DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 233 **SOME SOLUTIONS TO EXAM 1** **Fall 2018**

"Version A" refers to the regular exam and "Version B" to the make-up

1. Version A. Find the center and radius of the sphere given by the equation

$$x^2 - 4x + y^2 + 6y + z^2 - 2z = 11.$$

Solution. By completing squares we can rewrite the equation as follows

$$(x^2 - 4x + (-2)^2) + (y^2 + 6y + 3^2) + (z^2 - 2z + (-1)^2) = 11 + (-2)^2 + 3^2 + (-1)^2 = 25$$
$$(x-2)^2 + (y+3)^2 + (z-1)^2 = 5^2.$$

Hence, the center is $C = (2, -3, 1)$ and the radius is $r = 5$.

Version B. Find the center and radius of the sphere given by the equation

$$x^2 + y^2 - 6y + z^2 + 2z = -1.$$

Solution. By completing squares we can rewrite the equation as follows

$$x^2 + (y^2 - 6y + (-3)^2) + (z^2 + 2z + 1^2) = -1 + (-3)^2 + 1^2 = 9$$
$$x^2 + (y-3)^2 + (z+1)^2 = 3^2.$$

Hence, the center is $C = (0, 3, -1)$ and the radius is $r = 3$.

2. Version A. (a) Let **a** be the vector with initial point $P(0,2,2)$ and terminal point $Q(1,-1,-7)$ and let **b** be the vector with initial point $R(-3,0,1)$ and terminal point $S(-2,1,3)$. Find the unit vector in the direction of $3\mathbf{b} + \mathbf{a}$.

Solution.

$\mathbf{a} = \vec{PQ} = \langle 1-0, -1-2, -7-2 \rangle = \langle 1, -3, -9 \rangle$

$\mathbf{b} = \vec{RS} = \langle -2-(-3), 1-0, 3-1 \rangle = \langle 1, 1, 2 \rangle$

$3\mathbf{b} + \mathbf{a} = 3\langle 1, 1, 2\rangle + \langle 1, -3, 9\rangle = \langle 3, 3, 6\rangle + \langle 1, -3, -9\rangle = \langle 4, 0, -3\rangle$

$|3\mathbf{b} + \mathbf{a}| = \sqrt{(4)^2 + (0)^2 + (-3)^2} = \sqrt{25} = 5$

So the unit vector in the direction of $3\mathbf{b} + \mathbf{a}$ is $\mathbf{u} = \frac{\mathbf{3b+a}}{|\mathbf{3b+a}|} = \langle \frac{4}{5}, 0, -\frac{3}{5}\rangle$

(b) Let $\mathbf{a} = \langle -1, -3, 9\rangle$ and $\mathbf{b} = \langle 1, -1, 2\rangle$. The vector $3\mathbf{b} - \mathbf{a}$ lies in either the xy-, xz-, or yz-plane. Which one is it? Explain your reasoning.

Solution. $3\mathbf{b} - \mathbf{a} = 3\langle 1, -1, 2\rangle - \langle -1, -3, 9\rangle = \langle 3, -3, 6\rangle - \langle -1, -3, 9\rangle = \langle 4, 0, -3\rangle$

Since the y component is 0, it lies in the xz-plane.

Version B. (a) Let $\mathbf{a} = \langle -1, -3, 9\rangle$ and $\mathbf{b} = \langle 1, -1, 2\rangle$. Find the vector in the direction of $3\mathbf{b} - \mathbf{a}$ whose length is 2.

Solution.

$3\mathbf{b} - \mathbf{a} = 3\langle 1, -1, 2\rangle - \langle -1, -3, 9\rangle = \langle 3, -3, 6\rangle - \langle -1, -3, 9\rangle = \langle 4, 0, -3\rangle$

$|3\mathbf{b} - \mathbf{a}| = \sqrt{(4)^2 + (0)^2 + (-3)^2} = \sqrt{25} = 5$

The unit vector in the direction of $3\mathbf{b} - \mathbf{a}$ is $\mathbf{u} = \frac{\mathbf{3b+a}}{|\mathbf{3b+a}|} = \langle \frac{4}{5}, 0, -\frac{3}{5}\rangle$

