\mathbf{x}) \,|\, \mathbf{x} \text{ is a standard monomial}\}$$

is a linearly independent subset of $\mathbb{C}[a_{1i}, a_{2i}]_{1 \leq i \leq n}$. A cool idea is to order the variables as follows:

$$a_{11} < a_{12} < \ldots < a_{1n} < a_{21} < a_{22} < \ldots < a_{2n}$$

and to consider the corresponding *lexicographic ordering* of monomials in $\mathbb{C}[a_{1i}, a_{2i}]_{1 \leq i \leq n}$. For any polynomial f, let $\operatorname{init}(f)$ denote the *initial monomial* of f (i.e. the smallest monomial for lexicographic ordering). Notice that $\operatorname{init}(f)$ is multiplicative:

$$\operatorname{init}(fg) = \operatorname{init}(f) \operatorname{init}(g) \tag{1.6.3}$$

for any (non-zero) polynomials. We have

$$\operatorname{init} p_{ij} = a_{1i} a_{2j},$$

and therefore

$$\operatorname{init}(\psi(\mathbf{x})) = \operatorname{init}(p_{i_1 j_1} \ldots p_{i_k j_k}) = a_{1 i_1} a_{1 i_2} \ldots a_{1 i_k} a_{2 j_1} a_{2 j_2} \ldots a_{2 j_k}.$$

Notice that a standard monomial $\mathbf{x}$ is completely determined by $\operatorname{init}(\psi(\mathbf{x}))$. However, if the set of polynomials $\{\psi(\mathbf{x})\}$ is linearly dependent, then at least some of the initial monomials (namely, the smallest initial monomials) should cancel each other. $\square$

A lot of calculations in Algebraic Geometry can be reduced to manipulations with polynomials just like in the proof above. It is important to master these (quintessential algebraic!) techniques.

§1.7. **Hilbert polynomial.** It is rarely the case that equations of the moduli space are known so explicitly as in the case of the Grassmannian. But some numerical information about these equations is often available, as we now explain.

We start with a general situation: let $X \subset \mathbb{P}^n$ be a projective variety and let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal of polynomials that vanish on X. The algebra $R = \mathbb{C}[x_0, \ldots, x_n]/I$ is known as a *homogeneous coordinate algebra* of X. Note that R is graded (by degrees of polynomials):

$$R = \bigoplus_{j \geq 0} R_j, \quad R_0 = \mathbb{C},$$

and R is generated by R_1 as an algebra. The function

$$h(k) = \dim R_k$$

is called the *the Hilbert function* of X. Notice that knowing $h(k)$ is equivalent to knowing $\dim I_k$ for any k:

$$h(k) + \dim I_k = \binom{n+k}{k} \quad (why?)$$

We have the following fundamental theorem:

1.7.1. THEOREM. *There exists a polynomial $H(t)$ (called Hilbert polynomial) with*

$$h(k) = H(k) \quad \text{for} \quad k \gg 0.$$

This polynomial has degree $r = \dim X$ and has a form

$$\frac{d}{r!} t^r + (\text{lower terms}),$$

where d is the degree of X, i.e. the number of points in the intersection of X with a general projective subspace of codimension r (a subspace is general if it intersects $G(2, n)$ transversally in all intersection points).

We will prove this theorem later along with other important properties of the Hilbert function. But now let's use it!

1.7.2. PROPOSITION. *Let $n \geq 3$. The Hilbert function of $G(2, n)$ in the Plücker embedding is*

$$h(k) = \binom{n+k-1}{k}^2 - \binom{n+k}{k+1}\binom{n+k-2}{k-1}. \tag{1.7.3}$$

The degree of $G(2, n)$ in the Plücker embedding is the Catalan number

$$\frac{1}{n-1}\binom{2n-4}{n-2} = 1, 2, 5, 14, 42, 132, \ldots$$

Proof. During the proof of the Second Fundamental Theorem 1.6.1 we have established that $h(k)$ is equal to the number of standard monomials of degree k, i.e. to the number of standard tableaux with k columns. Let N_l be the number of non-decreasing sequences $1 \le i_1 \le \ldots \le i_l \le n$. Then we have

$$N_l = \binom{n+l-1}{l}:$$

this is just the number of ways to choose l objects from $\{1, \ldots, n\}$ with repetitions (so it is for example equal to the dimension of the space of polynomials in n variables of degree l). The number of tableaux

$$\begin{bmatrix} i_1 & i_2 & \ldots & i_k \\ j_1 & j_2 & \ldots & j_k \end{bmatrix},\ 1 \le i_1 \le \ldots \le i_k \le n,\ 1 \le j_1 \le \ldots \le j_k \le n$$

(but without the condition that $i_l < j_l$ for any l) is clearly equal to

$$N_k^2 = \binom{n+k-1}{k}^2.$$

Now we have to subtract the number of non-standard tableaux. We claim that there is a bijection between the set of nonstandard tableaux and the set of pairs (A, B), where A is a non-decreasing sequence of length $k+1$ and B is a non-decreasing sequence of length $k-1$. This will prove (1.7.3).

