DEPARTMENT OF MATHEMATICS AND STATISTICS
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MATH 233 **SOME SOLUTIONS TO THE FINAL EXAM**
Fall 2018

1. (a) Find a vector equation of the line passing through the points $(3, 1, 4)$ and $(1, 5, 9)$.

   Solution. The vector between the two points is $\langle 3-1, 1-5, 4-9\rangle = \langle 2, -4, -5\rangle$, so the line is given as $\vec{r}(t) = \langle 3, 1, 4\rangle + t\langle 2, -4, -5\rangle$

   (b) Find a parametric equation of the line in part (a).

   Solution. $x = 3 + 2t$, $y = 1 - 4t$, and $z = 4 - 5t$.

   (c) Find the value of k such that the line with the vector equation $\vec{r}(t) = t\langle 0, 14, k\rangle$ intersects the line in part (a) at exactly one point.

   Solution. Since the line is entirely in the yz plane, we find the point such that $x = 0$. This gives $t = -3/2$, so we have the point $(0, 7, 23/2)$. Setting this equivalent to the equation gives $t = 1/2$, so we have the equation $k/2 = 23/2$, or $k = 23$.

2. Find the equation of the tangent plane to the surface given by $xyz = \ln(x^2+y^2+z^2-1)$ at the point $(1, 1, 0)$. Simplify your answer to linear form (i.e. $ax + by + cz + d = 0$).

   Solution.
   Let $F(x, y, z) = xyz - \ln(x^2 + y^2 + z^2 - 1)$. Then $F_x(x, y, z) = yz - \dfrac{2x}{x^2 + y^2 + z^2 - 1}$, $F_y(x, y, z) = xz - \dfrac{2y}{x^2 + y^2 + z^2 - 1}$, and $F_z(x, y, z) = xy - \dfrac{2z}{x^2 + y^2 + z^2 - 1}$. Thus $F_x(1, 1, 0) = -2$, $F_y(1, 1, 0) = -2$, and $F_z(1, 1, 0) = 1$ and so the equation of the tangent plane is given by

$$-2(x - 1) - 2(y - 1) + 1(z - 0) = 0$$

$$-2x - 2y + z + 4 = 0$$

3. Consider the double integral

$$\iint_D (2 + y)dA,$$

   where D is the region bounded by the line $y = 2 - x$ and the parabola $x = y^2$.

   (a) Sketch the region of integration.

Solution.

(b) Set up iterated integrals for both orders of integration.

Solution.

Points of intersection:

$$y^2 = 2 - y \Leftrightarrow y^2 + y - 2 = 0 \Leftrightarrow y = -2, \ \text{ or } \ y = 1,$$

then for $y = -2$ we get $x = (-2)^2 = 4$, and for $y = 1$ we get $x = 1^2 = 1$. Thus, the points of intersection are $(1, 1)$ and $(4, -2)$.

Considering the region type I:

$$\iint_D (2+y)dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (2+y)dydx + \int_1^4 \int_{-\sqrt{x}}^{2-x} (2+y)dydx$$

Considering the region type II:

$$D = \left\{(x,y)| -2 \le y \le 1, \ y^2 \le x \le 2-y\right\}$$

$$\iint_D (2+y)dA = \int_{-2}^1 \int_{y^2}^{2-y} (2+y)dxdy$$

(c) Evaluate the double integral using the easier order of integration.

Solution.

$$\begin{aligned}
\iint_D (2+y)dA &= \textstyle\int_{-2}^1 \int_{y^2}^{2-y} (2+y)dxdy \\
&= \textstyle\int_{-2}^1 \left[2x + yx\right]_{x=y^2}^{x=2-y} dy \\
&= \textstyle\int_{-2}^1 \left[2(2-y) + y(2-y) - (2y^2 + y^3)\right] dxdy \\
&= \textstyle\int_{-2}^1 \left(-y^3 - 3y^2 + 4\right) dxdy \\
&= \left[-\frac{y^4}{4} - y^3 + 4y\right]_{-2}^1 \\
&= \left(-\frac{1^4}{4} - 1^3 + 4\right) - \left(-\frac{(-2)^4}{4} - (-2)^3 + 4(-2)\right) \\
&= \frac{27}{4}.
\end{aligned}$$

4. Find the volume of the solid under the surface $z = \frac{x^2}{x^2+y^2}$ and above the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution. We need to compute the integral:

$$\int_D \frac{x^2}{x^2+y^2} dA$$

where D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Polar coordiates: $x = r\cos(\phi)$, $y = r\sin(\phi)$; $1 \le r \le 2$ and $0 \le \phi \le 2\pi$.

