Mathematics in the late 16th - early 17th century

- Simon Stevin (1548 – 1620) wrote a short pamphlet *La Disme*, where he introduced decimal fractions to a wide audience.

- Transition to Hindu-Arabic from Roman numerals was initially resisted, hence the practice of writing amounts on checks
Beginnings of Calculus

- Archimedes (287 BC - 212 BC) “The Quadrature of the Parabola” area under the curve
- Kepler (1571-1630) “New solid geometry of wine bottles” - various optimization problems
- Napier (1550-1617) “Description of the wonderful canon of logarithms” - Laplace said that invention of logarithms “by shortening the labors, doubled the life of an astronomer”
- Descartes (1596 – 1650) “Discourse on the Method for Rightly Directing One's Reason and Searching for Truth in the Sciences” - Analytic Geometry (study of geometry using algebra and Cartesian coordinate system)
• Pascal (1623–1662) “Treatise on the sines of the quadrant” - area under the sine curve - integral of $\sin(x)$

• Gregory (1584-1667) - integral of $1/x$ is equal to $\ln(x)$

• Van Heuraut (1634-1660) - “Epistola de transmutatione curvarum linearum in rectas” - arc length formula

• Fermat (1601-1665)
  • “Methodus ad disquirendam maximam et minima” - Fermat’s principle - in modern day’s formulation, if $f(x)$ achieves extremum at $x=a$ then $f''(a)=0$
  • “De tangentibus linearum curvarum” - construction of tangents to curves
Barrow (1630-1677) “Geometrical Lectures” - Fundamental Theorem of Calculus - both its geometric manifestation - relationship between tangents (“derivatives”) and the area (“integrals”) and the physical interpretation - relationship between velocity (derivatives) and distance (integrals).

His father decided that he can’t become a successful merchant because of his temper and sent him to study theology (!), which in turn get him interested in chronology, then astronomy, and finally geometry (to design telescope lenses).

On a journey to the Holy Land, the ship was attacked by pirates. He was the only passenger to join the crew sword in hand to repel the pirates.

Barrow took a prestigious position as a Lucasian professor of geometry at Cambridge, where one of his students was Isaac Newton. He later took the same position and Barrow happily became a full-time theologian.
• It looks like everything that modern students study in Calculus was discovered in the 17th century before Newton.
• Why is he called an inventor of Calculus?
• Newton discovered a major method to solve any equation, algebraic or differential (the concept he introduced) using power series, also known as Taylor series (Taylor was a student of Newton)

• He was guided by an analogy

<table>
<thead>
<tr>
<th>numbers</th>
<th>decimal fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>functions</td>
<td>power series</td>
</tr>
</tbody>
</table>
Fluxions and Fluents

- For Newton, every variable \( x \) he considered depended on time \( t \). Nowadays we would use a function notation \( x = f(t) \). Newton called the quantity \( x \) the fluent.

- The fluxion \( \dot{x} \) is the speed with which \( x \) increases. In modern notation, this is \( f'(t) \), although Newton’s notation is still used, especially in physics.
Discussion of Essay proposals

• (I) Maria Hebert, Raabe Rocha, Sarah Mepham, Leah Herd, Maria Fernandes, Tanner Kranich, Yu Cheng

• (II) Sara Bajwa, Andy Marton, Prayush Pokharel, Pui Yi May, Max Trask, Andrew Preston

• (III) John Demoy, Michael Miceli, Karolyna Myca, Donald Mbanya, Jian Ma, Tim Doyle

• (IV) Brandon Nowakowski, Molly O’Neal, Andrea Watson, Shane Becker, Alex Martirosyan, Lucas Mastro

• (V) Michelle Kim, Garrett Greene, Nick Derian, Frederick Krauss, Fedor Arkhipov, Mustapha Ozdemir
Group discussion of Newton’s “The method of fluxions and infinite series”

• (I) Table of contents and Introduction: what’s in the book? Comparison of Newton’s and modern terminologies.

• (II) Problems 4 and 5.

• (III) Problems 6 and 7

• (IV) Problem 11

• (V) Problems 19 and 20.
Problem III.

To determine the Maxima and Minima of Quantities.

