§3. Icosahedron and $E_8$ (Feb 16, 18, 21, 23, 25)

It is time to study invariant theory and orbit spaces more systematically. We will start with a finite group $G$ acting linearly\(^3\) on a vector space $V$ and discuss the orbit space $V/G$ and the quotient morphism $V \to V/G$. There are several reasons to do this:

- The space $V/G$ is usually singular, and singularities of this form (the so-called quotient or orbifold singularities) form perhaps the most common and useful class of singularities.
- Moduli spaces are often constructed as quotients $X/G$. Here $X$ is usually not a vector space and $G$ is usually not a finite group. However, many results can be generalized and hold in this more general situation (albeit with some complications).
- In good situations, a moduli space near a point $p$ often “looks like” $V/G$, where $V$ is a vector space (a so-called versal deformation space of the geometric object that corresponds to $p$) and $G$ is an automorphism group of this object, which is usually finite.

To give us a concrete goal, we will try to understand geometry of the most enigmatic du Val singularity related to $E_8$ and the icosahedron. It is defined as follows. According to Plato, finite subgroups in $SO_3$ correspond to platonic solids. For example, $A_5$ embeds in $SO_3$ as a group of rotations of the icosahedron. In particular, $A_5$ acts on the circumscribed sphere of the icosahedron. This action is obviously conformal (preserves oriented angles), and so if we think about $S^2$ as a Riemann sphere $\mathbb{P}^1$ (for example by using the stereographic projection), we get an embedding $A_5 \subseteq PSL_2$ (since it is proved in complex analysis that conformal maps are holomorphic). The preimage of $A_5$ in $SL_2$ is called the binary icosahedral group $\Gamma$. What is the orbit space $\mathbb{C}^2/\Gamma$? Let’s start with more straightforward examples.

§3.1. Symmetric polynomials, one-dimensional actions. Let $G = S_n$ be a symmetric group acting on $\mathbb{C}^n$ by permuting the coordinates. Recall that our old recipe for computing the orbit space calls for studying the ring of invariants

$$ \mathbb{C}[x_1, \ldots, x_n]^{S_n}. $$

These invariant polynomials are called symmetric polynomials. By the classical Theorem on Symmetric Polynomials, they are generated by elementary symmetric polynomials

$$ \sigma_1 = x_1 + \ldots + x_n, $$

$$ \sigma_k = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}, $$

$$ \sigma_n = x_1 \ldots x_n. $$

Thus the candidate for the quotient map is

$$ \pi : \mathbb{A}^n \to \mathbb{A}^n, \ (x_1, \ldots, x_n) \mapsto (\sigma_1, \ldots, \sigma_n). $$

\(^3\)Recall that a linear action is given by a homomorphism $G \to GL(V)$. In this case we also say that $V$ is a representation of $G$. 
Notice the following interesting feature:

3.1.1. **Proposition.** The map $\pi$ is surjective and its fibers are the $S_n$-orbits.

**Proof.** By the Vieta formulas, we can recover $x_1, \ldots, x_n$ from $\sigma_1, \ldots, \sigma_n$ as roots of the polynomial $T^n - \sigma_1 T^{n-1} + \ldots + (-1)^n \sigma_n = 0$. □

If $G$ acts linearly on $\mathbb{C}$ then essentially we have a character, i.e. a homomorphism $G \to \mathbb{C}^\ast$. Its image is a subgroup $\mu_d$ of $d$-th roots of unity. For the purpose of computing invariants we can always assume that the action is faithful, so let’s just assume that $G = \mu_d$ acts on $\mathbb{C}$ by multiplication. A non-zero orbit has $d$ elements $x, \zeta x, \ldots, \zeta^{d-1} x$, where $\zeta \in \mu_d$ is a primitive root. There is an obvious invariant, namely $x^d$, and it is clear that

$$\mathbb{C}[x]^\mu_d = \mathbb{C}[x^d].$$

It is also clear that $x^d$ separates orbits and that the quotient morphism in this case is just

$$\pi : \mathbb{A}^1 \to \mathbb{A}^1, \ x \mapsto x^d.$$  

**§3.2. $A_1$-singularity.** Let $\mathbb{Z}^2$ act on $\mathbb{A}^2$ by $(x, y) \mapsto (-x, -y)$. Invariant polynomials are just polynomials of even degree, and so

$$\mathbb{C}[x, y]^\mathbb{Z}_2 = \mathbb{C}[x^2, y^2, xy]$$

and the quotient morphism is

$$\pi : \mathbb{A}^2 \to \mathbb{A}^3, \ (x, y) \mapsto (x^2, y^2, xy).$$

It is clear that invariants separate orbits. It is also clear that the quotient map is surjective onto the quadratic cone

$$(uv = w^2) \subset \mathbb{A}^3.$$  

So the quotient $\mathbb{A}^2/\mathbb{Z}_2$ is a quadratic cone. It has a very basic singularity called “$A_1$-singularity” (the easiest du Val singularity).

**§3.3. Chevalley–Shephard–Todd theorem.** Based on the examples above, one can ask: when is the algebra of invariants a polynomial algebra? It is true for the natural actions of $S_n$ on $\mathbb{A}^n$ and $\mu_d$ on $\mathbb{A}^1$, but fails for the action of $\mathbb{Z}_2$ on $\mathbb{A}^2$ by $\pm 1$. What is so special about the first two cases? The answer turns out to be very pretty, but a bit hard to prove:

3.3.1. **Theorem.** Let $G$ be a finite group acting linearly and faithfully on $\mathbb{C}^n$. Then $\mathbb{C}[x_1, \ldots, x_n]^G$ is isomorphic to a polynomial algebra if and only if $G$ is generated by pseudo-reflections, i.e. by elements $g \in G$ such that the subspace

$$\{v \in \mathbb{C}^n \mid gv = v\}$$

has codimension 1.

In other words, $g$ is a pseudo-reflection if and only if its matrix is some basis is equal to $\mathrm{diag}[\zeta, 1, 1, \ldots, 1]$, where $\zeta$ is a root of unity. If $\zeta = -1$ then $g$ is called a reflection. For example, if $S_n$ acts on $\mathbb{C}^n$ then any transposition $(ij)$ acts as a reflection with mirror $x_i = x_j$. Further examples of groups generated by reflections are Weyl groups of root systems. On the other hand, the action of $\mu_d$ on $\mathbb{C}$ is generated by a pseudo-reflection $z \mapsto \zeta z$. This is not a reflection (for $d > 2$) but the algebra of invariants is still polynomial.
The action of \( \mathbb{Z}_2 \) on \( \mathbb{C}^2 \) by \( \pm 1 \) is not a pseudoreflection (a fixed subspace has codimension 2).

§3.4. Finite generation.

3.4.1. Theorem. Let \( G \) be a finite group acting linearly on a vector space \( V \). The algebra of invariants \( \mathcal{O}(V)^G \) is finitely generated.

We will split the proof into two Lemmas. The second one will be later reused to proof finite generation for various important infinite groups.

3.4.2. Definition. A linear map

\[ R : \mathcal{O}(V) \to \mathcal{O}(V)^G \]

is called a Reynolds operator if

- \( R(1) = 1 \);
- \( R(fg) = fR(g) \) for any \( f \in \mathcal{O}(V)^G \) and \( g \in \mathcal{O}(V) \).

