MODELING THE OPTIMAL STRATEGY IN AN INCOMPLETE MARKET

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Abstract. We examine the optimal portfolio selection problem for a single agent who receives a unhedgeable endowment. The agent wishes to optimize his/her log-utility derived from his/her terminal wealth. We do not solve this problem analytically but rigorously prove that there exists a unique optimal portfolio strategy. We present a recursive computational algorithm which produces a sequence of portfolios converging to the optimal one. We present an "intelligent" initial portfolio which requires, numerically, about 25\% fewer corrective steps in the algorithm than a random initial portfolio, and outperforms the portfolio which ignores the unhedgeable risk of the endowment.

Key words: Utility maximization, incomplete markets, endowment uncertainty, numerical methods

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1. Introduction

A market is incomplete if the uncertainties in the economy cannot be hedged by trading the market’s financial instruments. Therefore, in an incomplete market there might exist uncontrollable risks in the cash flow of an agent. An example of this uncertainty is when the agent cannot hedge the uncertainties of his/her endowment process. This paper models the optimal portfolio process for an agent who receives such an unhedgeable endowment and wishes to optimize his/her log-utility from terminal wealth. The paper introduces a numerical technique to solve this problem.

An example of our incomplete market is an energy company which produces electricity by using a hydropower system and hedges the electricity price risk with electricity derivatives (e.g., Keppo, 2002). If appropriate weather derivatives do not exist, the electricity company cannot hedge the risk associated to the endowment process, namely, the rainfall uncertainty in the water inflow. Another example is a usual investor in financial market whose labor income is not perfectly correlating with financial assets (e.g., Duffie et al., 1997).


In this paper, we assume that financial markets are complete in the sense that there exists a unique linear pricing function for all tradable assets. However, the endowment process of the single agent contains risk that is uncorrelated with the tradable assets and, therefore, the market as a whole is incomplete. We extend the initial market to a complete market, where the endowment process can be hedged by creating a fictitious risky asset. We show that in one of the new complete market extensions, the agent does not want to hedge the endowment risks; thus, his/her strategy can be realized in the initial incomplete market. Because this extended market is complete we can employ the methods of complete markets.

This idea of “completing” the market begins with Karatzas et al. (1991). Using the same martingale methods, Cvitanic et al. (1996) generalizes the result to other types of constraints of the portfolio process. Cuoco (1997) studies a similar problem but with an incomplete endowment. Instead of equating the problem to a “dual” minimization problem, as done in Karatzas et al. (1991) and Cvitanic et al. (1996), Cuoco (1997) proves that a solution exists for the original maximization problem. Cvitanic et al. (1999) extends the results of Kramkov et al. (1999) to solve a more general problem where the incomplete markets with an unhedgeable endowment are only assumed to follow a semimartingale model. All these papers are similar in that they prove the existence of a solution to the problem, normally as the limit of some sequence.

Although our framework is not as general as the later papers, we actually derive an explicit equation for the market price of risk of the fictitious asset under which the investor does not invest in that asset. Unfortunately, the equation does not seem solvable. So instead, we alter the problem slightly so that the corresponding
equation is solvable, leading to a so-called “myopic” optimal portfolio strategy. The myopic strategy is suboptimal, but we use it as the first step in a recursive algorithm which approximates the unique optimal strategy. The myopic strategy is a good initial strategy in this algorithm because it requires fewer steps to converge to the optimal strategy than random initial strategies.

We show numerically that the optimal and myopic portfolio process are better than the portfolio for an agent who ignores the endowment risk he/she cannot hedge. For instance, if the expected return of a risky asset is higher than the risk-free rate, the agent invests less in the risky asset with our strategy than under the ignorant one. This difference can be seen as a method to decrease the portfolio’s variance that has been increased from the corresponding complete market case due to the endowment uncertainty.

Related results appear in Duffie et al. (1997) which uses the HJB method instead of the martingale method to provide a feedback formula for the solution in the case of HARA utility, and in El Karoui et al. (1998) which solves the problem when the endowment risk can be hedged with the tradable assets. Henderson (2004) solves an incomplete stochastic income model with negative exponent utility explicitly. Goll et al. (2001) presents an explicit answer to the log-utility problem where it assumes only the semi-martingale model for the price processes. That result does not supersede ours, however, as it does not consider the unhedgeable endowment. Another recent independent result addressing consumption optimization derives a similar explicit equation for the market price of risk, Schroder et al. (2002).