Thus the vector in the direction of $3\mathbf{b} - \mathbf{a}$ with length 2 is $2\mathbf{u} = \langle \frac{8}{5}, 0, -\frac{6}{5}\rangle$

(b) Let $\mathbf{a}$ be the vector with initial point $P(0, 5, 0)$ and terminal point $Q(2, -1, 7)$ and let $\mathbf{b}$ be the vector with initial point $R(1, 1, -7)$ and terminal point $S(3, -5, 0)$. Are $\mathbf{a}$ and $\mathbf{b}$ equivalent vectors? Why or why not?

Solution. Yes, $\mathbf{a}$ and $\mathbf{b}$ are equivalent vectors, since:

$\mathbf{a} = \vec{PQ} = \langle 2 - 0, -1 - 5, 7 - 0\rangle = \langle 2, -6, 7\rangle$

$\mathbf{b} = \vec{RS} = \langle 3 - 1, -5 - 1, 0 - (-7)\rangle = \langle 2, -6, 7\rangle$

3. Version A. (a). Consider the points $P = (1, 3, 4), Q = (2, -1, 3)$, and $R = (0, 1, 3)$. Find the vector projection of the vector $\overrightarrow{PQ}$ onto the vector $\overrightarrow{PR}$.

Solution. $\overrightarrow{PQ} = Q - P = \langle 1, -4, -1\rangle$, $\overrightarrow{PR} = R - P = \langle -1, -2, -1\rangle$.

$$\begin{aligned}\mathbf{proj}_{\overrightarrow{PR}}(\overrightarrow{PQ}) &= (\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PR}|^2})\overrightarrow{PR} \\ &= \frac{1(-1)+(-4)(-2)+(-1)(-1)}{(-1)^2+(-2)^2+(-1)^2}\langle -1,-2,-1\rangle \\ &= \langle \frac{-4}{3}, \frac{-8}{3}, \frac{-4}{3}\rangle\end{aligned}$$

(b). A particle moves in a straight line from the point $A = (1,2,3)$ to the point $B = (-1,0,7)$ with the help of a force $\overrightarrow{F}$ of magnitude 100 newtons that points in the direction $\langle -1,-1,1\rangle$. Find the work done by the force $\overrightarrow{F}$ on the particle as it moves from A to B. (Distance is measured in meters).

Solution. $\overrightarrow{F} = 100\frac{\langle -1,-1,1\rangle}{\sqrt{3}}$, $\overrightarrow{D} = B - A = \langle -2,-2,4\rangle$.

Work $= \overrightarrow{F}\cdot\overrightarrow{D} = \frac{100}{\sqrt{3}}((-1)(-2)+(-1)(-2)+1(4)) = \frac{800\sqrt{3}}{3}$ joules (or newton-meters).

Version B. (a). Consider the points $P = (-1,3,5), Q = (3,-1,3)$, and $R = (1,3,1)$. Find the scalar projection of the vector $\overrightarrow{PQ}$ onto the vector $\overrightarrow{PR}$.

Solution. $\overrightarrow{PQ} = Q - P = \langle 4,-4,-2\rangle$, $\overrightarrow{PR} = R - P = \langle 2,0,-4\rangle$.

$$\begin{aligned}\mathbf{comp}_{\overrightarrow{PR}}(\overrightarrow{PQ}) &= (\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PR}|}) \\ &= \frac{4(2)+(-4)(0)+(-2)(-4)}{\sqrt{(2)^2+(0)^2+(-4)^2}} \\ &= \frac{8\sqrt{5}}{5}\end{aligned}$$

(b). A force $\overrightarrow{F} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, measured in newtons, moves a particle in a straight line from the origin for 50 meters in the direction $(1,1,1)$. Find the work done by the force on the particle.

Solution. $\overrightarrow{F} = \langle 3,2,-1\rangle$, $\overrightarrow{D} = 50\frac{\langle 1,1,1\rangle}{\sqrt{3}}$.

Work $= \overrightarrow{F}\cdot\overrightarrow{D} = \frac{50}{\sqrt{3}}((3)(1)+(2)(1)+(-1)(1)) = \frac{200\sqrt{3}}{3}$ joules (or newton-meters).