In particular, the Reynolds operator is a projector onto \( \mathcal{O}(V)^G \):

\[ R(f) = R(f \cdot 1) = fR(1) = f \quad \text{for any} \quad f \in \mathcal{O}(V)^G. \]

3.4.3. Lemma. The Reynolds operator exists for any linear action of a finite group.

Proof. Since \( G \) acts on \( V \), it also acts on the polynomial algebra \( \mathcal{O}(V) \). It’s fun to check that the action has to be defined as follows: if \( p \in \mathcal{O}(V) \) then

\[ (g \cdot p)(x) = p(g^{-1}x). \]

This is how the action on functions is defined: if you try \( g \) instead of \( g^{-1} \), the group action axiom will be violated (why?). We define the Reynolds operator \( R \) as an averaging operator:

\[ R(p) = \frac{1}{|G|} \sum_{g \in G} g \cdot p. \]

It is clear that both axioms of the Reynolds operator are satisfied. This works over any field as soon as its characteristic does not divide \( |G| \).

3.4.4. Lemma. If \( G \) is any group acting linearly on a vector space \( V \) and possessing a Reynolds operator, then \( \mathcal{O}(V)^G \) is finitely generated.

It is obvious that Theorem 3.4.1 follows from these two lemmas.

Proof of Lemma 3.4.4. This is an ingenious argument that belongs to Hilbert. First of all, the action of \( G \) on polynomials preserves their degrees. So \( \mathcal{O}(V)^G \) is a graded subalgebra of \( \mathcal{O}(V) \). Let \( I \subseteq \mathcal{O}(V) \) be the ideal generated by homogeneous invariant polynomials \( f \in \mathcal{O}(V)^G \) of positive degree. By Hilbert’s basis theorem (proved in the same paper as the argument we are discussing), \( I \) is in fact generated by finitely many homogeneous invariant polynomials \( f_1, \ldots, f_r \) of positive degree. We claim that the same polynomials generate \( \mathcal{O}(V)^G \) as an algebra, i.e. any \( f \in \mathcal{O}(V)^G \) is a polynomial in \( f_1, \ldots, f_r \). Without loss of generality, we can assume that \( f \) is homogeneous and argue by induction on its degree. We have

\[ f = \sum_{i=1}^{r} a_i f_i, \]
where $a_i \in \mathcal{O}(V)$. Now apply the Reynolds operator:

$$f = R(f) = \sum_{i=1}^{r} R(a_i)f_i.$$ 

Each $R(a_i)$ is an invariant polynomial, and if we let $b_i$ be its homogeneous part of degree $\deg f - \deg f_i$, then we still have

$$f = \sum_{i=1}^{r} b_i f_i.$$ 

By inductive assumption, each $b_i$ is a polynomial in $f_1, \ldots, f_r$. This shows the claim. 

\[\square\]

§3.5. Basic properties of quotients.

3.5.1. Definition. The finite generation Theorem allows us to construct the quotient variety and the quotient map. Namely, $V/G$ is defined as an affine variety such that

$$\mathcal{O}(V/G) = \mathcal{O}(V)^G$$

and the quotient morphism $\pi : V \to V/G$ is a morphism with the pullback of regular functions given by the inclusion $\pi^* : \mathcal{O}(V)^G \subset \mathcal{O}(V)$. More concretely, we choose a system of generators $f_1, \ldots, f_r$ of $\mathcal{O}(V)^G$ and write

$$k[x_1, \ldots, x_r]/I \simeq \mathcal{O}(V)^G, \quad x_i \leftrightarrow f_i.$$ 

We define $V/G$ as an affine subvariety in $\mathbb{A}^r$ given by the ideal $I$ and let

$$\pi : V \to V/G \hookrightarrow \mathbb{A}^r, \quad v \mapsto f_1(v), \ldots, f_r(v).$$

A different system of generators gives an isomorphic affine variety.

3.5.2. To show that this definition is reasonable, let’s check two things:

- Fibers of $\pi$ are exactly the orbits, i.e. any two orbits are separated by polynomial invariants, and
- All points of $V/G$ correspond to orbits, i.e. $\pi$ is surjective.

3.5.3 (Separation of Orbits). To show the first property, we are going to use significantly that the group is finite. (In fact, we will see later on that separation of orbits fails for infinite groups such as $\text{SL}_n$.) Take two orbits, $S_1, S_2 \subset V$. Since they are finite, it is easy to see (why?) that there exists a polynomial $f \in \mathcal{O}(V)$ such that $f|_{S_1} = 0$ and $f|_{S_2} = 1$. Then the average

$$F = R(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f$$

is an invariant polynomial but we still have $F|_{S_1} = 0$ and $F|_{S_2} = 1$.

Now surjectivity:

3.5.4. Theorem. Let $G$ be a group acting linearly on a vector space $V$ and possessing a Reynolds operator. Then the quotient map $\pi : V \to V/G$ is surjective.
3.5.5. To prove this in style, let’s introduce another more intrinsic point of view on affine varieties based on Hilbert’s Nullstellensatz. Let $X$ be an affine variety. By the aforementioned theorem, we have equality of sets

$$ X = \text{MaxSpec} \mathcal{O}(X), \ x \mapsto \{ f \in \mathcal{O}(X) \mid f(x) = 0 \}. $$

The Zariski topology can also be completely recovered here:

$$ V(I) \leftrightarrow \{ m \in \text{MaxSpec} \mathcal{O}(X) \mid I \subseteq m \}. $$

Principal open sets? No problem:

$$ D(f) \leftrightarrow \{ m \in \text{MaxSpec} \mathcal{O}(X) \mid f \not\in m \}. $$

Functions? This is a fun part: suppose $f \in \mathcal{O}(X)$ and let $x \in X$. What is $f(x)$ in terms of the maximal ideal $m \subseteq \mathcal{O}(X)$ that corresponds to $x$? We have an isomorphism $\mathcal{O}(X)/m \cong \mathbb{C}$ and we simply have

$$ f(x) = f + m \in \mathcal{O}(X)/m \cong \mathbb{C}. $$

How to see a regular map $\pi : X \to Y$ in terms of maximal ideals? Suppose $\pi(x) = y$. We have a pull-back homomorphism on functions

$$ \pi^* : \mathcal{O}(Y) \to \mathcal{O}(X). $$

Now notice that a function $f \in \mathcal{O}(Y)$ vanishes at $y$ iff its pull-back $\pi^*(f)$ vanishes at $x$. In other words, in the language of maximal ideals we have

$$ \pi : \text{MaxSpec} \mathcal{O}(X) \to \text{MaxSpec} \mathcal{O}(Y), \quad \pi(m) = (\pi^* )^{-1}(m). $$

For instance, how would we check algebraically that a given regular map of affine varieties $\pi : X \to Y$ is surjective?

3.5.6. **Lemma.** A regular map $\pi : X \to Y$ of affine varieties is surjective if and only if $\mathcal{O}(X)\pi^*(n) \neq \mathcal{O}(X)$ for any maximal ideal $n \subseteq \mathcal{O}(Y)$.

**Proof.** For any point $y \in Y$ (i.e. a maximal ideal $n \subseteq \mathcal{O}(Y)$) we have to show existence of a point $x \in X$ (i.e. a maximal ideal $m \subseteq \mathcal{O}(X)$) such that $f(x) = y$ (i.e. $(\pi^* )^{-1}(m) = n$). So we have to show that there exists a maximal ideal $m \subseteq \mathcal{O}(X)$ that contains $\pi^*(n)$. The image of an ideal under homomorphism is not necessarily an ideal, so the actual condition is that the ideal $\mathcal{O}(X)\pi^*(n)$ is a proper ideal.

Of course this machinery will be successful only if a regular map $\pi$ is from the beginning defined not geometrically but algebraically, in terms of the pull-back of functions. But this is exactly the case for the quotient map!