The paper is organized as follows. Section 2 presents our model and discusses some technical conditions required to prove the existence and uniqueness of the optimal strategy. Section 3 constructs the optimal portfolio process using the market price of risk of the completed market. It also derives an equation the market price of risk must solve to ensure that the fictitious asset is unnecessary. Section 4 introduces our myopic optimization problem and outlines the algorithm for approximating the optimal strategy. Section 5 compares the myopic, the ignorant and the optimal strategies via numerical examples. Finally Section 6 concludes.

2. A fictitious completion of the market

2.1. The optimization problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space where $P$ is the real world probability. Let $t \in [0, T]$ denote time and let $W = (W_1, W_2)$ be a 2-dimensional Brownian motion with $W_1$ and $W_2$ independent. Let $\{\mathcal{F}_t\}$ denote the information generated by $W$. All equations involving random variables are assumed to hold $P$-almost surely.

Our financial market has one bond, priced at $P_0(t)$ (at time $t$) and one tradable risky assets, priced at $P_1(t)$. They are assumed to satisfy

\begin{align*}
  dP_0(t) &= P_0(t)rdt, \quad P_0(0) = 1, \\
  dP_1(t) &= P_1(t)(\mu_1 dt + \sigma_1 dW_1(t)), \quad P_1(0) = 1.
\end{align*}

We assume the interest rate $r$, the drift rate $\mu_1$, and the volatility $\sigma_1$ are constant and non-negative. We denote the market price of risk by $\theta_1 := \sigma^{-1}(\mu_1 - r)$. We remark that all the results in this paper hold when there are several risky assets, but we restrict to the case of one to simplify notation.
An agent receives a non-negative (instantaneous) stochastic endowment, \( y \), which we assume satisfies
\[
dy(t) = y(t) (\alpha_y dt + \sigma_y dW_2(t))
\]
where \( \alpha_y \) and \( \sigma_y \) are constant. Since \( W_2 \) is independent of \( W_1 \), \( y \) cannot be replicated by the two assets described in (1). For this reason, the market is said to be incomplete.

We say that the market can be \textit{fictitiously completed} because \( y \) is replicable if we introduce any asset whose price process obeys
\[
dP_2(t) = P_2(t) (\mu_2(t) dt + \sigma_2(t) dW_2(t)), \quad P_2(0) = 1.
\]
We assume the agent invests \( \pi_i(t) \) proportion of his/her wealth \( X(t) \) in asset \( i \) and the rest in the bond. Note that in the true market, we must have \( \pi_2(t) \equiv 0 \).

Let \( \pi^* = (\pi_1, \pi_2) \) denote the portfolio process in the extended market. (In this paper, we denote the adjoint of \( A \) by \( A^* \).

The wealth process obeys
\[
dX(t) = \sum_{i=1}^{2} \pi_i(t) X(t) (\mu_i(t) dt + \sigma_i(t) dW_i(t))
\]
where \( x \) is the initial wealth. Denote such a wealth process by \( X^\pi,x \).

We say that a portfolio process is \textit{feasible}, if
\[
\int_0^T \| \pi(s) \|^2 ds < \infty
\]
and denote by \( A(x) \) the set of feasible portfolios with initial wealth \( x \).

Let
\[
J(\pi, x) := E[\log(X^\pi,x(T))].
\]
The optimization problem is to find a feasible portfolio process in the fictitiously completed market which maximizes \( J \). That is, find \( \pi_0 \in A(x) \) such that
\[
J(\pi_0, x) = \sup_{\pi \in A(x)} J(\pi, x).
\]

\section*{2.2. A technical remark.} Denote the market price of risk of the fictitious asset by \( \theta_2(t) := (\mu_2(t) - \tau)/\sigma_2(t) \) and let
\[
\theta(t) := \begin{pmatrix} \theta_1 \\ \theta_2(t) \end{pmatrix}, \quad \sigma(t) := \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2(t) \end{pmatrix}
\]