4. Version A. Consider the lines

$$l_1 : \quad \frac{x-1}{2} = \frac{y-3}{3} = \frac{z-1}{2}$$

$$l_2 : \quad \frac{x}{4} = \frac{y-3}{6} = \frac{z-2}{4}$$

(a) Show that the lines l_1 and l_2 are parallel.

Solution.

The direction vector of l_1 is $\vec{v}_1 = \langle 2,3,2\rangle$. The direction vector of l_2 is $\vec{v}_2 = \langle 4,6,4\rangle = 2\mathbf{v}_1$. Therefore these two lines are parallel.

(b) Find the equation of the plane which passes through l_1 and l_2.

Solution.

We pick a point p_1 on l_1, for example $p_1 = (1,3,1)$. Similarly we pick a point p_2 on l_2, for example $p_2 = (0,3,2)$. Consider the vector $p_1\vec{p}_2 = \langle -1,0,1\rangle$. We compute the cross product $\vec{n} = \vec{v}_1 \times p_1\vec{p}_2 = \langle 3,-4,3\rangle$. The plane P with normal vector $\vec{n}$ which passes through p_1 is the desired plane. The vector equation of the plane P is

$$\langle 3,-4,3\rangle \cdot \langle x-1, y-3, z-1\rangle = 0$$
$$3(x-1) - 4(y-3) + 3(z-1) = 3x - 4y + 3z + 6 = 0.$$

Version B. Consider the lines

$$\begin{aligned} l_1 : &\quad x = -t+1,\ y = t+2,\ z = t+1 \\ l_2 : &\quad x = 4t,\ y = 6t+3,\ z = 4t+2 \end{aligned}$$

(a) Show that the point $p = (0,3,2)$ is both on l_1 and l_2.

Solution.

For the parameter $t = 1$ on l_1 we get the point p. For the parameter $t = 0$ on l_2 we get the point p.

(b) Show that l_1 and l_2 are not the same lines.

Solution. The point $p_1 = (1,2,1)$ is on l_1 but not l_2. Indeed if $(1,2,1)$ is on l_1, for some t we should have $1 = 4t,\ 2 = 6t+3,\ 1 = 4t+2$ which is not possible.

(c) Find the vector equation of the plane which contains both lines l_1 and l_2.

Solution. The direction vector of l_1 is $\vec{v}_1 = \langle -1,1,1\rangle$. The direction vector of l_2 is $\vec{v}_2 = \langle 4,6,4\rangle$. We compute the cross product $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -2,8,-10\rangle$. The plane P with normal vector $\vec{n}$ which passes through p_1 is the desired plane. The vector equation of the plane P is

$$\langle -2,8,-10\rangle \cdot \langle x, y-3, z-2\rangle = 0$$
$$-2(x) + 8(y-3) - 10(z-2) = -2x + 8y - 10z + 4 = 0.$$

5. Version A. Consider the points below

$$P(0,1,2),\ Q(2,4,5),\ R(-1,0,1) \text{ and } S(6,-1,4).$$

(a) Find a unit vector orthogonal to the plane through the points P, Q, and R.

Solution. The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore to the plane through the points P, Q, and R. Since $\overrightarrow{PQ} = 2\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{PR} = -\mathbf{i}-\mathbf{j}-\mathbf{k}$ we obtain

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} = &\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ -1 & -1 & -1 \end{vmatrix} = \\ &= \begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix}\mathbf{k} = \\ &= -\mathbf{j} + \mathbf{k}, \end{aligned}$$

and $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |-\mathbf{j} + \mathbf{k}| = \sqrt{\mathbf{0^2 + (-1)^2 + 1^2}} = \sqrt{\mathbf{2}}$.

Thus, the unit vector orthogonal to both is

$$\boxed{\mathbf{u} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(-\mathbf{j} + \mathbf{k}) = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\mathbf{j} + \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\mathbf{k}}.$$

(b) Find the area of the triangle PQR.