**Proof of Theorem 3.5.4.** Let $n \subseteq \mathcal{O}(V)^G$ be a maximal ideal. We have to show that

$$ \mathcal{O}(V)n \neq \mathcal{O}(V) $$

(recall that a pull-back of functions for the quotient map $\pi : V \to V/G$ is just the inclusion $\mathcal{O}(V)^G \subseteq \mathcal{O}(V)$). Arguing by contradiction, suppose that $\mathcal{O}(V)n = \mathcal{O}(V)$. Then we have

$$ \sum a_if_i = 1, $$

where $a_i \in \mathcal{O}(V)$ and $f_i \in n$. Applying the Reynolds operator, we see that

$$ \sum b_if_i = 1, $$
where $b_i \in \mathcal{O}(V)^G$. But $n$ is a proper ideal of $\mathcal{O}(V)^G$, contradiction. □

This argument only uses the existence of a Reynolds operator, and so has a wider range of applications then the case of finite groups. But for finite groups we can do a little bit better:

3.5.7. Lemma. $\mathcal{O}(V)$ is integral over $\mathcal{O}(V)^G$.

Proof. Indeed, any element $f \in \mathcal{O}(V)$ is a root of the monic polynomial

$$\prod_{g \in G} (T - g \cdot f).$$

Coefficients of this polynomial are in $\mathcal{O}(V)^G$ (by Vieta formulas). □

3.5.8. Definition. Let $\pi : X \to Y$ be a regular map of affine varieties such that the pull-back map is an inclusion $\pi^* : \mathcal{O}(Y) \subset \mathcal{O}(X)$. Geometrically this means that $\pi$ is dominant, i.e. $\pi(X)$ is dense in $Y$ (otherwise we would find a non-trivial function in $\mathcal{O}(Y)$ that vanishes on $\pi(X)$ – this function will be in $\ker \pi^*$). We say that $\pi$ is finite if $\mathcal{O}(X)$ is integral over $\mathcal{O}(Y)$.

3.5.9. Remark. The reason finite maps are called finite is because they have finite fibers: suppose we fix a point $y \in Y \subset \mathbb{A}^n$. Suppose that $\pi(x) = y$ for some $x \in X \subset \mathbb{A}^m$. Then any linear function $l$ on $\mathbb{A}^m$ is a regular function on $X$, and therefore satisfies a monic equation

$$l^s + a_1 l^{s-1} + \ldots + a_s = 0,$$

where $a_i \in \mathcal{O}(Y)$. So we have

$$l(x)^s + a_1(y)l(x)^{s-1} + \ldots + a_s(y) = 0,$$

which implies that $l(x)$ can have at most $s$ different values, and so $\pi^{-1}(y)$ contains at most $s$ points.

3.5.10. Remark. Of course not any map with finite fibers is finite: the typical counterexample is an inclusion

$$\mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1.$$

The pull-back homomorphism is an embedding $\mathbb{C}[x] \subset \mathbb{C}[x, \frac{1}{x}]$. Notice that $1/x$ is not integral over $\mathbb{C}[x]$ (why?) . To resolve this confusion, regular maps with finite fibers are called quasi-finite.

We have the following

3.5.11. Theorem. A finite map is surjective.

Algebraically, by Lemma 3.5.6, this follows from the following fact:

3.5.12. Proposition. Let $A \subset B$ be rings. Suppose $B$ is integral and finitely generated over $A$. If $M \subset A$ is a proper ideal then $BM \subset B$ is a proper ideal.

Proof. Arguing by contradiction, suppose $BM = B$. Since $B$ is finitely generated as an $A$-algebra and integral over it, $B$ is a finite $A$-module (why?) . Let $b_1, \ldots, b_s$ be generators. Then we have a system of equation

$$\sum_j m_{ij} b_j = b_i \quad\text{for some } m_{ij} \in M \quad\text{for any } i,$$
or
\[ \sum_j (\delta_{ij} - m_{ij})b_j \quad \text{for any } i. \]
Multiplying by the adjoint matrix (i.e. by Cramer’s rule), we have
\[ Db_i = 0 \quad \text{for any } i, \quad \text{where} \quad D = \det(\delta_{ij} - m_{ij}). \]
But then \( D \cdot 1 = 0. \) Since \( D = 1 - m \) for some \( m \in M, \) this implies that \( 1 \in M, \) contradiction. \( \Box \)

§3.6. Quotient singularity \( \frac{1}{r}(1, a) \) and continued fractions. How to compute the algebra of invariants? In general it can be quite complicated but things become much easier if the group is Abelian. Let’s focus on the most useful example of a cyclic quotient singularity \( \frac{1}{r}(1, a) \). It is defined as follows: consider the action of \( \mu_r \) on \( \mathbb{C}^2, \) where the primitive generator \( \zeta \in \mu_r \) acts via the matrix
\[
\begin{bmatrix}
\zeta & 0 \\
0 & \zeta^a
\end{bmatrix}
\]
The cyclic quotient singularity is defined as the quotient
\[ \mathbb{C}^2 / \mu^r = \text{MaxSpec} \mathbb{C}[x, y]^{\mu_r}. \]

How to compute this algebra of invariants? Notice that the group acts on monomials diagonally as follows:
\[ \zeta \cdot x^i y^j = \zeta^{-i} x^i y^j. \]
So a monomial \( x^iy^j \) is contained in \( \mathbb{C}[x, y]^{\mu_r} \) if and only if \( i + ja \equiv 0 \pmod{r}. \)

There are two cases when the answer is immediate:

3.6.1. Example. Consider \( \frac{1}{r}(1, r-1). \) Notice that this is the only case when \( \mu_r \subset \text{SL}_2. \) The condition on invariant monomials is that \( i \equiv j \pmod{r} \)
(draw). We have
\[ \mathbb{C}[x, y]^{\mu_r} = \mathbb{C}[x^r, xy, y^r] = \mathbb{C}[U, V, W]/(V^r - UW). \]
So we see that the singularity \( \frac{1}{r}(1, r-1) \) is a hypersurface in \( \mathbb{A}^3 \) given by the equation \( V^r = UW. \) It is called an \( A_{r-1} \)-singularity.

3.6.2. Example. Consider \( \frac{1}{r}(1, 1). \) The condition on invariant monomials is \( i + j \equiv 0 \pmod{r} \)
(draw). We have
\[ \mathbb{C}[x, y]^{\mu_r} = \mathbb{C}[x^r, x^{r-1}y, x^{r-2}y^2, \ldots, y^r]. \]
The quotient morphism in this case is
\[ \mathbb{A}^2 \rightarrow \mathbb{A}^{r+1}, \quad (x, y) \mapsto (x^r, x^{r-1}y, x^{r-2}y^2, \ldots, y^r). \]
The singularity \( \frac{1}{r}(1, 1) \) with a cone over a rational normal curve
\[ [x^r : x^{r-1}y : x^{r-2}y^2 : \ldots : y^r] \subset \mathbb{P}^{r-1}. \]

3.6.3. Example. Here is a more random example:
To describe a general cyclic quotient singularity $\frac{1}{r}(1, a)$ we need an amusing concept called Hirzebruch–Young continued fractions. It looks just like an ordinary continued fraction but with minuses instead of pluses. More precisely, we have

3.6.4. Definition. Let $r > b > 0$ be coprime integers. The following expression is called the Hirzebruch–Jung continued fraction:

$$\frac{r/b}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}}} = [a_1, a_2, ..., a_k].$$

For example,

$$\frac{5}{1} = [5],$$
$$\frac{5}{4} = [2, 2, 2],$$
$$\frac{5}{2} = [3, 2].$$

Here’s the result:

3.6.5. Theorem. Suppose $\mu_r$ acts on $\mathbb{A}^2$ with weights 1 and $a$, where $a$ and $r$ are coprime. Let $\frac{r}{a-r} = [a_1, a_2, ..., a_k]$ be the Hirzebruch–Jung continued fraction expansion. Then $\mathbb{C}[x, y]^{\mu_r}$ is generated by

$$f_0 = x^r, \quad f_1 = x^{r-a}y, \quad f_2, \ldots, \quad f_k, \quad f_{k+1} = y^r,$$

where the monomials $f_i$ are uniquely determined by the following equations:

$$f_{i+1} = f_i^a f_{i-1}^{-1} \quad \text{for} \quad i = 1, \ldots, k. \quad (3.6.6)$$

3.6.7. Remark. We see that the codimension of $\mathbb{A}^2/\Gamma$ in the ambient affine space $\mathbb{A}^{k+2}$ is equal to the length of the Hirzebruch–Young continued fraction. This is a good measure of the complexity of the singularity. From

4Notice that we are expanding $r/(r - a)$ and not $r/a!$
The intersection of the most complicated singularity: the Hirzebruch–Young continued fraction
\[ \frac{r}{r-1} = [2, 2, 2, \ldots, 2] \quad (r-1 \text{ times}) \]
uses the smallest possible denominators. It is analogous to the “standard” continued fraction of the ratio of two consecutive Fibonacci numbers, which uses only 1’s as denominators.

**Proof of Theorem 3.6.5.** This is completely combinatorial: invariant monomials in \( \mathbb{C}[x, y]^\mathbb{Z}_r \) correspond to the intersection of the first quadrant
\[ \{(i, j) | i, j \geq 0\} \subset \mathbb{Z}^2 \]
with the lattice
\[ L = \{(i, j) | i + aj \equiv 0 \pmod{r}\} \subset \mathbb{Z}^2. \]
This intersection is a semigroup and we have to find its generators. The lattice \( L \) contains the sublattice \( r\mathbb{Z}^2 \) (generated by \( \begin{bmatrix} 0 \\ r \end{bmatrix} \) and \( \begin{bmatrix} r \\ 0 \end{bmatrix} \)) and modulo this sublattice \( L \) is generated by \( \begin{bmatrix} r \quad a \\ -a \\ 1 \end{bmatrix} \) (draw the “torus” \( \mathbb{Z}_r \times \mathbb{Z}_r \)).

So \( L \) is generated by \( \begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ r \end{bmatrix} \), and by the monomials inside the square \( \{(i, j) | 0 < i, j < r\} \), which are precisely the monomials
\[ ((r-a)j \ mod \ r, \ j), \quad j = 1, \ldots, r-1. \]

Of course many of these monomials are unnecessary. The first monomial in the square that we actually need is \( \begin{bmatrix} r-a \\ 1 \end{bmatrix} \). Now start taking multiples of \( \begin{bmatrix} r-a \\ 1 \end{bmatrix} \). The next generator will occur when \( (r-a)j \) goes over \( r \), i.e. when
\[ j = \left\lfloor \frac{r}{r-a} \right\rfloor = a_1 \]
(in the Hirzebruch–Young continued fraction expansion for \( \frac{x}{r-a} \)). Since \( (r-a)a_1 \mod r = (r-a)a_1 - r \), the next generator is
\[ \begin{bmatrix} (r-a)a_1 - r \\ a_1 \end{bmatrix}. \]
Notice that so far this confirms our formula (3.10.1). We are interested in the remaining generators of \( L \) inside the \( r \times r \) square. Notice that they all lie above the line spanned by \( \begin{bmatrix} r-a \\ 1 \end{bmatrix} \). So we can restate our problem: find generators of the semigroup obtained by intersecting \( L \) with points lying in the first quadrant and above the line spanned by \( \begin{bmatrix} r-a \\ 1 \end{bmatrix} \).

Next we notice that
\[ \det \begin{bmatrix} r & r-a \\ 0 & 1 \end{bmatrix} = \det \begin{bmatrix} r-a & (r-a)a_1 - r \\ 1 & a_1 \end{bmatrix} = r. \]
It follows that lattice $L$ is also spanned by $\begin{bmatrix} 0 \\ r \end{bmatrix}, \begin{bmatrix} (r - a)a_1 - r \\ a_1 \end{bmatrix}$, and $\begin{bmatrix} r - a \\ 1 \end{bmatrix}$. We are interested in generators of the semigroup obtained by intersecting this lattice with the “angle” spanned by vectors $\begin{bmatrix} 0 \\ r \end{bmatrix}$ and $\begin{bmatrix} r - a \\ 1 \end{bmatrix}$.

Consider the linear transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that
$$
\psi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{r} \end{bmatrix}, \quad \psi \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{r}{r-a} \end{bmatrix}.
$$
Then we compute
$$
\psi \begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r-a \end{bmatrix}, \quad \psi \begin{bmatrix} r - a \\ 1 \end{bmatrix} = \begin{bmatrix} r - a \\ 0 \end{bmatrix}, \quad \psi \begin{bmatrix} (r - a)a_1 - r \\ a_1 \end{bmatrix} = \begin{bmatrix} (r - a)a_1 - r \\ 1 \end{bmatrix}.
$$
So we get the same situation as before with a smaller lattice. Notice that if
$$
\frac{r}{r-a} = a_1 - \frac{1}{q}
$$
then
$$
q = \frac{r-a}{(r-a)a_1 - r},
$$
so we will recover all denominators in the Hirzebruch–Jung continued fraction as we proceed inductively. 

§3.7. **Zariski tangent space.** How do we know that $\frac{1}{n}(1, a)$ is a singularity? Recall that our current definition of the tangent space to an algebraic variety is as follows. Suppose $X = V(I) \subset \mathbb{A}^k$ is an affine variety and choose generators $h_1, \ldots, h_s$ of $I$. Let $p \in X$. Then $T_pX$ is a kernel of the Jacobian matrix $\{ \frac{\partial h_i}{\partial x_j} \}$, i.e. a linear subspace of vectors $(a_1, \ldots, a_k)$ such that
$$
\sum_j \frac{\partial h_i}{\partial x_j} a_j = 0 \quad \text{for any } i.
$$
This definition is very convenient if we know the generators $h_i$ of the ideal, but what if we don’t? For example, in our case we have generators $f_1, \ldots, f_k$ of the algebra $\mathbb{C}[x, y]^{\mu_\mu}$, which allows us to write
$$
\mathbb{C}[x, y]^{\mu_\mu} = \mathbb{C}[x_1, \ldots, x_k]/I.
$$
But to compute generators of $I$, we would have to describe all relations between $f_1, \ldots, f_k$. This is possible but it would be nice to have a description of the tangent space entirely in terms of the algebra of functions. It turns out that this description is available and incredibly simple:

3.7.1. LEMMA. Let $p \in X$ be a point of an affine algebraic variety that corresponds to the maximal ideal $\mathfrak{m} \subset \mathcal{O}(X)$. Then the cotangent vector space $T^*_pX$ is canonically identified with the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$.

3.7.2. EXAMPLE. In the setup of Theorem 3.6.5, take the ideal $\mathfrak{m} \subset \mathbb{C}[x, y]^{\mu_\mu}$ generated by $f_0, \ldots, f_{n+1}$. Since $\mathbb{C}[x, y]^{\mu_\mu}/\mathfrak{m} \cong \mathbb{C}$, this ideal is maximal and hence gives a point $p \in X = \text{MaxSpec} \mathbb{C}[x, y]^{\mu_\mu}$. Notice that the ideal $\mathfrak{m}^2$ is generated by pairwise products of generators $f_i f_j$. From our description of
generators $f_0, \ldots, f_{n+1}$, it is clear that they are minimal, i.e. none of them can be written as a non-trivial product of invariant monomials. Hence
\[ T^*_p X = m/m^2 = \langle f_0 + m^2, \ldots, f_{n+1} + m^2 \rangle = \mathbb{C}^{n+1}. \]
Since $\dim X = 2$, it follows that $X$ is always singular at $p$ (with the exception of the trivial case $r = 1$).