Suppose that l is the number of the first column where $i_l \ge j_l$. Then we can produce two sequences:

$$j_1 \le \ldots \le j_l \le i_l \le \ldots \le i_k$$

of length $k+1$ and

$$i_1 \le \ldots \le i_{l-1} \le j_{l+1} \le j_k$$

of length $k-1$. In an opposite direction, suppose we are given sequences

$$i_1 \le \ldots \le i_{k+1} \quad \text{and} \quad j_1 \le \ldots \le j_{k-1}.$$

Let l be the minimal index such that $i_l \le j_l$ and take a tableaux

$$\begin{bmatrix} j_1 & \ldots & j_{l-1} & i_{l+1} & i_{l+2} & \ldots & i_k \\ i_1 & \ldots & i_{l-1} & i_l & j_l & \ldots & j_{k-1} \end{bmatrix}$$

If $i_l > j_l$ for any $l \le k-1$, then take the tableaux

$$\begin{bmatrix} j_1 & \ldots & j_{k-1} & i_k \\ i_1 & \ldots & i_{k-1} & i_{k+1} \end{bmatrix}.$$

After some manipulations with binomial coefficients, (1.7.3) can be rewritten as

$$\frac{1}{(n-1)!(n-2)!}(k+n-1)(k+n-2)^2 \ldots (k+2)^2(k+1).$$

This is a polynomial in k of degree $2n-4$ with a leading coefficient $\frac{1}{(n-1)!(n-2)!}$. Since the degree of $G(2, n)$ is equal to $(2n-4)!$ multiplied by the leading coefficient of $h(k)$, we see that this degree is indeed the Catalan number. $\square$

§1.8. **Enumerative geometry.** Why do we need moduli spaces? One reason is that their geometry reflects delicate properties of parametrized geometric objects. As an example, let's try to relate the degree of the Grassmannian (i.e. the Catalan number) to geometry of lines.

1.8.1. THEOREM. *The number of lines in* $\mathbb{P}^{n-1}$ *that intersect* $2n-4$ *general codimension* 2 *subspaces is equal to the Catalan number*

$$\frac{1}{n-1}\binom{2n-4}{n-2}$$

For example, there is only one line in $\mathbb{P}^2$ passing through 2 general points, 2 lines in $\mathbb{P}^3$ intersecting 4 general lines, 5 lines in $\mathbb{P}^4$ intersecting 6 general planes, and so on. This is a typical problem from the classical branch of Algebraic Geometry called *enumerative geometry*, which was described by H. Schubert (around 1870s) as a field concerned with questions like: How many geometric figures of some type satisfy certain given conditions? If these figures are lines (or projective subspaces), the enumerative geometry is nowdays known as *Schubert calculus*, and is more or less understood. Recently enumerative geometry saw a renaissance (*Gromov–Witten invariants*, etc.) due to advances in moduli theory.

Proof. The degree is equal to the number of points in

$$G(2,n)\cap L,$$

where $L\subset\mathbb{P}^{\binom{n}{2}-1}$ is a general subspace of codimension $2n-4$, i.e. a subspace that intersects $G(2,n)$ transversally in all intersection points. In other words, it is the number of points in the intersection

$$G(2,n)\cap H_1\cap\ldots\cap H_{2n-4},$$

where H_i's are hyperplanes, as long as this intersection is transversal.

Notice that the set of lines intersecting a fixed codimension 2 subspace can be described as the intersection with a hyperplane

$$D=G(2,n)\cap H.$$

For example, lines intersecting $W=\langle e_3,\ldots,e_n\rangle$ are exactly the lines that do not surject onto $\langle e_1,e_2\rangle$ when projected along W. This is equivalent to vanishing of the Plücker coordinate p_{12}. So D is exactly the complement of the chart U_{12}! It is called a special *Schubert variety*.