We impose some weak integrability conditions which have been adopted elsewhere (e.g., Karatzas et al., 1991). We say that a fictitious completion is \textit{admissible} if \( \mu_2 \) and \( \sigma_2 \) are \( \mathcal{F}_t \)-adapted and satisfy
\[
\int_0^T \theta_2(s)^2 ds < \infty \quad \text{or, equivalently,} \quad \int_0^T \| \theta(s) \|^2 ds < \infty.
\]
Given an admissible fictitious completion define the state-price deflator

\[ H(t) := \exp \left( -\int_0^t \theta^*(s)dW(s) - \frac{1}{2} \int_0^t (\|\theta(s)\|^2 + 2r)\,ds \right) . \tag{9} \]

If we considered only fictitious assets whose market price of risk satisfied the Novikov condition

\[ E \left[ \exp \left( \frac{1}{2} \int_0^T \theta^2(s)\,ds \right) \right] < \infty \]

instead of (8), then \( H(t)e^{rt} \) would be a martingale, and hence by Girsanov’s theorem, we could define the (unique) risk-neutral probability \( Q \) by \( \frac{dQ}{dP} = H(t)e^{rt} \) for the fictitiously completed market. Instead, (8) only provides a local martingale property for \( H(t)e^{rt} \) (see Karatzas et al., 1991).

For those completed markets which satisfied the more restricted Novikov condition, we could apply the usual Lagrange multiplier method to derive the optimal strategy. That is, the martingale property would imply the so-called budget equation

\[ E[H(T)(X^{\pi,x}(T))] = x + E \left[ \int_0^T H(s)y(s)\,ds \right] . \tag{10} \]

And so the unconstrained optimization problem would become

\[
\sup_{\{X^{\pi,x} \mid \pi \in A(x)\}} E[\log(X^{\pi,x}(T))] - \lambda \left( E[H(T)X^{\pi,x}(T)] - x - E \left[ \int_0^T H(s)y(s)\,ds \right] \right) \tag{11}
\]

where \( \lambda > 0 \) is a Lagrange multiplier.

In the next section we wish to find a fictitiously completed market whose optimal strategy does not call for the agent to invest in the fictitious asset. In such a market we do not have the Novikov condition, but only the weaker condition (8). In Appendix A, we see that deriving the optimal portfolio in this situation is not as straightforward as in the previous paragraph. In particular, compare (10) with Lemma 12.

3. The optimal portfolio for a well-chosen fictitious completion

In this section we prove that for any admissible completion of the market, there exists a unique optimal trading strategy which maximizes log-utility from terminal wealth. We then derive an SDE for the market price of risk, \( \theta_2 \), which when satisfied implies that the agent does not invest in the fictitious asset. This leads to our ultimate goal, finding the optimal portfolio in the true incomplete market.

Fix any admissible completion of the market. Introduce auxiliary variables

\[
\xi_0 := \frac{x + E \left[ \int_0^T H(s)y(s)\,ds \right]}{H(T)}, \quad Y(t) := \frac{1}{H(t)} \left( E \left[ \int_t^T H(s)y(s)\,ds \big| \mathcal{F}_t \right] + H(T)\xi_0 \right). \tag{12}
\]

The discounted \( \xi_0 \) equals the initial wealth plus the expected discounted cumulative endowment.
By the (local) Martingale Representation Theorem, Karatzas et al. (1988), there exists a \( \psi = (\psi_1, \psi_2)^* \) such that

\[
E \left[ -\int_0^T H(s)y(s)ds \middle| \mathcal{F}_t \right] = E \left[ -\int_0^T H(s)y(s)ds \right] + \int_0^t \psi^*(s)dW(s)
\]

with

\[
\int_0^T \| \psi(s) \|^2 ds < \infty.
\]

(15)

\( \psi \) can be thought of as the volatility of the discounted endowment.

Consider the portfolio

\[
\pi_0(t) := (\sigma(t))^{-1} \left( \frac{\psi}{Y(t)H(t)} + \theta(t) \right).
\]

(16)

**THEOREM 1.** For the given admissible completion, \( \pi_0 \in A(x) \) and

\[
J(\pi_0, x) = \sup_{\pi \in A(x)} J(\pi, x).
\]

This trading strategy is unique. Moreover, the corresponding optimal wealth process, \( X_{\pi_0, x} \), equals \( Y \) from (13).

We prove Theorem 1 in Appendix A. An equation similar to (16) has been previously derived for the complete market with or without consumption; see Cvitanic et al. (1996), Cuoco (1997) for example. In fact, our proof of Theorem 1 is similar to their approaches. However, we are unaware of such an explicit derivation (using Martingale methods) in the case of unhedgeable stochastic income (2) and weak integrability conditions (8).