When a quantity is the greatest or the least that it can, be at that moment it neither flows backwards nor forwards: for if it flows forwards or increases then it was less, and will presently be greater than it is; and on the contrary if it flows backwards or decreases, then it was greater, and will presently be less than it is. Wherefore find its Fluxion by Prob. I. and suppose it to be equal to nothing.

Example. i. If in the equation \( x^3 - ax^2 + axy - y^3 = 0 \) the greatest value of \( x \) be required, find the relation of the Fluxions of \( x \) and \( y \), and you will have \( 3xx' - 2axx' + axy' - 3yy' + ayx' = 0 \), then make \( x' = 0 \) there will remain \( 3yy' + ayx = 0 \), or \( 3y^2 = ax \), by the help of which you may exterminate either \( x \) or \( y \) out of the primary equation; and by the resulting equation you may determine the other, and in like manner solve the rest.
Newton applied his new mathematics to solve a staggering number of mathematical and physical problems, many of them contained in “Principia” (Mathematical Principles of Natural Philosophy). The crowning achievement of Principia was introduction of the Law of Gravity and demonstration that

- Law of Gravity is the only possible central force field where Laws of Kepler can hold (this is a very difficult theorem)

Analysis went through a period of rapid expansion in the 18th century based primarily on Newton’s revolutionary ideas.
Re-evaluation of rigor in analysis

• Toward the end of the 18th century, with the increasing necessity for mathematicians to teach rather than just do research, there was an increasing concern with how mathematical ideas should be presented to students.

• There was concern that analysis became too much of an “art”. The grand masters would apply it to obtain remarkable results, but for the beginners it was difficult to separate valid methods from imprecise speculation.
Re-evaluation of rigor in analysis

• Following Newton, Lagrange and others attempted to base all of calculus on the notion of a power series.

• The derivative was defined using linearization:

\[ f(a + x) \approx f(a) + f'(a)x \]

• Further terms of the power series give the second derivative, etc. (like taking consecutive digits in the decimal fraction)

\[ f(a + x) \approx f(a) + f'(a)x + \frac{f''(a)}{2}x^2 \]

• Integral was defined as an antiderivative followed by the proof of the “fundamental theorem” relating areas to antiderivatives.
A familiar definition of derivative as a limit of a fraction was popularized by d’Alembert, and later taken as definition by Lacroix in his textbook *Traité élémentaire du calcul différentiel et du calcul integral*, which went through many editions in the 19th century.

For example, he showed that if \( u=ax^2 \) and \( u_1=a(x+h)^2 \) then the derivative \( 2ax \) is “the limit of the ratio \( (u_1-u)/h \), or, is the value towards which this ratio tends in proportion as the quantity \( h \) diminishes, and to which it may approach as near as we choose to make it.”

After calculating several other limits of ratios, Lacroix writes that “the differential calculus is the finding of the limit of the ratios of the simultaneous increments of a function and of the variables on which it depends.”

Lacroix then returned to power series expansions and used them to develop the differentiation formulas for various functions and the methods for determining maxima and minima, even for functions of several variables.

Lacroix proved a familiar “second derivatives test” that a function of two variables \( f(x,y) \) has an extreme value if \( f_{xx}f_{yy}-(f_{xy})^2>0 \).
Augustin-Louis Cauchy (1789–1857)

- Cauchy received an excellent education at the Ecole Polytechnique. While working as an engineer from 1810 to 1813 on military projects of the Napoleonic government, he showed a strong interest in mathematics.

- Cauchy was encouraged by Laplace and Lagrange to leave engineering. With their help he secured a teaching position at the Ecole Polytechnique.

- Cauchy wrote so many mathematical papers that the journal of the Paris Academy was forced to limit the contributions of any one person. Cauchy got around these restrictions by establishing his own journal.

- As Abel wrote in a letter to a friend in 1826 during his visit to Paris, “there is no way to get along with him. … Cauchy is immoderately Catholic and bigoted, a very strange thing for a mathematician. Otherwise he is the only one [in Paris] who at present works in pure mathematics.”
• Cauchy found the method based on power series lacking in “rigor.” He was not satisfied with what he believed were unfounded manipulations of algebraic expressions, especially infinitely long ones.