We can describe D explicitly by writing down the minor $p_{12}=0$ in other charts X_{ij}: in most charts (when $i>2$) D is a quadric of rank 4, in particular it is a singular hypersurface. This looks a bit worrisome for us because it implies that H is not everywhere transversal to $G(2,n)$, (because at transversal intersection points the intersection is non-singular). In particular, H is not really a general hyperplane (general hyperplanes intersect $G(2,n)$ everywhere transversally by *Bertini's Theorem*). However, at any point p of a smooth locus $D^0\subset D$ the hyperplane H is transversal to $G(2,n)$ and the intersection of tangent spaces

$$T_pD\cap T_pG(2,n)=T_pD.$$

Notice that any codimension 2 subspace in $\mathbb{P}^{n-1}$ is GL_n-equivalent to W. So the claim that we have to check is that if $g_1, \ldots, g_{2n-4} \in G = \mathrm{GL}_n$ are sufficiently general group elements then any point in

$$g_1 D \cap \ldots \cap g_{2n-4} D$$

is

- away from the singular locus of each $g_i D$;
- a transversal intersection point of $g_i D$'s.

Quite remarkably, the proof relies only on the fact that GL_n acts transitively on $G(2, n)$ and on nothing else. It is known as the *Kleiman–Bertini transversality* argument. But first we have to explain a very powerful technique in algebraic geometry called *dimension count*. □

Firstly, let's discuss *regular morphisms* and *rational maps* between algebraic varieties. This is very straightforward but there are several delicate points. We start with an affine case: suppose we have affine varieties

$$X \subset \mathbb{A}^n \quad \text{and} \quad Y \subset \mathbb{A}^m.$$

A *regular function* on X is a restriction of a polynomial function. Regular functions form a coordinate algebra $\mathcal{O}(X)$. A *morphism* (or a regular morphism)

$$f : X \to \mathbb{A}^m$$

is a map given by m regular functions $f_1, \ldots, f_m \in \mathcal{O}(X)$. If the image lies in $Y \subset \mathbb{A}^m$ then we have a morphism $X \to Y$. It defines a *pull-back homomorphism* of coordinate algebras $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ and is completely determined by it. An *isomorphism of algebraic varieties* is a morphism that has an inverse. This happens if and only if the pullback f^* is an isomorphism of algebras.

Now suppose, in addition, that X is irreducible. Suppose $f \in \mathbb{C}(X)$ is a rational function. Recall that this means that f is a ratio of two regular functions, and so we can write $f = p/q$, where p and q are some polynomials. Notice, however, that of course different polynomials can define the same function on X, so the presentation $f = p/q$ is not unique. Let $x \in X$ be a point. We say that *f is defined at x* if it can be written as a ratio of two regular functions such that $q(x) \neq 0$. The set of points where f is defined is obviously a Zariski open subset of X.

Suppose we have m rational functions $f_1, \ldots, f_m$ such that if all of them are defined at $x \in X$ then $(f_1(x), \ldots, f_m(x)) \in Y$. In this case we say that we have a rational map

$$f : X \dashrightarrow Y$$

(so a rational map is not everywhere defined). Here is a good exercise on comparing definitions of regular and rational maps:

1.8.2. LEMMA. *If a rational map $f : X \dashrightarrow Y$ is everywhere defined then in fact f is a regular morphism.*

Proof. For any point $x \in X$, choose a presentation $f = p_x/q_x$ with $q_x(x) \neq 0$. Thus X is covered by *principal Zariski open sets*

$$D(q_x) := \{a \in X \mid q_x \neq 0\}.$$

Consider the ideal $I = (q_x) \subset \mathcal{O}(X)$ generated by all denominators and choose its finite basis $q_1, \ldots, q_r$ out of them. These functions don't have any zeros on X, and therefore $I = \mathcal{O}(X)$ by Hilbert's Nullstellensatz. Thus we can write

$$1 = a_1 q_1 + \ldots + a_r q_r.$$

It follows that

$$f = a_1 f q_1 + \ldots + a_r f q_r = a_1 p_1 + \ldots + a_r p_r$$

is a regular function. □

Finally, let's discuss rational maps of projective varieties. Even more generally, a (Zariski) open subset U of a projective variety X is called a *quasi-projective* variety. If $x \in U \subset X \subset \mathbb{P}^n$ and $f = P/Q$ is a rational function in $x_0, \ldots, x_n$ such that $\deg P = \deg Q$ then f defines a rational function on U regular at x. (We can also just work in the affine chart $\mathbb{A}^n$). A rational function is regular on U if it is regular at any point. A rational function $U \to \mathbb{A}^n$ is a mapping with rational components. A regular morphism $f : U \to V \subset \mathbb{P}^m$ is a map such that for any point $x \in X$ and $y = f(x)$ there exists a Zariski neighborhood U' of x and an affine chart $y \in \mathbb{A}^m$ such that the induced map $U' \to \mathbb{A}^m$ is regular.