The differential: $dA = rdrd\phi$.

The integral is:

$$\int_0^{2\pi} \int_1^2 r\cos^2(\phi)drd\phi. \tag{1}$$

The inner integral is:

$$\int_1^2 r\cos^2(\phi)dr = \cos^2(\phi)\left[\frac{r^2}{2}\right]_1^2 = \frac{3\cos^2(\phi)}{2}.$$

Thus (1) becomes:

$$\int_0^{2\pi} \frac{3\cos^2(\phi)}{2}d\phi = \frac{3\pi}{2}.$$

5. Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 2$. Let $A(1,0,1)$ and $B(0,1,2)$ be two points on the curve C.

(a) Write a parametrization for the curve C. Find values of the parameter which correspond to points A and B.

(b) Calculate the work done by the force field $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ in moving an object from the point $A(1,0,1)$ to the point $B(0,1,2)$ along the curve C.

Solution.

(a) A parametrization of C is given by $x = \cos(t)$, $y = \sin(t)$ and $z = 2 - \cos(t)$, with $0 \le t \le 2\pi$ or

$$\mathbf{r}(t) =< \cos(t), \sin(t), 2 - \cos(t) >$$

We have $A(1,0,1) = \mathbf{r}(0)$and $B(0,1,2) = \mathbf{r}(\pi/2)$

(b) We have

$$\mathbf{r}'(t) =< -\sin(t), \cos(t), \sin(t) >$$

therefore

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt \quad = \textstyle\int_0^{\pi/2} < \cos(t), \sin(t), 1 > \cdot < -\sin(t), \cos(t), \sin(t) > dt$$

$$= \textstyle\int_0^{\pi/2} \sin(t)dt = -\cos(t)|_{t=0}^{t=\pi/2} = 1$$

6. Let $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field, where

$$P(x,y) = y^2\sin(xy) \quad \text{and} \quad Q(x,y) = (1 + xy)\sin(xy).$$

(a) Compute P_y and Q_x and decide based on that information whether or not $\vec{F}$ is a conservative vector field.

Solution. $P_y = 2y\sin(xy) + xy^2\sin(xy)$ and $Q_x = y\sin(xy) + (1+xy)y\cos(xy)$. These functions are different, for example when $x = y = 1$ because $3\sin 1 \neq 3\cos 1$. Since $P_y \neq Q_x$, $\vec{F}$ is not a conservative vector field by the conservative vector field test.

(b) Formulate the fundamental theorem for line integrals and explain if one can use it to show that the line integral $\int_C \vec{F}\cdot d\vec{r}$ has the same value for every curve C that starts at the point $(1,0)$ and ends at the point $(\frac{\pi}{2}, 1)$.

Solution. The fundamental theorem for line integrals says that if $\vec{F}$ is a conservative vector field with potential function f then $\int_C \vec{F}\cdot\vec{r} = f(x_1,y_1) - f(x_0,y_0)$ for every curve C which starts at (x_0,y_0) and ends at (x_1,y_1). In our case $\vec{F}$ is not a conservative vector field by part (a) and so the theorem does not apply.

7. a) State the formula in Green's Theorem under the following assumptions: "Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. Suppose that two-variable functions $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D. Then the following integrals are equal:"

Solution.

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

b) Consider the line integral

$$\int_C y^3 dx - x^3 dy,$$

where C is the circle $x^2 + y^2 = 4$ oriented *clockwise*. Using Green's Theorem, rewrite the line integral as a double integral and then evaluate it.

Solution.

Let D be the disk bounded by C. Since C is negatively oriented we need to consider $-C$ in order to apply Green's Theorem. This results in

$$\begin{aligned}\int_C y^3 dx - x^3 dy &= -\int_{-C} y^3 dx - x^3 dy \\ &= -\iint_D \left(\frac{\partial(-x^3)}{\partial x} - \frac{\partial(y^3)}{\partial y}\right) dA \\ &= -\iint_D -3x^2 - 3y^2 dA = 3\iint_D x^2 + y^2 dA.\end{aligned}$$

We then change to polar coordinates. Since $D = \{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$, $x^2 + y^2 = r^2$ and $dA = r dr d\theta$, we have

$$\begin{aligned}3\iint_D x^2 + y^2 dA &= 3\int_0^{2\pi}\int_0^2 r^2 r dr d\theta \\ &= 3\int_0^{2\pi} d\theta \int_0^2 r^3 dr \\ &= 3\cdot 2\pi \cdot \left[\frac{1}{4}r^4\right]_{r=0}^{r=2} = 3\cdot 2\pi\cdot 4 = 24\pi.\end{aligned}$$

Scratch paper