Gauss’s Work on Geometry and Geodesy

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What is Geodesy?

- Branch of applied mathematics, engineering, and earth sciences
- Deals with the measurement and representation of the Earth (or any planet)
- Includes its curvature, coordinate systems, gravitational field, tidal forces and plate movement

Before we jump into Gauss, we should first explain what Geodesy is. Geodesy is a branch of applied math that studies measurements of the Earth, or any planet for that matter. Geodesy is Greek for divisions of the Earth, which is very fitting. There are 2 ways of thinking of geodesy, on a small scale, which includes surveying land and measurements and on a larger scale, which is less engineering based, and focuses on the entire earth on a global scale, including coordinate systems, tides, and magnetic fields.
These are 10 Deutsche Mark banknotes from 1999 honoring Gauss. As we’ll see in the next class, Gauss was exceptionally gifted in number theory. However, the focus of this presentation is his contributions to geometry and geodesy, an area of science which he also made massive contributions to. This can be seen in these bank notes. On the front we can see Gauss’s portrait as well as his famous distribution. His idea of normal distributions helped him accurately predict where Ceres would reappear after disappearing right when it was first discovered. If you look at the back of the bank note, you can see a sextant, which is a tool used to measure angles between two objects (usually an astronomical object and the horizon to estimate latitude). Cliff Stoll believes this is a heliotrope, which is an invention of Gauss’s that we discuss later, but I think he is wrong. You can also see on the bottom right a triangular shaped figure. This is a diagram of the area of Hannover, which Guass participated in surveying. So, it is clear that his study of the Earth was very influential and important.
Gauss was born in 1777 in a place called the Dutchy of Braunschweig. This is located in modern day Germany, but about 100 years before these various fiefdoms combined to form Germany. He was born in a relatively poor family, teetering between lower middle class and peasant class. His intelligence was noticed early on. Anecdotally, he was familiar with numbers before words. He corrected his father’s ledger once at the age of 3. Another famous tale is his summation of the numbers from 1 to 100 in school. Not only was Gauss the fastest at finding this solution, he was the only one who got the correct answer.

There is a common trick taught that by lining up the list of numbers and it’s reverse, you can see that each term adds up to 101, then from there you can do 100*101/2. However, another way, which leads back to the stones in the Greeks. You can see that any 1+2+…+n will be a triangle number. Then you can “complete the square” then divide by 2.

His intelligence attracted the attention of Charles Ferdinand who gave him a stipend to continue his studies in 1791 (Gauss was just 14). He went on to study at Collegium Carolinum for 3 years then to University of Göttingen for 3 more years. At the University of Göttingen he made a significant discovery in Euclidean geometry, which Nick will go into. He then wrote *Disquisitiones arithmeticae*, which is considered his magnum opus. Then in his thesis, he proved the fundamental theorem of algebra, while critiquing other incorrect proofs. All of this by the age of 21! As you can see this early work was very much number theory/algebra focused. So, we are going to stay away from these topics. Interesting to note is that in 1802 he gets an offer at St. Petersburg, but declines due to his allegiance to his patron, Charles
Ferdinand, which I think is important because people have the impression that if he went to St. Petersburg, he may have focused more on number theory. He finally ends up at The University of Göttingen in 1807.
Heptadecagon

- Part of Gauss’s work included the compass and ruler construction of a heptadecagon.
- In order to find out how to construct this shape using only a ruler and compass, Gauss needed to know the measure of one angle of the shape (and thus all measures, since the shape itself is regular).
- This angle is \( \frac{2\pi}{17} \approx 21.18^\circ \)

- Gauss was interested in compass and ruler constructions, a medium of geometric proofs invented by the Greeks. These constructions were important to the Greeks, as all of their proofs were geometric. One of the problems in compass and ruler constructs was how to construct different regular polygons with \( n \) sides. The Greeks were able to do this with smaller constructions, but Gauss was able to construct a 17 sided shape.
- The first step to constructing this shape is to draw a circle, and then find a point on that circle that, using only a ruler and starting at that point, can be traced around the circle to create 17 sides equal in length (We’ll see this later with an animation). Thus, Gauss wanted to find the angle of this starting point (The length was arbitrary). **Draw on the board a circle, describe radians and 2\pi.** We can see then that the angle for the starting point must be 2 pi over 17
- Calculating this in degrees, its 21.18
Heptadecagon

- Gauss-Wantzel theorem
- \( F_n = 2^{2^n} + 1 \)
- \( F_2 = 2^{2^2} + 1 = 2^4 + 1 = 16 + 1 = 17 \)

- To show that 17 was even constructible with ruler and compass, Pierre Wantzel came up with the theorem around twenty years after Gauss’ construction that states a prime number \( p \) is \( p \)-constructible if and only if \( p \) is a Fermat prime. This tied in with Gauss’s construction of the heptadecagon years before, so today the theorem is called the Gauss-Wantzel theorem.
- A Fermat prime is a number in the form \( 2^{2^n} + 1 \). You may think wow, this theorem is really useful, except when you realize that the equation itself grows rapidly, and that the shapes that are proved for constructability are 3, then 5, then 17, then 257.
- Using this Fermat prime equation, we can see that 17 is just \( F_2 \). The Greeks had already proved constructability of a triangle (3 sided) and a pentagon (5 sided), so the next logical step in this theorem was the actual construction of the 17-gon.
Proof of Theorem