**THEOREM 2.** Assume a positive initial wealth \( x > 0 \). There exists an admissible completion such that the optimal portfolio process, (16), does not invest in the fictional asset.

Moreover, the market price of risk in this completion satisfies

\[
\theta_2(t) = -\frac{\psi_2(t)}{X_{\pi_0, x}(t)H(t)}.
\]

(17)

In the proof of this Theorem, presented in Appendix A, we show why (17) is in fact an SDE and not a solution for \( \theta_2 \). We also explain why we must consider all \( \theta_2 \) which satisfy (8) and not just those which satisfy the Novikov condition.

**COROLLARY 3.** The optimal strategy for the original incomplete market is given by the portfolio (16) whose market price of risk \( \theta_2 \) satisfies (17).

**PROOF.** Let \( A'(x) = \{ \pi \in A(x) \mid \pi_2 \equiv 0 \} \). Clearly, for the admissible completion of Theorem 2

\[
J(\pi_0, x) = \sup_{\pi \in A(x)} J(\pi, x) \geq \sup_{\pi \in A'(x)} J(\pi, x) \geq J(\pi_0, x).
\]
Remark 4. From (17), the market price of risk equals the ratio of the unhedgable volatility of the discounted endowments to the discounted level of wealth. At this equilibrium the agent does not buy or sell the fictitious asset. The completion can be characterized by the fictitious price process whose market price of risk level causes the agent to neither increase nor decrease his/her exposure to the risk associated to the endowment. Alternatively, if the fictitious asset can be thought of as a contingent claim, the aforementioned price process would represent the “fair price” (see Davis, 1994).

4. Approximating the optimal portfolio

Ideally, we would like an explicit formula for $\theta_2$. This is because $\theta_2$ appears implicitly in the first coordinate of $\pi_0$ in (16); thus, the agent adopting our strategy needs to know $\theta_2(t)$ to select his/her portfolio at time $t$. Unfortunately, the market price of risk is not deterministic. Indeed, we prove in Appendix B that

Proposition 5. $\theta_2$ is deterministic if and only if the market is complete, that is $\sigma_y = 0$.

Instead, we develop a numerical method which approximates the optimal portfolio. The first subsection describes our initial guess at an optimal portfolio. The second subsection describes how to computational refine this guess.

4.1. A myopic optimization problem. Since we cannot solve (17), we begin by solving a simplified “myopic” problem. The optimal solution to this myopic problem provides a initial guess to the original optimization problem. At time $t$, we (temporarily) guess that $\theta_2(s) = \hat{\theta}_2(t)$ for all $s \in [t, T]$. We denote this myopic market price of risk by $\hat{\theta}_2(t)$.

Note that in this myopic setting, the discount factor $H$ is now a martingale. Thus, the classical Lagrangian-multiplier optimization method described in Section 2.2 applies to our setting. Specifically, the optimization problem we consider is a modification of (11) and we assume that for all $t \in [0, T]$ the agent solves

$$
\sup_{\{X^{\pi, x} \mid \pi \in A(x)\}} E[\log(X(T)) | \mathcal{F}_t] - \lambda(t) \left( E[H(T)X^{\pi, x}(T) | \mathcal{F}_t] - H(t)X^{\pi, x}(t) - E \left[ \int_t^T H(s)y(s)ds \right] | \mathcal{F}_t \right)
$$

where $H$ is determined by a myopic market price of risk $\hat{\theta}^* = (\hat{\theta}_1, \hat{\theta}_2)$ and $\lambda(t)$ is the Lagrange multiplier corresponding to the problem at time $t$. In the myopic problem at each time $t$ we incorrectly guess that the market price of risk is myopic. However, because the true market price of risk is not myopic we have to recalculate our problem for each time $t$.