**Proof of Lemma.** We can realize $X = V(I) \subset \mathbb{A}^k$. By shifting coordinates, for simplicity let’s also assume that $p \in X$ is at the origin. Then $T^*_p X$ is a quotient vector space:
\[ T^*_p X = \mathbb{C}^k_{x_1, \ldots, x_k}/\langle dh_1(0), \ldots, dh_k(0) \rangle, \]
where
\[ dh_i(0) = \sum_j \frac{\partial h_i}{\partial x_j}(0) x_j \]
is a differential (or linearization of $h_i$). Notice that if $h = \sum g_i h_i$ is some other function in the ideal $I$ then we have
\[ dh(0) = \sum_i g_i(0) dh_i(0) + h_i(0) dg_i(0) = \sum_i g_i(0) dh_i(0) \in \langle dh_1(0), \ldots, dh_k(0) \rangle \]
So we can also write
\[ T^*_p X = \mathbb{C}^k_{x_1, \ldots, x_k}/\langle dh(0) \rangle_{h \in I}. \]
Now let’s untangle the other side of our formula. If $M = (x_1, \ldots, x_k)$ is the maximal ideal of the origin, then we have
\[ m/m^2 = M/(I + M^2) = (x_1, \ldots, x_k)/(I + \sum x_i x_j) = \mathbb{C}^k_{x_1, \ldots, x_k}/\langle dh(0) \rangle_{h \in I}. \]
So the formula is proved. \qed

As an extra bonus, this also shows that our definition of the tangent space is intrinsic to the variety $X$, i.e. does not depend on the choice of embedding in $\mathbb{A}^k$.

§3.8. $E_6$-singularity. Let’s return to computation of $\mathbb{A}^2/\Gamma$, where $\Gamma \subset \text{SL}_2$ is the binary icosahedral group. Recall that $\Gamma$ is the preimage of $A_5 \subset \text{PSL}_2$ thought of as a group of conformal transformations of $\mathbb{P}^1 \simeq S^2$ preserving the inscribed icosahedron\(^5\).

We have to compute $\mathbb{C}[x, y]^\Gamma$. There is a miraculously simple way to write down some invariants: $A_5$ has three special orbits on $S^2$: 20 vertices of the icosahedron, 12 midpoints of faces (vertices of the dual dodecahedron), and 30 midpoints of the edges. Let $f_{12}$, $f_{20}$, and $f_{30}$ be polynomials in $x, y$ that factor into linear forms that correspond to these special points.

We claim that these polynomials are invariant. Since $\Gamma$ permutes their roots, they are clearly semi-invariant, i.e. any $\gamma \in \Gamma$ can only multiply them by a scalar. Since they all have even degree, the element $-1 \in \Gamma$ does not change these polynomials. But $\Gamma/\{ \pm 1 \} \simeq A_5$ is a simple group, hence has no characters at all, hence the claim.

\(^5\)It is well-known that the image of this embedding $\text{SO}_3(\mathbb{R}) \subset \text{PSL}_2(\mathbb{C})$ is $\text{PSU}_2(\mathbb{C})$. This is a geometric interpretation of a famous 2 : 1 homomorphism $\text{SU}_2(\mathbb{C}) \twoheadrightarrow \text{SO}_3(\mathbb{R})$. We are not going to use this.
3.8.1. **Theorem.** \( \mathbb{C}[x, y]^\Gamma = \mathbb{C}[f_{12}, f_{20}, f_{30}] \cong \mathbb{C}[U, V, W]/(U^5 + V^3 + W^2). \)

**Proof.** Let’s try to prove this using as few explicit calculations as possible. The key is to analyze a chain of algebras

\[
\mathbb{C}[x, y] \supset \mathbb{C}[x, y]^\Gamma \supset \mathbb{C}[f_{12}, f_{20}, f_{30}] \supset \mathbb{C}[f_{12}, f_{20}].
\]

We claim that \( \mathbb{C}[f_{12}, f_{20}] \subset \mathbb{C}[x, y] \) (and hence all other inclusions in the chain) is an integral extension. In other words, we claim that a regular map

\[
\mathbb{A}^2 \rightarrow \mathbb{A}^2, \quad (x, y) \mapsto (f_{12}, f_{20})
\]

is finite. We are only going to use the fact that \( f_{12} \) and \( f_{20} \) have no common zeros in \( \mathbb{P}^1 \). By Nullstellensatz, this implies that

\[
\sqrt{(f_{12}, f_{20})} = (x, y),
\]

i.e. \( x^n, y^n \in (f_{12}, f_{20}) \) for some large \( n \). This implies that \( \mathbb{C}[x, y] \) is finitely generated as an \( \mathbb{C}[f_{12}, f_{20}] \)-module by monomials \( x^iy^j \) with \( i, j < n \). We have to check that any monomial \( x^iy^j \) can be written as a linear combination of monomials with \( i, j < n \) with coefficients in \( \mathbb{C}[f_{12}, f_{20}] \). But using the fact that \( x^n, y^n \in (f_{12}, f_{20}) \), we can repeatedly rewrite \( x^iy^j \) as a linear combination of smaller and smaller monomials with coefficients in \( \mathbb{C}[f_{12}, f_{20}] \) until all the monomials have degrees less than \( n \). It follows that \( \mathbb{C}[x, y] \Gamma \) is integral over \( \mathbb{C}[f_{12}, f_{20}] \).

Now let’s consider the corresponding chain of fraction fields

\[
\mathbb{C}(x, y) \supset \text{Quot} \mathbb{C}[x, y]^\Gamma \supset \mathbb{C}(f_{12}, f_{20}, f_{30}) \supset \mathbb{C}(f_{12}, f_{20}).
\]

(You will show in the Exercises that \( \text{Quot} \mathbb{C}[x, y]^\Gamma = \mathbb{C}(x, y)^\Gamma \) but we are not going to use this.)

Here are some basic definitions, and a fact.

3.8.3. **Definition.** Let \( f : X \rightarrow Y \) be a dominant map of irreducible affine varieties of the same dimension. It induces an embedding of fields

\[
f^* : \mathbb{C}(Y) \subset \mathbb{C}(X).
\]

Since the dimension is equal to the transcendence degree of the field of functions, this embedding is algebraic, hence finite (because \( \mathbb{C}(X) \) is finitely generated). We define the **degree of \( f \)** as follows:

\[
\deg f = [\mathbb{C}(X) : \mathbb{C}(Y)].
\]

Now suppose, in addition, that \( f \) is a finite map. Then the previous definition applies, but we also know that \( f \) has finite fibers. We want to compare \( \deg f \) with the number of points in each fiber. Here we face one subtlety: if \( C \subset \mathbb{A}^2 \) is a nodal cubic then its normalization \( \mathbb{A}^1 \rightarrow C \) has degree 1, but the fiber over a node has two points in it. To avoid this sort of situation, here is another definition:

3.8.4. **Definition.** An irreducible affine variety \( X \) is called **normal** if \( \mathcal{O}(X) \) is integrally closed in \( \mathbb{C}(X) \).

Now the fact:
3.8.5. Theorem. Let \( f : X \to Y \) be a finite map of irreducible affine varieties. Suppose that \( Y \) is normal. Then any fiber \( f^{-1}(y) \) has at most \( \deg f \) points. Let
\[
U = \{ y \in Y \mid f^{-1}(y) \text{ has exactly } \deg f \text{ points} \}.
\]
Then \( U \) is open and non-empty.