- \( e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta) \)
- \( z = e^{\frac{2\pi i}{p}} \)
- \( e^{2\pi i} = 1 \rightarrow \left( e^{\frac{2\pi i}{p}} \right)^p = 1 \rightarrow z^p = 1 \rightarrow z^p - 1 = 0 \)
- \( z^p - 1 = 0 \rightarrow (z - 1)(z^{p-1} + z^{p-2} + \ldots + 1) = 0 \)
- \( z \) has a root in the cyclotomic polynomials, denoted \( \phi_p \)

- First off, we need to know a couple of things. The equation you see above encodes angles in a polar coordinates equation. In other words, it’s a compact way of placing points on the complex plane (Quickly Explain Complex plane). If we increase \( \theta \), it changes the value of the \( \sin \) and \( \cos \), and another point is plotted on the complex plane. However, the beauty in this is that the points are plotted out in an exact circle.
- We want to find the point of where to start our ruler and compass drawing. We start this at angle \( \frac{2\pi}{p} \) where \( p \) is the prime number, so we plug that in for the angle. We set the term equal to \( z \) for convenience.
- We know that on the complex plane, if we plug in \( 2\pi \) for theta, it equals one. Therefore, we can raise \( z \) to the \( p \), and it should be equal to 1. Taking one to the other side we get a
- Since \( p \) is prime, it follows that 1 is one of its square roots. The other root to the left of it is known as a cyclotomic polynomial.
- Since \( z \) has a root in the cyclotomic polynomials, then it is constructible. This is a result of a couple of theorems dealing with fields over the rational numbers, so we’ll omit the proof.
Proof of Theorem

- If \( z = e^{\frac{2\pi i}{p}} \) is constructible, then \( p - 1 = 2^s \)
- Claim: \( s \) is of the form \( 2^k \) for some \( k \geq 1 \)
  - By Contradiction: If not, then it is of the form \( s = km \), where one factor, let’s say \( k \), is odd
    \[ p = 2^s + 1 = (2^m)^k + 1 = (2^m + 1)((2^m)^{k-1} + (2^m)^{k-2} + \ldots + 1) \]
- Plugging in our new \( s \), we get that \( p = 2^{2^k} + 1 \), which is by definition a Fermat Prime ■

- Proving the other way, assume \( z \) is constructible. If so, there is another corollary that states that \( p - 1 = 2^s \) (Again, this is a proof that requires some knowledge of some pretty advanced maths, so we’ll leave it to you to find the proof.)
- Let’s make a claim, and say that \( s \) is of the form \( 2^k \) for some \( k \) greater than or equal to 1.
- We do proof by contradiction. If \( s \) is not in the claimed form than it is in the form \( km \), where one of the factors is odd (let’s say \( k \) for convenience)
- If we plug in our new \( s \), we get a new polynomial, where \( 2^m \) is a new variable and \( k \) is odd. However, we know from before if \( k \) is odd, then it can be factored out like above. However \( p \) is prime, so it doesn’t have these factors. We find a contradiction, and prove \( s \) is of the form \( 2^k \)
- Plug in our new \( s \), and the proof is done.
Gauss wrote a book *Disquisitiones Arithmeticae*, which was about finding roots to polynomials. He was trying to find a cosine of the angle $\frac{2\pi}{17}$, since this would provide him with the “$x$” of the starting point.

This is a graph showing that exact point.
So it was proven to be constructible, it just took a while to construct...
Although Gauss was already a very renowned mathematician, he was still very much interested in the practical side of his profession. He had already used his skills to help predict where Ceres would appear after going from behind the sun by a technique called the method of least squares. Andrea is writing an essay on Gauss and his work with discovering Ceres, so please read that for the very fascinating tale of his work. He felt that doing survey work and observations was not below him and in fact better use of his time then teaching “to which I have always had an antipathy, which is increased, though not caused by the feeling of throwing my time away”. He made this clear in a letter to his friend Bessel.

The land in question was a new area of land that Hannover received as a result of the Vienna of Congress, which was set up to distribute land to provide peace after the end of The Napoleonic Wars. Gauss decided to do this work to fill his leisure time and collect data that he could then use in his work on surfaces. This survey was a multi-year project where field-work was conducted in the summer and number crunching was conducted in the winter.
Gaussian Curvature

- Definition: In differential geometry, the Gaussian curvature ($K$) of a surface at a point is the product of the principal curvatures $k_1$ and $k_2$ at the given point.
  \[ K = k_1 \times k_2 \]
- The two principal curvatures at a given point on a surface are the eigenvalues of the shape operator at the point. They measure how the surface bend by different amount.
Theorema Egregium

• This Theorem states that the Gaussian curvature of a surface does not change if one bends the surface without stretching it.
• This means that the Gaussian curvature can be calculated just by measuring angles, distances and their rates on the surface itself.
Types of Gaussian Curvatures