Define $\hat{\psi}(t)$ similar to $\psi(t)$ using (14) with $\hat{\theta}_2$ instead of $\theta_2$. The exact same arguments (Lemmas 13 and 14) then show that the optimal trading strategy for (18), which we call the myopic strategy, is given by

$$
\hat{\pi}_0(t) = (\sigma(t))^{-1} \left( \frac{\theta_1}{\hat{\theta}_2(t)} + \frac{\hat{\psi}(t)}{X^{\pi_0, x}(t)H(t)} \right).
$$
To ensure that the agent following this strategy does not invest in the fictitious asset, we must find a myopic market price of risk which solves the analogue of (17):

$$\hat{\theta}_2(t) = -\frac{\hat{\psi}_2(t)}{X^{\pi_0,x}(t)H(t)}.$$  \hfill (20)

In Appendix B, we prove Proposition 6. Solving $\hat{\theta}_2$ in (20) reduces to solving the following fixed point problem:

$$\hat{\theta}_2(t) = -y(t) \left( e^{-(r-\alpha_y)(T-t)-\sigma_y(T-t)\hat{\psi}_2(t)} - 1 \right) \left( \sigma_y - \hat{\theta}_2(t) \right) \frac{1}{X^{\pi_0,x}(t)(r-\alpha_y + \sigma_y\hat{\psi}_2(t))}. \hfill (21)
$$

Moreover, for all model parameter values, the solution exists (although it may not be unique) and must between $0$ and $\sigma_y$ exclusively.

See Allgower et al. (1990), for example, for algorithms on how to solve fixed point problems. For consistency, we choose the largest solution to the fixed point problem; that is, the one closest to $\sigma_y$.

Having found the myopic market price of risk, we can now explicitly solve the myopic optimal strategy. We prove in Appendix B Proposition 7.

The optimal strategy for the myopic problem is given by

$$\hat{\pi}_0(t) = \left( 1 + \frac{y(t)}{X^{\pi_0,x}(t)} \left( 1 - e^{(t-T)(r-\alpha_y + \sigma_y\hat{\psi}_2(t))} \right) \right) \sigma^{-1} \begin{pmatrix} \theta_1 \\ 0 \end{pmatrix}. \hfill (22)
$$

4.2. A numerical algorithm for the optimal strategy. Although we have a strategy in Proposition 7, we must improve that to approximate the optimal strategy of the original problem. We outline below a numerical method for doing this based on Monte Carlo Simulations.

Step 1, sample paths and convergence criteria:

Generate a sample space for $W_{M \times N \times 2}$, where $M$ is the number of sample paths and $N$ is the number of time intervals on $[0, T]$. This space is the sample paths of $W_1(\cdot)$ and $W_2(\cdot)$. Assign a small value $\epsilon > 0$ and two large values $K_1, K_2 > 0$ for the convergence criteria.

Step 2, endowment:

For the sample path $i, i = 1, \ldots, M$, set $y(i, 0) = y_0$ and use (2) to compute $dy(i, 0)$ where $dt = T/N$. At time $j, j = 1, \ldots, N$, set $y(i, j) = y(i, j-1) + dy(i, j-1)$ and use (2) to compute $dy(i, j)$.

Step 3, initial market price of risk, portfolio and wealth at time 0:

At time 0, set $X^{\pi_0,x}(i, 0) = x$. Determine $\hat{\theta}_2(i, 0)$ by solving the fixed point problem (21), where wealth on the right hand side is $x$. Then use (22) to get $\hat{\pi}_0(i, 0)$, plugging in $\hat{\theta}_2(i, 0), x$, and $y_0$ on the right hand side. Finally use (3) to get $dX^{\pi_0,x}(i, 0)$.

Step 4, initial market price of risk, portfolio and wealth at time $j$:

At time $j, j = 1, \ldots, N$, set $X^{\pi_0,x}(i, j) = X^{\pi_0,x}(i, j-1) + dX^{\pi_0,x}(i, j-1)$. 


Determine $\hat{\theta}_2(i, j)$ by solving the fixed point problem (21), where wealth on the right hand side is $X^{\pi_{0.2}}(i, j)$. Then use (22) to get $\hat{\pi}(i, j)$, plugging in $\hat{\theta}_2(i, j), X^{\pi_{0.2}}(i, j)$, and $y(i, j)$ on the right hand side. Finally use (3) to get $dX^{\pi_{0.2}}(i, j)$.

**Step 5, initial state-price deflator and endowment volatility:**
Given $\theta_2(i, j)$, use (9) to get $H(i, j)$. Then use (14) to get $\psi(i, j)$. Set initial values as $H^0 = H$, $\pi^0 = \hat{\pi}_0$, $X^0 = X^{\pi_{0.2}}$, $\psi^0 = \psi$ and $\theta_2^0 = \theta_2$.