Solution. The area of the parallelogram with adjacent sides is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{2}$. Therefore, the are of the triangle PQR is half the area of this parallelogram, that is, $\frac{\sqrt{2}}{2}$.

(c) Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS. Do the points P, Q, R and S lie in the same plan? Justify your answer.

Solution. We have that $\overrightarrow{PS} = 6\mathbf{i} - \mathbf{2j} + \mathbf{2k}$ and

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 2 & 3 & 3 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{vmatrix} = 4.$$

Thus, the volume of the parallelepiped

$$V = |\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})| = |4| = 4.$$

Since the volume of the parallelepiped is different from 0 then the points P, Q, R and S do not lie in the same plan.

Version B. Consider the points below

$$A = (1, 1, 1),\ B = (2, 0, 3),\ C = (4, 1, 7) \text{ and } D = (3, -1, -2).$$

(a) Find two vectors perpendicular to the plane through the points A, B, and C.

Solution. Since $\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + \mathbf{2k}$ and $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{6k}$, we obtain

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & 0 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix}\mathbf{k} = -\mathbf{6i} + \mathbf{3k}.$$

Thus, the two vectors orthogonal to both are $-6\mathbf{i} + \mathbf{3k}$ and $6\mathbf{i} - \mathbf{3k}$.

(b) Find the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution. The area of the parallelogram with adjacent sides is

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |-6\mathbf{i} + \mathbf{3k}| = \sqrt{(\mathbf{-6})^\mathbf{2} + \mathbf{0^2} + \mathbf{3^2}} = \sqrt{\mathbf{45}}.$$

(c) Find the volume V of the parallelepiped such that the A, B, C and D are vertices and the vertices B, C, D are all adjacent to the vertex A. Are the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ coplanar? Justify your answer.

Solution. We have that $\overrightarrow{AD} = 2\mathbf{i} - \mathbf{2j} - \mathbf{3k}$ and

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} = -21.$$

Thus, the volume of the parallelepiped

$$V = |\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})| = |-21| = 21.$$

Since the volume of the parallelepiped is different from 0 the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are non coplanar.

6. Version A. Sketch the surface in $\mathbb{R}^3$ with equation $z = 4 - x^2$. Label the coordinate axes, and include and label the trace at $y = 3$.

   Solution. The equation $z = 4 - x^2$ factors as $z = (2 - x)(2 + x)$, and thus gives a parabolic cylinder, the $z = 0$ trace of which is a downward opening parabola with roots $x = \pm 2$. The $y = 3$ trace appears as an identical downward opening parabola in the plane $z = 3$.

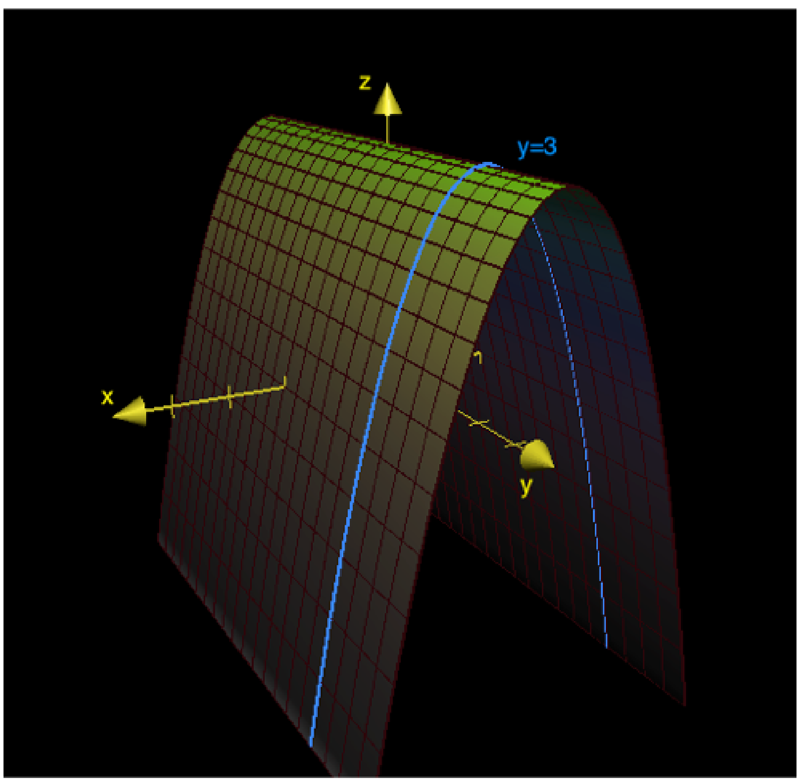

Figure 1: The trace at $y = 3$ is the light blue parabolic curve.