Let’s postpone the proof and see how we can use it. First of all, any UFD is integrally closed, hence \( \mathbb{C}[x, y] \) and \( \mathbb{C}[f_{12}, f_{20}] \) are integrally closed.

Secondly, \( \mathbb{C}[x, y]^\Gamma \) is integrally closed. Indeed, if \( f \in \text{Quot } \mathbb{C}[x, y]^\Gamma \) is integral over \( \mathbb{C}[x, y]^\Gamma \), then it is also integral over \( \mathbb{C}[x, y] \), but the latter is integrally closed, hence \( f \) is actually a polynomial, and so \( f \in \mathbb{C}[x, y]^\Gamma \).

It follows that \( [\mathbb{C}(x, y) : \text{Quot } \mathbb{C}[x, y]^\Gamma] = 120 \) (of course if we know that the second field is \( \mathbb{C}(x, y)^\Gamma \), this formula also follows from Galois theory). The fibers of the map (3.8.2) are level curves of \( f_{12} \) and \( f_{20} \), and therefore contain at most 240 points by Bezout theorem. One can show geometrically that general fibers contain exactly 240 points or argue as follows: if this is not the case then \( \text{Quot } \mathbb{C}[x, y]^\Gamma = \mathbb{C}(f_{12}, f_{20}, f_{30}) = \mathbb{C}(f_{12}, f_{20}) \) and in particular \( f_{30} \in \mathbb{C}[f_{12}, f_{20}] \). But \( f_{30} \) is integral over \( \mathbb{C}[f_{12}, f_{20}] \), and the latter is integrally closed, so \( f_{30} \in \mathbb{C}[f_{12}, f_{20}] \). But this can’t be the case because of the degrees! So in fact we have

\[
\text{Quot } \mathbb{C}[x, y]^\Gamma = \mathbb{C}(f_{12}, f_{20}, f_{30}) \quad \text{and} \quad \mathbb{C}(f_{12}, f_{20}, f_{30}) : \mathbb{C}(f_{12}, f_{20}) = 2.
\]

The latter formula implies that the minimal polynomial of \( f_{30} \) over \( \mathbb{C}(f_{12}, f_{20}) \) has degree 2. The second root of this polynomial satisfies the same integral dependence as \( f_{20} \), and therefore all coefficients of the minimal polynomial are integral over \( \mathbb{C}[f_{12}, f_{20}] \), by Vieta formulas. But this ring is integrally closed, and therefore all coefficients of the minimal polynomial are in fact in \( \mathbb{C}[f_{12}, f_{20}] \). So we have an integral dependence equation of the form
\[
f_{30}^2 + af_{30} + b = 0,
\]
where \( a, b \in \mathbb{C}[f_{12}, f_{20}] \). Looking at the degrees, there is only one way to accomplish this (modulo multiplying \( f_{12}, f_{20}, \) and \( f_{30} \) by scalars), namely
\[
f_{12}^2 + f_{20}^2 + f_{30}^2 = 0.
\]

It remains to prove that \( \mathbb{C}[x, y]^\Gamma = \mathbb{C}[f_{12}, f_{20}, f_{30}] \). Since they have the same quotient field, it is enough to show that the latter algebra is integrally closed, and this follows from the following extremely useful theorem that we are not going to prove, see [M3, page 198].

3.8.6. Theorem. Let \( X \subset \mathbb{A}^n \) be an irreducible affine hypersurface such that its singular locus has codimension at least 2. Then \( X \) is normal.

For example, a surface \( S \subset \mathbb{A}^3 \) with isolated singularities is normal. It is important that \( S \) is a surface in \( \mathbb{A}^3 \), it is easy to construct examples of non-normal surfaces with isolated singularities in \( \mathbb{A}^4 \).

Proof of Theorem 3.8.5. Let \( y \in Y \) and choose a function \( a \in \mathcal{O}(X) \) that takes different values on points in \( f^{-1}(y) \). The minimal polynomial \( F(T) \) of \( a \) over \( \mathbb{C}(Y) \) has degree at most \( \deg f \). Since \( Y \) is normal, all coefficients of the minimal polynomial are in fact in \( \mathbb{C}(Y) \). Arguing as in Remark 3.5.9, we see that \( f^{-1}(y) \) has at most \( n \) points. Since we are in characteristic 0, the extension \( \mathbb{C}(X)/\mathbb{C}(Y) \) is separable, and hence has a primitive element.
Let \( a \in \mathcal{O}(X) \) be an element such that its minimal polynomial (=integral dependence polynomial) has degree \( n \):

\[
F(T) = T^n + b_1 T^{n-1} + \ldots + b_n, \quad b_i \in \mathcal{O}(Y).
\]

Let \( D \in \mathcal{O}(Y) \) be the discriminant of \( F(T) \) and let \( U = \{ y \in Y \mid D \neq 0 \} \) be the corresponding principal open set. We claim that \( f \) has exactly \( n \) different fibers over any point of \( U \). Indeed, the inclusion \( \mathcal{O}(Y)[a] \subset \mathcal{O}(X) \) is integral, hence induces a finite map, hence induces a surjective map. But over a point \( y \in Y \), the fiber of

\[
\text{MaxSpec} \mathcal{O}(Y)[a] = \{(y, t) \in Y \times \mathbb{A}^1 \mid t^n + b_1(y)t^{n-1} + \ldots + b_n(y) = 0\}
\]

is just given by the roots of the minimal polynomial, and hence consists of \( n \) points. Thus the fiber \( f^{-1}(y) \) also has \( n \) points. \( \square \)

§3.9. Resolution of singularities: cylindrical resolution. No discussion of \( \frac{1}{2}(1, 1) \) would be complete without describing its resolution of singularities. We are looking for a smooth surface \( S \) and a birational surjective map \( \pi : S \to \mathbb{A}^2 / \mu_p \). To avoid fake resolutions which essentially just remove singular points, we have to assume that \( \pi \) is proper, which in analytic geometry means “has compact fibers”. In algebraic geometry the definition is slightly more technical, and we are going to skip it for now.

As a warm-up, let’s look at \( \frac{1}{2}(1, 1) \), i.e. let’s resolve singularities of the quadratic cone

\[
Z = (V^2 - UW) \subset \mathbb{A}^3.
\]

Notice that the locus in \( \mathbb{P}^2 \) given by the equation above is just a smooth conic. Our quadratic cone is a cone over this conic: over each point \( p \) of the conic we have the corresponding ruling \( L_p \), which is a line in \( \mathbb{A}^3 \). All these lines of course pass through the origin, which is exactly the point that creates singularity. The idea of the “cylinder” resolution is simply to take the disjoint union of lines \( L_p \) (draw the picture). More precisely, the resolution will be a line bundle \( S \to C \) with the map \( S \to Z \subset \mathbb{A}^3 \) defined as follows.