• Cauchy discovered that all derivatives of the function \( f(x) = e^{-\frac{1}{x^2}} \) vanish at 0. Its power series is identically zero but the function is not!

• Cauchy began to rethink the basis of the calculus entirely. In 1821, at the urging of his colleagues, he published a textbook, in which he introduced new methods into the foundations of the calculus. His approach provided the model for calculus and analysis texts ever since.
Limit

• Leibniz (1684): If any continuous transition is proposed terminating in a certain limit, then it is possible to form a general reasoning, which covers also the final limit.

• Newton (1687): The ultimate ratio of evanescent quantities … [are] limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished \textit{in infinitum}.

• D’Alembert (1754): The ratio \(a/(2y+z)\) is always smaller than \(a/(2y)\), but the smaller \(z\) is, the greater the ratio will be and, since one may choose \(z\) as small as one pleases, the ratio \(a/(2y+z)\) can be brought as close to the ratio \(a/(2y)\) as we like. Consequently, \(a/(2y)\) is the limit of the ratio \(a/(2y+z)\).

• Lacroix (1806): The limit of the ratio \((u_1-u)/h\) . . . is the value towards which this ratio tends in proportion as the quantity \(h\) diminishes, and to which it may approach as near as we choose to make it.

• Cauchy (1821): If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others.
• As an example, Cauchy noted that an irrational number is the limit of the various fractions that approach it.

• Cauchy defined an infinitely small quantity to be a variable whose limit is zero.

• Cauchy’s definition may appear different from the modern epsilon-delta definition. But reading Cauchy’s proofs reveals that he used it in the same way, by translating statements arithmetically using the language of inequalities.

• Let’s look at one proof from the Cauchy’s textbook.
THEOREM  If, for increasing values of $x$, the difference $f(x + 1) - f(x)$ converges to a certain limit $k$, the fraction $\frac{f(x)}{x}$ converges at the same time to the same limit.\(^7\)

Cauchy began by translating the hypothesis of the theorem into an arithmetic statement: Given any value $\epsilon$, as small as one wants, one can find a number $h$ such that if $x \geq h$, then $k - \epsilon < f(x + 1) - f(x) < k + \epsilon$. He then proceeded to use this translation in his proof. Because each of the differences $f(h + i) - f(h + i - 1)$ for $i = 1, 2, \ldots, n$ satisfies the inequality, so does their arithmetic mean

$$\frac{f(h + n) - f(h)}{n}.$$\[\text{It follows that}\]

$$\frac{f(h + n) - f(h)}{n} = k + \alpha,$$

where $-\epsilon < \alpha < \epsilon$ or, setting $x = h + n$, that

$$\frac{f(x) - f(h)}{x - h} = k + \alpha.$$\[\text{But then } f(x) = f(h) + (x - h)(k + \alpha) \text{ or }\]

$$\frac{f(x)}{x} = \frac{f(h)}{x} + \left(1 - \frac{h}{x}\right)(k + \alpha).$$\[\text{Because } h \text{ is fixed, Cauchy concluded that as } x \text{ gets large, } f(x)/x \text{ approaches } k + \alpha, \text{ where } -\epsilon < \alpha < \epsilon. \text{ Because } \epsilon \text{ is arbitrary, the conclusion of the theorem holds. Cauchy also proved the theorem for the cases } k = \pm \infty \text{ and then used it to conclude, for example, that as } x \text{ gets large } \log x/x \text{ converges to } 0 \text{ and } a^x/x (a > 1) \text{ has limit } \infty.\]
Continuity

• The geometric notion of a continuous curve, one without any breaks, was understood but Cauchy sought to find a rigorous analytic definition.

• Lagrange had attempted such a definition in the specific case of a function “continuous at 0” and having value 0 there: “We can always find an abscissa h corresponding to an ordinate less than any given quantity; and then all smaller values of h correspond also to ordinates less than the given quantity.”

• Cauchy generalized Lagrange’s idea and gave his own definition: “The function \( f(x) \) will be, between two assigned values of the variable \( x \), a **continuous function** … if for each value of \( x \) between these limits, the numerical [absolute] value of the difference \( f(x+\alpha) - f(x) \) decreases indefinitely with \( \alpha \). In other words, the function \( f(x) \) will remain continuous with respect to \( x \) between the given values if, between these values, an infinitely small increment of the variable always produces an infinitely small increment of the function itself.”

