Who Discovered 17 Crystallographic Groups?

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Plane Crystallographic Groups a.k.a Wallpaper Groups are the possible symmetries of 2 dimensional “crystals”
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Examples in Early History

Bingata patterns are traditional to Okinawa and can be traced as far back as the 15th century.
Examples in Early History

A window of the 9th century mosque Ibn Tulun in Cairo.
Examples in Early History

Roman mosaic found in Terme di Diocleziano from 1\textsuperscript{st} century BCE.
Nature: The Original Geometer
Possible Transformations

There are 4 nontrivial transformations we can do with a pattern on a lattice. They are:
Possible Transformations

Translation:

[Image of airplanes forming a pattern]
Possible Transformations

Rotation:

Point A

Point A
Possible Transformations

Reflection:
Possible Transformations

Glide Reflection:
In the crystallographic notation the first symbol represents the lattice type.
A *p* represents a primitive (rectangular) lattice and a *c* represents a centered (rhombic) lattice.  

\[4\]
The second symbol represents the highest order of rotation. We will see that this will be either 1, 2, 3, 4, or 6. 4
The third symbol tells us whether there is a reflection or glide reflection. This is denoted by an $m$ for reflection, $g$ for glide-reflection, or 1 for neither.\textsuperscript{4}

A fourth symbol tells us whether there is reflection or glide-reflection along another axis of reflection.\textsuperscript{4}
0 indicates identity
* indicates reflection
X indicates a glide-reflection
n indicates rotational symmetry of order n

Note: If a number n comes after the * the center of the corresponding rotation is on mirror lines.
Center of Half Turn

Axis of Reflection

Axis of Glide Reflections
p6m; *632
p6m; *632

120 Degrees Rotation

30 Degrees inclined to each axis of reflection
### Table With All 17

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<th>Conway</th>
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<tr>
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The group of rotations about a fixed point, $O$, by multiples of $360^\circ/n$ is called the cyclic group of size $n$, $C_n$.

- $C_1$ is the trivial group and means no symmetry.
- $C_2$ has rotation by $180^\circ$ and the identity. It is the symmetry group of the letter $S$.
- $C_4$ has rotation by $90^\circ$, $180^\circ$, $270^\circ$, and the identity. It is the symmetry group of a swastika.
$D_n$: The Dihedral Groups

The group of rotations about a fixed point by multiples of $\alpha = \frac{360^\circ}{n}$ along with reflections in $n$ axes forming the angles $\alpha/2$ is called the dihedral group of size $2n$, $D_n$.

- $D_1$ has bilateral symmetry and the identity. It is the symmetry group of the letter W.
- $D_2$ has rotation by $180^\circ$, symmetry along two axes, and the identity. It is the symmetry group of the letter H.
- $D_n$ in general is the symmetry group of a regular $n$ sided polygon.

Note: You may have seen $D_n$ denoted as $D_{2n}$ in the past. We will not be using the latter notation during this talk.
This is a square.

I claim that eight moves will get me every permutation of the vertices.
These moves will be rotations from red line to red line or reflections about a green line.
$D_4$: Symmetries of a Square

We’ll label the vertices A, B, C, and D.

\[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

Lets call this the identity.
$D_4$: Symmetries of a Square

I can rotate it by $90^\circ$ four times before I get back to my identity.
$D_4$: Symmetries of a Square

I can also reflect.
Then I can spin this reflection thrice before getting back to where I was.
It is important to note that any other reflections would result in a previous labelling. To demonstrate this I will put all 8 permutations side by side.
Lattices

Definition

Given two linearly independant vectors, $e_1$ and $e_2$, a \textit{two-dimensional lattice} is all of the linear combinations $xe_1 + ye_2$ where $x$ and $y$ are integer coefficients. Given an origin point, $O$, all of the points into which $O$ goes is called a \textit{lattice basis}.
Essentially a lattice is a grid of points separated by a fixed integer sized gap. It looks like such:
It is vital to note that lattices need not be uniform and can be skewed like this one:
Recall that a lattice is generated by linear combinations of basis vectors, $e_1$ and $e_2$, with integral coefficients. It follows that if vectors $e'_1$ and $e'_2$ form another basis for the same lattice then we must have

$$e'_1 = a_{11} e_1 + a_{21} e_2$$

$$e'_2 = a_{12} e_1 + a_{22} e_2$$

where $a_{ij}$ are integers.
This change of basis must be a two way relation. Therefore the inverse transformation back to $e_1$ and $e_2$ must also have integer coefficients. We then have the following mutually inverse unimodular linear transformations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}$$
Equivalence between lattices (and, as we will see, between patterns) are then reduced to unimodular equivalence. This is to say that my transformation has determinant $\pm1$ and they satisfy:

\[
e_1' = a_{11} e_1 + a_{21} e_2
\]
\[
e_2' = a_{12} e_1 + a_{22} e_2
\]
\[
e_1 = a_{11}' e_1' + a_{21}' e_2'
\]
\[
e_2 = a_{12}' e_1' + a_{22}' e_2'
\]
The symmetry groups \((C_n \text{ and } D_n)\) can be algebraically expressed as a group of orthogonal transformations.