• A sphere of radius \( r \) has Gaussian curvature \( \frac{1}{r^2} \) everywhere, meaning this has positive curvature
• A cylinder or a flat surface has Gaussian curvature 0
• Negative curvature would be in an object just as a saddle or hyperboloid.
Computing the curvature of a cycloid

\[ x = a(\theta - \sin \theta) \]
\[ y = a(1 - \cos \theta) \]

\((\pi a, 2a)\)
Constant Curvature

- Curvature of a Helix

\[ y = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \text{ at } t = \frac{\pi}{2} \]

- \[ K = \frac{25}{26} \]
Different Curvatures on the Same Surface

- Interactive 3D visuals.
Early on in Gauss’ surveying in 1818 he got the idea to make something called a heliotrope. He was surveying some land and the sun from a window in St. Michael’s tower in Hamburg distracted him, which made him realize he can harness the sun to help with signaling, and creating lines of sight. The idea is to take the tool and line up the sights to a point on the earth. Then use a mirror to reflect the sun’s rays. Someone on the other end of the rays can then use his or her surveying tools to get an accurate measurement, even if it’s foggy, cloudy, or some other obstruction is hindering the measurement. Previously, these difficult surveys were performed at night using lanterns, which was inconvenient and more dangerous.

Gauss sketched and designed the mirror shape and had a local builder make him some for surveying. Heliotrope is Greek for sun turn.
Like I mentioned earlier, Gauss wanted to use this survey to further his knowledge of surfaces and geometry. One such survey he performed was finding a triangle among three mountains in Hannover. This particular survey is mentioned a lot in literature due a rumor that he made this survey to measure the curvature of space-time (many years before Einstein). The rumor goes that he was trying to see if the angles of the triangle did not add to 180 degrees. Gauss of course knew that a triangle on the surface of the Earth wouldn’t add up to 180. His measurement though is of the plane triangle from the mountain tops. If those didn’t add up to 180, so the rumor goes, then Gauss would have shown that space-time is curved. However, his angles add up to 180 degrees, within error. This rumor is not backed by any letters or other credible evidence. This triangle was probably surveyed as a check to ensure smaller triangles were correct.

“I have from time to time in jest expressed the desire that Euclidean geometry would not be correct” – Guass -- https://www.aps.org/publications/apsnews/201306/physicshistory.cfm
Results of Survey

- Practical difficulties made it impossible to achieve the accuracy he had expected
  - even with his improvements in instrumentation and the mathematical skills
  - Results were used in making rough geographic and military maps, but they were unsuitable for precise land surveys nor for measurement of the earth.
- Change of Pace
  - Getting older, heart disease and illness
  - Got a Raise


In spite of these failures and dissatisfactions, the period of preoccupation with geodesy was in fact one of the most scientifically creative of Gauss’s long career. The difficulties of mapping the terrestrial ellipsoid on a plane led him in 1816 to formulate his ideas on curved surfaces and mapping one surface to another so that the two are “similar in their smallest parts.”

https://books.google.com/books?id=Mkph45PQ3AAC&pg=PA35&lpg=PA35&dq=Disquisitiones+generates+circa+superficies+curvas&source=bl&ots=rM-y0CzAZ7&sig=Gp6QcWFmIN-sJDucHazy3E_S2d8&hl=en&sa=X&ved=0ahUKEwjahJjm7IPTAhUs1oMKHfpJCu8Q6AEIzAC#v=onepage&q=Disquisitiones%20generates%20circa%20superficies%20curvas&f=false

Almost from the beginning of his surveying work Gauss had misgivings, which proved to be well founded.

He never ceased trying to overcome these faults, but his virtuosity as a mathematician and surveyor could not balance the factors beyond his control.

In July 1825, Gauss wrote to Olbers that he wondered whether other activities might have been more fruitful. Not only did the results seem questionable but he felt during these years, even
more than usual, that he was prevented from working out many ideas that still crowded his mind
Earth’s Magnetic Field

- Earth is a large magnet due to its iron core
- Every point on near earth is affected by this magnet
- Gauss because interested after finishing surveying
- New physics professor at Göttingen named Wilhelm Weber
- Together they investigated magnetism
Declination
East: Positive
West: Negative
True north is determined using celestial bodies (Polaris for north)

Source: https://www.youtube.com/watch?v=94iK27PFqDE
Inclination
Relative Intensity

- Observe the number of oscillations of a needle within a set time period
- The strength is assumed to be proportional to the square of the number of oscillations
- Humboldt made many of these measurements

Values are always changing
Needle could lose potency
Long distances use different strengthed needles
Not enough time to verify value hasn’t changed
Need absolute values
Interest In Extraterrestrial Life on Mars and the Moon

- Gauss’s Pythagorean Right Triangle Proposal
- "With 100 separate mirrors, each of 16 square feet, used conjointly, one would be able to send good heliotrope-light to the moon .... This would be a discovery even greater than that of America, if we could get in touch with our neighbors on the moon.” – Gauss 1822

Picture Source: http://madscientistjournal.org/2012/08/gausss-invitation/


Titans of science 122

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