MATH 475. SPRING 2016. HOMEWORK 2.

All solutions has to be written in “essay format” using complete English sentences to explain what you are doing and what’s going on. The homework should be written neatly (or typed) and stapled.

1. Show algebraically that any square number is a sum of two “consecutive” triangle numbers. Draw a picture with pebbles to illustrate this fact.
2. Show using pebbles that eight times any triangular number plus one is a square number. Check the same fact algebraically.
3. A lune of Hippocrates is the upper left shaded area in this diagram:

![Diagram of a lune of Hippocrates]

Show that it has the same area as the lower right shaded triangle. The lune is historically the first “curved” shape to have its exact area calculated without using $\pi$. Its existence is one reason people believed squaring the circle should be possible.

4. In your own words, restate and give a complete proof of the Proposition I-34 from the “Elements”. Do not quote anything from the Elements.

5. In your own words, give a complete proof of the Proposition VIII-14 from the “Elements”. Do not quote anything else from the Elements.

6. Let $ABC$ be an arbitrary triangle. Place weights $p, q$ and $r$ at vertices $A, B$ and $C$. Use Archimedes’ axioms for the center of gravity to conclude that lines $AA', BB'$ and $CC'$ intersect at the center of gravity, where $A'$ divides the segment $BC$ in ratio $r:q$, $B'$ divides the segment $CA$ in ratio $p:r$, and $C'$ divides the segment $AB$ in ratio $q:p$.

7. Use the previous problem to prove the following statement: Given a triangle $ABC$, let the lines $AA', BB'$ and $CC'$ be drawn from the vertices to points $A', B'$ and $C'$ on opposite sides. Suppose $\frac{AC'}{CB'} = \frac{BA'}{AC} = \frac{CB'}{BA} = 1$. Then lines $AA', BB'$ and $CC'$ intersect in one point.

8. Prove Lemma 1 from Archimedes’ “Measurement of the Circle”: Suppose $OA$ is the radius of a circle and $CA$ is tangent to the circle at $A$. Let $DO$ bisect $\angle COA$ and intersect the tangent at $D$. Then

\[ \frac{DA}{OA} = \frac{CA}{(CO + OA)}. \]
You may find useful the Law of Sines (probably first proved in the 13th century by Nasir al-Din al-Tusi in his work “On the Sector Figure”):

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},
\]

where \(a\), \(b\) and \(c\) are sides of a triangle with opposite angles \(A\), \(B\) and \(C\).

9. The Archimedean spiral is the curve given in modern polar coordinates by the equation \(r = a\theta\).

Use calculus to prove Proposition 24 from Archimedes’ book “On Spirals”: the area bounded by one complete turn of the spiral given in polar coordinates by \(r = a\theta\) is one-third of the area of the circle with radius \(2\pi a\).

10. Archimedes’ calculation of the area of a parabolic segment is based on the following beautiful and non-trivial observation discussed in class: the area of each white triangle is equal to the one eighth of the area of the blue triangle (see the figure). Prove this.