We cover the conic \( C \) by two copies of \( \mathbb{A}^1 \): the piece \( C_0 = (W = 0) \) is parametrised by \([x^2 : x : 1]\) and the piece \( C_1 = (U \neq 0) \) is parametrised by \([1 : y : y^2]\). On the overlap we have \( x = y^{-1} \) (this of course agrees with the usual identification of a conic with \( \mathbb{P}^1 \)). We take trivial line bundles \( C_0 \times \mathbb{C} \) (with coordinate \( w \)) and \( C_1 \times \mathbb{C} \) (with coordinate \( u \)). The map \( C_0 \times \mathbb{C} \to \mathbb{A}^3 \) is defined by \((x, w) \mapsto (x^2 w, x w, w)\). The map \( C_1 \times \mathbb{C} \to \mathbb{A}^3 \) is defined by \((y, u) \mapsto (u, yu, y^2u)\). Finally, we want to define a line bundle \( E \) on \( C \) trivialized as above and a morphism \( E \to \mathbb{A}^3 \) that on the cover restricts to morphisms above. There is only one way to do this: on the overlap \( C_1 \cap C_2 \), we have \( x, y \neq 0 \), and the transition function should be

\[
(x, w) \mapsto (y, u) = (x^{-1}, x^2 w).
\]

Notice that the preimage of the singular point is nothing but the zero section of this line bundle, which is obviously isomorphic to \( \mathbb{P}^1 \). One can show that its self-intersection number is \(-2\).
§3.10. **Resolution of cyclic quotient singularities.** Next we are going to resolve a cyclic quotient singularity $\mathbb{C} / (1, a)$. We will first discuss how the resolution should look like and then state a theorem.

In the quadratic cone example above, the surface $S$ was a line bundle. This turns out to be a bit of a fluke: the fact of importance is that $S$ is covered by two charts $C_0 \times \mathbb{C}$ and $C_1 \times \mathbb{C}$ both isomorphic to $\mathbb{A}^2$. This is how we are going to construct our resolution in general: $S$ will be the union

$$S = U_1 \cup \ldots \cup U_r$$

of $r$ charts each isomorphic to $\mathbb{A}^2$. We will construct regular maps

$$\psi_i : U_i \to \mathbb{C}$$

for each $i$ first. Then we will describe how these charts should be glued to give a regular map $S \to \mathbb{C}$: this will be our resolution. The surface $S$ we are going to construct will be neither affine nor projective. In fact it will be quasi-projective but we won’t prove this as no realization of $S$ as a subset of the projective space is important enough to justify our attention. We will describe a general definition of an algebraic variety obtained by gluing after we finish the discussion of our example: it will provide a useful illustration.

So how to construct maps these maps $\psi_i : U_i \to \mathbb{C}$? Recall that we want each chart $U_i$ to be isomorphic to $\mathbb{A}^2$, so let’s process one chart at a time and denote coordinates in the chart by $\zeta_i, \eta_i$. Algebraically, we have to define the pullback homomorphisms

$$\psi_i^* : \mathbb{C}[x, y]^{\mu_r} \to \mathbb{C}[\zeta_i, \eta_i].$$

A simple way to do this would be to send “monomials to monomials”: for any invariant monomial $x^\alpha y^\beta \in \mathbb{C}[x, y]^{\mu_r}$, we want $\psi_i(x^\alpha y^\beta)$ to be equal to some monomial $c^\alpha_i \eta_i^\beta \in \mathbb{C}[\zeta_i, \eta_i]$. The condition of being a homomorphism then simply means that the map

$$(\alpha, \beta) \mapsto (\alpha', \beta')$$

is a linear map. Recall that the semigroup of invariant monomials can be described as

$$\Lambda = M \cap (\mathbb{Z}_{\geq 0})^2,$$

where $M = \{ (\alpha, \beta), \quad \alpha + a \beta \equiv 0 \mod (r) \} \subset \mathbb{Z}^2$

is a lattice. So we have to define a linear map of semigroups

$$A_i : \Lambda \to (\mathbb{Z}_{\geq 0})^2.$$

We want each $\psi_i$ to be dominant, so we want each pull-back homomorphism $\psi_i^*$ to be injective, so we want our linear map $A_i$ to be injective. In fact, we want each $\psi_i$ to be birational, hence we will impose a condition that $A_i$ is a restriction of an isomorphism of lattices

$$A_i : M \simeq \mathbb{Z}^2,$$

which we will denote by the same letter.

The trick is to construct a dual linear isomorphism $A^*$ instead

$$A_i^* : \mathbb{Z}^2 \simeq N,$$
where \( N \) is the lattice dual to \( M \). We can then restrict \( A_i^* \) and get a map of semigroups
\[
A_i^* : (\mathbb{Z}_{\geq 0})^2 \to \Lambda^*,
\]
where \( \Lambda^* \) is the dual semigroup. All these dual objects are defined as follows. We will identifying \( \mathbb{R}^2 \) and \((\mathbb{R}^2)^*\) by means of the standard inner product. Then
\[
N = \{(a, b) \in \mathbb{R}^2 | a\alpha + b\beta \in \mathbb{Z} \text{ for any } (\alpha, \beta) \in M\}.
\]
Since \( M \) is generated by \((r, 0), (0, r), \) and \((r - a, 1), \) this gives
\[
N = \frac{1}{r} \{(i, j) \in \mathbb{Z}^2 | i + (r - a)j \equiv 0 \mod r\} \subset \mathbb{R}^2
\]
and
\[
\Lambda^* = N \cap (\mathbb{Z}_{\geq 0})^2.
\]
For example, for \( \frac{1}{12} (1, 5) \), we will get the following semigroup \( \Lambda^* \):

\[
\begin{align*}
  \frac{12}{7} &= 1 + \frac{1}{2 - \frac{1}{7 - \frac{1}{4 - \frac{1}{2}}}}, \\
  e_0 &= (1, 0), \quad e_1 = \left(\frac{a}{r}, \frac{1}{r}\right), \quad e_2, \ldots, e_k, \quad e_{k+1} = (0, 1),
\end{align*}
\]

where vectors \( e_i \) are uniquely determined by the following equations:
\[
e_{i+1} = b_ie_i - e_{i-1} \quad \text{for} \quad i = 1, \ldots, k, \tag{3.10.1}
\]

where \( \frac{a}{r} = [b_1, b_2, \ldots, b_k] \) is the Hirzebruch–Jung continued fraction\(^6\).

It is clear from this inductive description that the lattice \( N \) is generated by \( e_i, e_{i+1} \) for any \( i \). So finally we can define our maps: the map \( \psi_i : U_i \to \mathbb{C}^2/\mu_r \) is defined by its pull-back \( \psi^* : \mathbb{C}[x, y]^{\mu_r} \to \mathbb{C}[\zeta_i, \eta_i] \), which is defined by the linear map of semigroups \( A_i : \mathbb{Z}_{\geq 0}^2 \to \Lambda^* \), which sends basis vectors of \( \mathbb{Z}_{\geq 0}^2 \) to \( e_i \) and \( e_{i+1} \).

This looks scary but in fact it’s just linear algebra. For example, let’s work out the gluing. The rational map
\[
\psi_{i+1}^{-1} \circ \psi_i : U_i \to U_{i+1}
\]

---

\(^6\)Notice that here we are expanding \( r/a \).
should be the identity on the overlap. This is a “monomial” map, and chasing the definition above, we see that the linear map of lattices dual to lattices of monomials is given by a matrix
\[
\begin{bmatrix}
 b_i & 1 \\
 -1 & 0
\end{bmatrix}.
\]

It follows that the linear map of lattices of monomials is given by a transposed matrix
\[
\begin{bmatrix}
 b_i & -1 \\
 1 & 0
\end{bmatrix}, \quad \text{i.e.} \quad \zeta_{i+1} = \zeta_i^b \eta_i, \quad \eta_{i+1} = \zeta_i^{-1}.
\]

So the gluing is just defined by these formulas above and goes like this:
\[
U_i \setminus \{\zeta_i = 0\} \simeq U_{i+1} \setminus \{\eta_{i+1} = 0\}.
\]

Now we can visualize the resolution as follows:

We would like to have a framework of “algebraic varieties” where the gluing above makes sense. We will do in the next chapter after working out another important example.
§3.11. Homework due on March 11.