Here is an important theorem:

1.8.3. THEOREM ([Ш1, 1.6.3]). *Let $f : X \to Y$ be a regular map between irreducible quasiprojective varieties. Suppose that f is surjective, $\dim X = n$, and $\dim Y = m$. Then $m \leq n$ and*

- $\dim F \geq n - m$ *for any irreducible component F of any fibre $f^{-1}(y)$, $y \in Y$.*
- *there exists a non-empty Zariski-open subset $U \subset Y$ such that* $\dim f^{-1}(y) = n - m$ *for any $y \in U$.*

Sketch of the Kleiman–Bertini transversality argument. The first part is a *dimension count*: denoting by Z the singular locus of D, consider a subset

$$W = \{g_1 z = g_2 d_2 = \ldots = g_{2n-4} d_{2n-4}\} \subset (G \times Z) \times (G \times D) \times \ldots \times (G \times D)$$

Clearly, g_1 and z can be arbirary, as are d_i for $i \geq 2$, and then for each g_i with $i \geq 2$ we have $\dim G(2, n) = 2n - 4$ independent conditions. So

$$\dim W = (2n-4)\dim G + \dim Z + (2n-5)(2n-5) - (2n-5)(2n-4) < (2n-4)\dim G.$$

It follows that the projection

$$W \to G \times \ldots \times G \quad (2n-4 \text{ times}) \tag{1.8.4}$$

has empty general fibers, i.e. the general translates of D intersect away from their singular points.

So we can throw away the singular locus and assume that D is nonsingular (but not compact now). The rest of the argument repeats itself but now we have to analyze tangent spaces a little bit, which is a part of the argument that we will skip. Consider a subset

$$W = \{g_1 d_1 = g_2 d_2 = \ldots = g_{2n-4} d_{2n-4}\} \subset (G \times D) \times (G \times D) \times \ldots \times (G \times D).$$

Now $\dim W = (2n-4)\dim G$. A little reflection shows that W is also nonsingular. So generic fibers of the projection (1.8.4) are either empty or a bunch of *non-critical* points. A little local calculation (that we skip) shows that this is equivalent to the transversality of the corresponding translates $g_1D, \dots, g_{2n-4}D$. □

HOMEWORK DUE ON FEB 7

Write your name and sign here:

Please turn in this problem sheet along with your solutions. If you have submitted any solutions orally, please check that you have my signature near these problems in the worksheet.

Problem 1. (a) Let $L_1, L_2, L_3 \subset \mathbb{P}^3$ be three general lines. Show that there exists a unique quadric surface $S \subset \mathbb{P}^3$ containing them all and that lines that intersect L_1, L_2, L_3 are exactly the lines from the ruling of S. Try to be as specific as possible about the meaning of the word "general". (b) Use the previous part to give an alternative proof of the fact that 4 general lines in $\mathbb{P}^3$ have exactly two common transversals (2 points).

Problem 2. Show that the Plücker vector space $\mathbb{C}^{\binom{n}{2}}$ can be identified with the space of skew-symmetric $n \times n$ matrices and $G(2,n)$ with the projectivization of the set of skew-symmetric matrices of rank 2. Show that the Plücker relations in this language are 4×4 Pfaffians (1 point).

Problem 3. For any line $L \subset \mathbb{P}^3$, let $[L] \in \mathbb{C}^6$ be the corresponding Plücker vector. The Grassmannian $G(2,4) \subset \mathbb{P}^5$ is a quadric, and therefore can be described as the vanishing set of a quadratic form Q, which in turn has an associated inner product such that $Q(v) = v \cdot v$. Describe this inner product and show that $[L_1] \cdot [L_2] = 0$ if and only if lines L_1 and L_2 intersect (1 point).

Problem 4. 4 In the notation of the previous problem, show that five lines $L_1, \ldots, L_5$ have a common transversal if and only if

$$\det \begin{bmatrix} 0 & [L_1]\cdot[L_2] & [L_1]\cdot[L_3] & [L_1]\cdot[L_4] & [L_1]\cdot[L_5] \\ [L_2]\cdot[L_1] & 0 & [L_2]\cdot[L_3] & [L_2]\cdot[L_4] & [L_2]\cdot[L_5] \\ [L_3]\cdot[L_1] & [L_3]\cdot[L_2] & 0 & [L_3]\cdot[L_4] & [L_3]\cdot[L_5] \\ [L_4]\cdot[L_1] & [L_4]\cdot[L_2] & [L_4]\cdot[L_3] & 0 & [L_4]\cdot[L_5] \\ [L_5]\cdot[L_1] & [L_5]\cdot[L_2] & [L_5]\cdot[L_3] & [L_5]\cdot[L_4] & 0 \end{bmatrix} = 0$$

(4 points)

Problem 5. Prove (1.6.3). (1 point)

Problem 6. Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface of degree d (i.e. a vanishing set of an irreducible homogeneous polynomial of degree d). Compute its Hilbert polynomial (1 point).