• Cauchy demonstrated how to use his definition by showing that \( \sin(x) \) is continuous. For \( \sin(x+\alpha) - \sin(x) = 2\sin(\alpha/2)\cos(x+\alpha/2) \), and the right side clearly “decreases indefinitely” with \( \alpha \).
Convergence

• One difference between decimal expansion of numbers and power series expansion of functions is that not every series is convergent. Cauchy gave a rigorous definition of convergence: “Let $s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$ be the sum of the first $n$ terms. If, for increasing values of $n$, the sum $s_n$ approaches indefinitely a certain limit $s$, the series will be called convergent, and the limit in question will be called the sum of the series. On the contrary, if, as $n$ increases indefinitely, the sum $s_n$ does not approach any fixed limit, the series will be divergent and will not have a sum.”

• Cauchy developed various convergence tests, especially for series of positive terms, using the comparison test, that if a given series is term-by-term bounded by a convergent series, then the given series is itself convergent.

  – His most common comparison was to a geometric series with ratio less than 1. Cauchy proved the root test, that if the limit of the values $u_n^{1/n}$ is a number $k<1$, then the series converges. Choosing a number $U$ such that $k<U<1$, Cauchy noted that for $n$ sufficiently large, $u_n<U^n$. It then follows by comparison with the convergent geometric series $1+U+U^2+U^3+\ldots$ that the given series also converges.

• Cauchy also proved the ratio test and the alternating series test.
Derivatives

• Cauchy’s *Cours d’Analyse* provided a treatment of the basic ideas of functions and series. In his follow-up 1823 textbook, Cauchy applied his new ideas on limits to the study of the derivative and the integral.

• Cauchy defined the derivative of a function as the limit of \([f(x + i) - f(x)]/i\) as \(i\) approaches the limit of 0, as long as this limit exists. He noted that this limit has a definite value for each value of \(x\), and therefore is a new function of that variable, a function for which he used Lagrange’s notation \(f'(x)\).

• Cauchy used this definition to rigorously calculate derivatives of several elementary functions. For example, if \(f(x) = \sin(x)\), then the quotient of the definition reduces to \([\sin(i/2)\cos(x+i/2)]/(i/2)\) whose limit is seen to be \(\cos x\).

• There was nothing new about Cauchy’s calculations of derivatives or about the theorems Cauchy was able to prove about derivatives. But because previous definitions of a derivative rested on the false assumption that any function could be expanded into a power series, the significance of Cauchy’s works lies in his explicit use of the modern definition of a derivative, translated into the language of inequalities through his definition of limit, to prove theorems.
Cauchy’s treatment of the derivative was closely related to the works of Euler and Lagrange. Cauchy’s treatment of the integral, on the other hand, was entirely new.

In the 18th century, integration was defined as the inverse of differentiation, although Leibniz had developed his notation to remind one of the integral as an infinite sum.

Cauchy constructed a theory of definite integrals based on approximation techniques: “it seems to me that this manner of conceiving a definite integral ought to be adopted … because it is equally suitable to all cases, even to those in which we cannot pass generally from the function placed under the integral sign to the primitive function.”

Supposing that $f(x)$ is continuous on $[x_0, X]$, Cauchy took $n-1$ new intermediate values $x_1 < x_2 < \ldots < x_{n-1}$ between $x_0$ and $x_n = X$ and formed the sum

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (X - x_{n-1})f(x_{n-1}).$$

Cauchy noted that $S$ depends both on $n$ and on the particular values $x_i$ selected. But “it is important to observe that if the numerical values of the elements $[x_{i+1} - x_i]$ become very small and the number $n$ very large, the method of division will have only an insensible influence on the value of $S$.”
• We call these sums “Riemann sums” nowadays (and the integral the “Riemann integral”) because Riemann slightly generalized Cauchy’s sums in his later treatment of integrals. Rather than computing the values of $f(x)$ at the left ends of subintervals as Cauchy did, Riemann allowed computing $f(x)$ at any point of a subinterval. This extra degree of freedom made the definition more flexible for applications.