Write your name here:

Problem 1. For the cyclic quotient singularity \( \frac{1}{7}(1,3) \), compute generators of \( \mathbb{C}[x,y]/\mu^7 \), realize \( \mathbb{A}^2/\mu^7 \) as an affine subvariety of \( \mathbb{A}^3 \), and compute the ideal of this subvariety. (2 points)

Problem 2. For the cyclic quotient singularity \( \frac{1}{5}(1,4) \), show how to write down explicitly 5 affine charts \( Y_i \approx \mathbb{A}^2 \), regular morphisms \( Y_i \rightarrow \mathbb{A}^2/\mu_5 \), and gluing maps between affine charts such that \( Y = \cup Y_i \) is a resolution of singularities of \( \mathbb{A}^2/\mu_5 \subset \mathbb{A}^3 \). (2 points)

Problem 3. Let \( G \) be a finite group acting on an affine variety \( X \) by automorphisms. (a) Show that there exists a closed embedding \( X \subset \mathbb{A}^r \) such that \( G \) acts linearly on \( \mathbb{A}^r \) inducing the original action on \( X \). (b) Show that the restriction homomorphism \( \mathcal{O}(V)^G \rightarrow \mathcal{O}(X)^G \) is surjective. (c) Show that \( \mathcal{O}(X)^G \) is finitely generated. (2 points).

Problem 4. Let \( G \) be a group acting linearly on a vector space \( V \). Let \( L \subset V \) be a linear subspace. Let \( Z = \{ g \in G \mid g|_L = \text{Id}|_L \} \), \( N = \{ g \in G \mid g(L) \subset L \} \), and \( W = N/Z \).

(a) Show that there exists a natural homomorphism \( \pi : \mathcal{O}(V)^G \rightarrow \mathcal{O}(L)^W \).

(b) Suppose there \( G \cdot L = V \). Show that \( \pi \) is injective. (2 points).

Problem 5. Let \( G = \text{SO}_n(\mathbb{C}) \) be an orthogonal group preserving a quadratic form \( f = x_1^2 + \ldots + x_n^2 \). Show that \( \mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[f] \). (Hint: apply the previous problem to \( L = \mathbb{C}e_1 \).) (1 point).

Problem 6. Let \( G = \text{GL}_n \) be a general linear group acting on \( \text{Mat}_n \) by conjugation. (a) Let \( L \subset \text{Mat}_n \) be a subspace of diagonal matrices. Show that \( G \cdot L = \text{Mat}_n \). (b) Show that \( \mathcal{O}(\text{Mat}_n)^G \) is generated by coefficients of the characteristic polynomial (2 points).

Problem 7. (a) In the notation of the previous problem, describe all fibers of the quotient morphism \( \pi : \text{Mat}_n \rightarrow \text{Mat}_n/G \). (b) Show that not all orbits are separated by invariants and find all orbits in the fiber \( \pi^{-1}(0) \).

(c) Describe all fibers of \( \pi \) that contain only one orbit (3 points).

Problem 8. (a) Let \( G \) be a finite group acting linearly on a vector space \( V \). Show that \( \mathbb{C}(V)^G \) (the field of invariant rational functions) is equal to the quotient field of \( \mathcal{O}(V)^G \). (b) Show that (a) can fail for an infinite group.

(c) Show that if \( G \) is any group acting linearly on a vector space \( V \) then any invariant rational function \( f \in \mathbb{C}(V)^G \) can be written as a ratio of two semi-invariant functions of the same weight. (3 points).

Problem 9. Consider the standard linear action of the dihedral group \( D_n \) in \( \mathbb{R}^2 \) (by rotating the regular \( n \)-gon) and tensor it with \( \mathbb{C} \). Compute generators of the algebra of invariants \( \mathbb{C}[x,y]^{D_n} \). (2 points)

Problem 10. Let \( R \) be an integral finitely generated graded algebra. Show that \( \text{MaxSpec } R \) is smooth if and only if \( R \) is isomorphic to a polynomial algebra. (1 point)

Problem 11. Show that a finite map between affine varieties takes closed sets to closed sets. (1 point)
Problem 12. Let $f : X \to Y$ be a regular map of affine varieties such that every point $y \in Y$ has an affine neighborhood $U \subset Y$ such that $f^{-1}(U)$ is affine and the restriction $f : f^{-1}(U) \to U$ is affine. Show that $f$ itself is finite. (2 points)

Problem 13. Let $R = \sum_{k \geq 0} R_k$ be a finitely generated graded algebra. Recall that a function

$$h(k) = \dim_{\mathbb{C}} R_k$$

is called the Hilbert function of $R$ and its generating function

$$P(t) = \sum h(k) t^k$$

is called the Poincare series. (a) Show that $P(t) = \frac{1}{(1-t)^n}$ for $R = \mathbb{C}[x_1, \ldots, x_n]$. (b) Let $R = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$. Show that $P(t) = \frac{1}{(1-t)(1-t^2) \cdots (1-t^n)}$. (1 point)

Problem 14. (a) Suppose that $R$ is a graded algebra generated by homogeneous generators $r_1, \ldots, r_s$ of degrees $k_1, \ldots, k_s$. Show that the Poincare series $P(t)$ is a rational function of the form

$$f(t) \frac{(1 - t^{k_1}) \cdots (1 - t^{k_s})}{(1 - t)^s},$$

where $f(t)$ is a polynomial with integer coefficients. (b) Compute Poincare series of the algebra $\mathbb{C}[x, y]^t$, where $\mu_t$ acts as $\frac{1}{r}(1, r - 1)$. (c) Compute Poincare series of $\mathbb{C}[x, y]^\Gamma$, where $\Gamma$ is a binary icosahedral group. (3 points)

Problem 15. Suppose that $R$ is a graded algebra generated by finitely many generators all in degree 1. Show that there exists a Hilbert polynomial $H(t)$ such that $h(k) = H(k)$ for $k \gg 0$. (2 points)

Problem 16. (a) Let $F : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism given by homogeneous polynomials $f_1, \ldots, f_n$ such that $V(f_1, \ldots, f_n) = \{0\}$. Show that $F$ is finite. (b) Give example of a dominant morphism $\mathbb{A}^2 \to \mathbb{A}^2$ which is not finite (2 points).

Problem 17. Let $G$ be a group acting by automorphisms on a normal affine variety $X$. Show that the algebra of invariants $O(X)^G$ is integrally closed. (1 point)

Problem 18. Let $A$ be an integrally closed domain with field of fractions $K$ and let $A \subset B$ be an integral extension of domains. Let $b \in B$ and let $f \in K[x]$ be its minimal polynomial. Show that in fact $f \in A[x]$. (1 point)

Problem 19. Consider the action of $\text{SL}_2$ on homogeneous polynomials in $x$ and $y$ of degree 6 written as follows:

$$\zeta_0 x^6 + 6\zeta_1 x^5 y + 15\zeta_2 x^4 y^2 + 20\zeta_3 x^3 y^3 + 15\zeta_4 x^2 y^4 + 6\zeta_5 x y^5 + \zeta_6 y^6.$$ 

Show that the function

$${\det \left[ \begin{array}{cccc} \zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 \\ \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 \end{array} \right]}$$

belongs to the algebra of invariants $\mathbb{C}[\zeta_0, \zeta_1, \ldots, \zeta_6]^\text{SL}_2$. (2 points)

Problem 20. Consider the action of $A_n$ on $\mathbb{A}^n$ by permutations of coordinates. Show that $\mathbb{C}[x_1, \ldots, x_n]^{A_n}$ is generated by elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ and the discriminant $D = \prod_{1 \leq i < j} (x_i - x_j)$ (2 points).