Problem 7. Let $X \subset \mathbb{P}^n$ be a hypersurface and let $F_X \subset G(2,n)$ be the subset of lines contained in X. Show that F_X is a projective algebraic variety (2 points).

Problem 8. (a) For any point $p \in \mathbb{P}^3$ (resp. plane $H \subset \mathbb{P}^3$) let $L_p \subset G(2,4)$ (resp. $L_H \subset G(2,4)$) be a subset of lines containing p (resp. contained in H). Show that each L_p and L_H is isomorphic to $\mathbb{P}^2$ in the Plücker embedding of $G(2,4)$. (b) Show that any $\mathbb{P}^2 \subset G(2,4)$ has a form L_p or L_H for some p or H. (3 points)

Problem 9. Consider the d-th Veronese map

$$\mathbb{P}^1 \to \mathbb{P}^d,$$

$$[x:y] \mapsto [x^d : x^{d-1}y : \ldots : y^d],$$

and its image, the rational normal curve. (a) Show that this map is an embedding of complex manifolds. (b) Show that the ideal of the rational normal curve is generated by 2×2 minors of the matrix

$$\det \begin{bmatrix} z_0 & z_1 & \ldots z_{n-1} \\ z_1 & z_2 & \ldots z_n \end{bmatrix},$$

and compute its Hilbert polynomial. (2 points)

Problem 10. Consider the Segre map

$$\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n^2-1} = \mathbb{P}(\mathrm{Mat}_{nn}),$$

$$([x_1 : \ldots : x_n], [y1 : \ldots : y_n]) \mapsto [x_1y_1 : \ldots : x_iy_j : \ldots : x_ny_n]$$

(a) Show that this map is an embedding of complex manifolds. (b) Show that the ideal of the Segre variety in $\mathbb{P}(\mathrm{Mat}_{nn})$ is generated by 2×2 minors $a_{ij}a_{kl} - a_{il}a_{kj}$ (c) Compute the Hilbert polynomial of the Segre variety. (d) Compute the degree of the Segre variety. (3 points)

Problem 11. In the notation of the previous problem, give a geometric interpretation of the degree of the Segre variety in the spirit of Theorem 1.8.1. What is the analogue of a special Schubert variety? (1 point)

Problem 12. Consider 4 lines $L_1, L_2, L_3, L_4 \subset \mathbb{P}^3$. Suppose no three of them lie on a plane. Show that if 5 pairs of lines L_i, L_j intersect then the sixth pair of lines intersects as well. (1 point)

Problem 13. Check a "little local calculation" at the end of the proof of Theorem 1.8.1. (3 points)

Problem 14. Let $I \subset R = \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal and let $V(I) \subset \mathbb{P}^n$ be the corresponding projective variety. (a) Show that $V(I)$ is empty if and only if there exists $s > 0$ such that I contains all monomials of degree s. (b) Show that there exists an inclusion-reversing bijection between projective subvarieties of $\mathbb{P}^n$ and radical homogeneous ideals of R different from $R_+ := (x_0, \ldots, x_n)$. (2 points)

Problem 15. An alternative way of thinking about a $2 \times n$ matrix

$$X = \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ x_{21} & \ldots & x_{2n} \end{bmatrix}$$

is that it gives n points $p_1, \ldots, p_n$ in $\mathbb{P}^1$ (with homogeneous coordinates $[x_{11} : x_{21}]$, …, $[x_{1n} : x_{2n}]$), at least as soon as X has no zero columns. Suppose $n = 4$ and consider the rational normal curve (twisted cubic) $f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. (a) Show that points $f(p_1), \ldots, f(p_4)$ lie on a plane if and only

$$F(X) = \det \begin{bmatrix} x_{11}^3 & x_{11}^2x_{21} & x_{11}x_{21}^2 & x_{21}^3 \\ x_{12}^3 & x_{12}^2x_{22} & x_{12}x_{22}^2 & x_{22}^3 \\ x_{13}^3 & x_{13}^2x_{23} & x_{13}x_{23}^2 & x_{23}^3 \\ x_{14}^3 & x_{14}^2x_{24} & x_{14}x_{24}^2 & x_{24}^3 \end{bmatrix} = 0.$$

(b) Show using the first fundamental theorem of invariant theory that $F(X)$ is a polynomial in 2×2 minors of the matrix X. (c) Do the same thing without using the first fundamental theorem (3 points).