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- Let $L$ be a lattice of points produced by the transformations in $\Delta$ carrying $O$. 

Proof

History

The People

Beyond Mathematics

Conclusion

Bibliography
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- Any operation on $\Delta$ can be regarded as a rotation around $O$ followed by a translation (i.e. a symmetry group along with translation).
- It follows that the rotations must carry the lattice into itself.
- The Symmetry group in $\Delta$ has $n = 1, 2, 3, 4, 6$
We will argue that $n \neq 8$ or larger by contradiction.

- Suppose $n = 8$ and that the points $A_1, \ldots, A_8$ form an octagon by sequentially rotating $A_1$ by $360^\circ / 8$ around $O$. 
Since the vectors from $O$ to $A_1$ and $O$ to $A_2$ must be in our lattice, so too must the vector from $A_1$ to $A_2$.

- But a vector of the length $A_1$ to $A_2$ from $O$ must be shorter than a lattice vector. Contradiction!

- Thus $\Delta$ has potential to be acted on by $C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6$. 

\[ n \geq 8 \]
All that is left is to consider how these symmetry groups act on particular lattices.

Consider $D_1$ which has reflection on an axis through $O$.

This immediately leaves only two types of lattices invariant by the group.
The rectangular lattice$^3$: 

![Rectangular Lattice](image)
The rhombic lattice$^3$: 

![Rhombic lattice image]
In counting the groups $\Delta$ we must notice that regardless of which lattice $C_1$ and $C_2$ act the same.

This, along with translation, leads us to the desired number of 17 unimodularly inequivalent discontinuous groups of linear transformations, $\Delta^2$. 
In the late 18\textsuperscript{th} century crystallography was a field positioned for a breakthrough. In the last 200 years, three of its fundamental laws were discovered and new technology allowed for more precise measurements than ever. Mathematics could now provide tools for looking at symmetries like never before.
In 1830 J.F.C Hessel discovered “symmetry classes”; the potential for a maximum of 32 unique combinations of axial symmetry while also abiding by a requirement that points (atoms) are equidistant.\(^5\)

In 1850 Auguste Bravais discovered the maximum number of types of geometric figures formed by points evenly distributed to be 14. \(^5\)
E.S. Fedorov began his work on symmetry at the age of 16, with his first published work *An Introductory Study of Figures*. He “Derived topological features of the convex polygons and polyhedra that fill the plane and space in parallel orientation”\(^6\)

Worked with analytic geometry “which he had adapted to the study of the regular systems of points.”\(^6\)
Fedorov demonstrated there were 230 space groups. The text was published in Russian so it remained obscure. Fedorov’s research, Schoenflies’ verification, and Barlow’s contributions were eventually published in cooperation.\textsuperscript{6}

While researching this he discovered the 17 symmetry classes in 1891.\textsuperscript{6}
George Polya discussed the groups in his work: On the Analog of Crystallographic Symmetry in the Plane.\textsuperscript{7}

Polya wrote the piece without knowing that Fedorov had proved the 17 “wallpaper groups” (as they are now known) in 1891.\textsuperscript{7}
As Polya knew, the symmetries of patterns extended to many facets of aesthetics.

M.C. Escher famously had great interest in tessellations and found great inspiration in them.

Throughout his life he interacted with several famous mathematicians (including Polya).
Wallpaper groups in Escher’s Work
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In his book on Symmetry, the mathematician Herman Weyl said the following about the 17 wallpaper groups:

“One can hardly overestimate the depth of geometric imagination and inventiveness reflected in the patterns. Their construction is far from being mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us.”
Bibliography

2. Symmetry by Hermann Weyl
3. Wallpaper Group webpage by Dr. D.E. Joyce; http://www.clarku.edu/djoyce/wallpaper/index.html
4. The Classification of Wallpaper Patterns: From Group Cohomology to Escher’s Tessellations by P.J. Morandi
5. Origins of the Science of Crystals by John G. Burke
7. On the analog of crystallographic symmetry in the plane By George Polya
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Questions?