Version B. Sketch the surface with equation $y^2 + z^2 = 9$. Label the coordinate axes and include and label the trace at $x = 4$.

Solution. The equation describes a right circular cylinder, with central axis the x-axis, and with radius 3. The trace at $x = 4$ is a circle of radius 3.

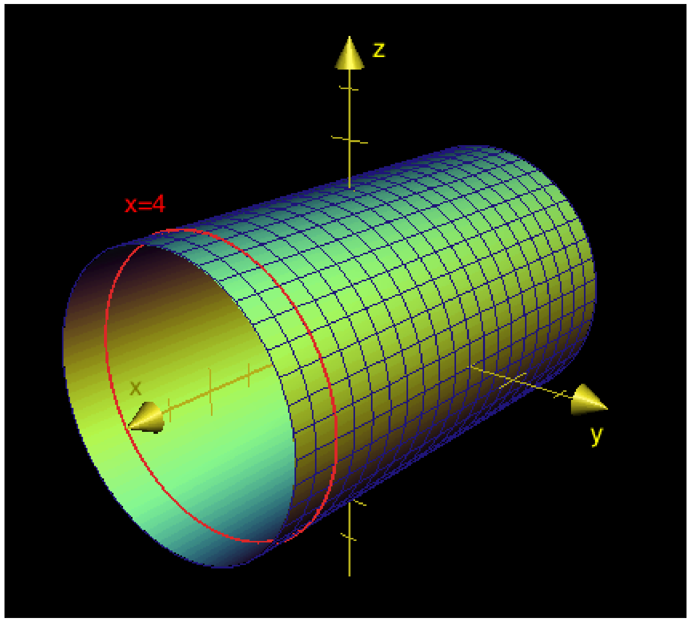

Figure 2: The trace at $x = 4$ is the red circle.

7. Version A. The path $\boldsymbol{r}(t)$ of a particle satisfies $\dfrac{d\boldsymbol{r}}{dt} = \langle 8, 5-3t, 4t^2\rangle$. If $\boldsymbol{r}(0) = \langle 1, 6, 0\rangle$, where is the particle located when $t = 4$?

Solution. The general solution is

$$\begin{aligned}\boldsymbol{r}(t) &= \int \langle 8, 5-3t, 4t^2\rangle dt\\ &= \langle 8t, 5t - \frac{3}{2}t^2,\ \frac{4}{3}t^3\rangle + C\end{aligned}$$

$$\begin{aligned}\boldsymbol{r}(0) &= \langle 8(0), 5(0) - \frac{3}{2}(0)^2,\ \frac{4}{3}(0)^3\rangle + C\\ &= \langle 0, 0, 0\rangle + C\end{aligned}$$

Since $\boldsymbol{r}(0) = \langle 1, 6, 0\rangle$, we have $C = \langle 1, 6, 0\rangle$. Then

$$\begin{aligned}\boldsymbol{r}(t) &= \langle 8t, 5t - \frac{3}{2}t^2, \frac{4}{3}t^3\rangle + \langle 1, 6, 0\rangle\\ &= \langle 8t+1, 5t - \frac{3}{2}t^2 + 6,\ \frac{4}{3}t^3\rangle\end{aligned}$$

Therefore, when $t = 4$, the position of the particle is

$$\begin{aligned}\boldsymbol{r}(4) &= \langle 8(4)+1, 5(4) - \frac{3}{2}(4)^2 + 6,\ \frac{4}{3}(4)^3\rangle\\ &= \langle 33, 2, \frac{256}{3}\rangle\end{aligned}$$

Version B. The path $\boldsymbol{r}(t)$ of a particle satisfies $\dfrac{d\boldsymbol{r}}{dt} = \langle 8t, 5 - 8\sin(2t)\rangle$. If $\boldsymbol{r}(0) = \langle 0, 10\rangle$, where is the particle located when $t = 3$?

Solution. The general solution is

$$\begin{aligned} \boldsymbol{r}(t) &= \int \langle 8t, 5 - 8\sin(2t)\rangle dt \\ &= \langle 4t^2, 5t + 4\cos(2t)\rangle + C \end{aligned}$$

$$\begin{aligned} \boldsymbol{r}(0) &= \langle 4(0)^2, 5(0) + 4\cos(0)\rangle + C \\ &= \langle 0, 4\rangle + C \end{aligned}$$

Since $\boldsymbol{r}(0) = \langle 0, 10\rangle$, we have $C = \langle 0, 6\rangle$. Then

$$\begin{aligned} \boldsymbol{r}(t) &= \langle 4t^2, 5t + 4\cos(2t)\rangle + \langle 0, 6\rangle \\ &= \langle 4t^2, 5t + 4\cos(2t) + 6\rangle \end{aligned}$$

Therefore, when $t = 3$, the position of the particle is

$$\begin{aligned} \boldsymbol{r}(3) &= \langle 4(3)^2, 5(3) + 4\cos(6) + 6\rangle \\ &= \langle 36, 21 + 4\cos(6)\rangle \end{aligned}$$

8. Version A. (a) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the surface $z = 2xy$.

Solution. Let $x = 3\cos(t)$ and $y = 3\sin(t)$, with $0 \le t \le 2\pi$. Then

$$z = 18\cos(t)\sin(t) = 9\sin(2t).$$

Thus the vector equation is

$$\mathbf{r} = \langle 3\cos(t), 3\sin(t), 9\sin(2t)\rangle.$$

(b) Two particles are traveling along the space curves: $\mathbf{r_1} = \langle t, t^2, t^3\rangle$ and $\mathbf{r_2} = \langle 1 + t/2, 2 + t, 4 + 2t\rangle$. Do the particles collide? If they do then find the position at which they collide.

Solution. The particles collide if we can find t such that $\mathbf{r_1}(t) = \mathbf{r_2}(t)$. That means: $t = 1 + t/2$, $t^2 = 2 + t$ and $t^3 = 4 + 2t$. There is only one solution $t = 2$, thus two particles will collide at t = 2. The position of collision is $(2, 4, 8)$.

(c) Find the limit

$$\lim_{t\to\infty} \left(\frac{t^2}{\sqrt{2t^4 + 1}}, \frac{t + 1}{e^t}, \arctan t\right)$$

Solution. By taking limit for each component, as $t \to \infty$, the vector tends to $(\frac{1}{\sqrt{2}}, 0, \frac{\pi}{2})$.

Version B. (a) Find a vector function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

Solution. Since $y = x^2$, we have that $z = 4x^2 + x^4$. Thus let $x = t$ with $-\infty < t < \infty$ we have that $y = t^2$ and $z = 4t^2 + t^4$. The vector equation is:

$$\mathbf{r} = \langle t, t^2, 4t^2 + t^4 \rangle.$$

(b) Two particles travelling along the space curves:

$\mathbf{r_1} = (t^2 - 2, 7t, t^3 - 1)$ and $\mathbf{r_2} = (1 + 2t, 6 + 5t, 4 + 3t)$. Do the particles collide? If they do then find the position at which they collide.

Solution. The particles collide if we can find t such that $\mathbf{r_1}(t) = \mathbf{r_2}(t)$. That means:

$t^2 - 2 = 1 + 2t$, $7t = 6 + 5t$ and $t^3 - 1 = 4 + 3t$. There is only one solution to the first two equations: $t = 2$. However, $t = 2$ does not satisfy the third equation, thus two particles will not collide.

(c) Find the limit

$$\lim_{t \to \infty} (te^{-t}, \frac{t^3 + t}{2t^3 - 1}, t \sin \frac{1}{t})$$

Solution. By taking limit of each component, as $t \to \infty$, the vector tends to $(0, \frac{1}{2}, 1)$.

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