**Step 6, state-price deflator and endowment volatility:**
At the $k^{th}$ iteration, plug $\theta_2^k(i, j)$ into (9) to calculate $H^{k+1}(i, j)$. Calculate
\[
E \left[ -\int_0^T H^{k+1}(s) y(s) ds \right] \mid \mathcal{F}_{jT/N} \] for all $(i, j)$ by using $W_c(a, b)$ for $a = 1, \ldots, M$, $b = j+1, \ldots, N$ and $c = 1, 2$. Given these values, calculate $\psi^{k+1}(i, j)$ by solving (14).

**Step 7, market price of risk, portfolio and wealth:**
Calculate $\theta_2^{k+1}$,
\[
\theta_2^{k+1}(i, j) = -\frac{\psi^{k+1}(i, j)}{X^{\pi^{k+1}}(i, j) H^{k+1}(i, j)}. \]
Then, obtain $\pi^{k+1}$ from
\[
\pi^{k+1}(i, j) := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 & \theta_2^k(i, j) \end{pmatrix}^{-1} \left( \frac{\psi^{k+1}(i, j)}{X^{\pi^{k+1}}(i, j) H^{k+1}(i, j)} + \left( \begin{pmatrix} \theta_1 \\ \theta_2^{k+1}(i, j) \end{pmatrix} \right) \right), \]
where $\sigma_2^{k+1}(i, j) = (\mu_2 - r)/\theta_2^{k+1}(i, j)$. Compute the optimal wealth $X^{\pi^{k+1}}$ using (3).

**Step 8, test of convergence:**
Given $\pi^{k+1}$, $\theta_2^{k+1}(i, j)$, if
\[
\frac{1}{MN} \sum_{i,j}^{i=M, j=N} \left( \pi^{k+1}(i, j) - \pi^k(i, j) \right)^2 + \left( \theta_2^{k+1}(i, j) - \theta_2^k(i, j) \right)^2 \geq \epsilon \]
or $|\theta_2^l(i, j)| > K_2$ for some $l \in \{k - K_2, \ldots, k + 1\}$
where $K_2$ is an integer, go to Step 6. Otherwise, stop.

The following proposition, proved in Appendix B, demonstrates that the above algorithm approximates the optimal portfolio.

**Proposition 8.** If $\lim_{k \to \infty} |\theta_2^{k+1}(i, j) - \theta_2^*(i, j)| = 0$ for all $(i, j) \in (1, \ldots, M) \times (1, \ldots, N)$ and there exists $K_1, K_2 > 0$ such that for all $k \geq K_2$ and for all $(i, j) \in (1, \ldots, M) \times (1, \ldots, N)$, $|\theta_2^l(i, j)| < K_1$, then the strategy $\pi^k(i, j)$ approaches the optimal strategy as $k \to \infty$.

Note that even if we know the rate of convergence for $\theta_2^k(i, j)$, we do not know the rate of convergence of $\pi^k(i, j)$. Therefore, in Step 8, we must also numerically confirm the convergence rate of $\pi^k(i, j)$.

Next we demonstrate numerically the robustness of our numerical method. First, we confirm that after the 300th iteration the upper boundary of $\{\theta_2^k(i, j)\}$ is 19.35. This implies that the boundary condition of Proposition 8 is satisfied with $K_1 = 19.35$ and $K_2 = 300$. Second, we test the rate of convergence for different model parameters by using the following measure:
\[
\max_{\theta_1, \sigma_2, \alpha} \max_{i,j} |\theta_2^{k+1}(i, j) - \theta_2^*(i, j)| \]
where the maximum is over a large parameter space. Figure 1 confirms that over a large variation of the parameters the measure converges to zero.

Figure 1. The convergence measure. The parameters values: \( T = 1, 0 \leq \theta_1 \leq 1, 0 \leq \sigma_y \leq 10, 0 \leq \alpha_y \leq 1, r = 0.05, \sigma_1 = 0.3, \) and \((i, j) \in (0, \ldots, 1000) \times (1, \ldots, 100).\)

The following table shows the average number of iterations needed to calculate the optimal strategy with different initial guesses and convergence criterions. As can be seen, the myopic strategy improves the algorithm significantly.

Table 1: Average number of iterations with different initial strategies. Parameter values: \( x = y = 1, T = 1, 0 \leq \theta_1 \leq 1, 0 \leq \sigma_y \leq 10, 0 \leq \alpha_y \leq 1, \sigma_1 = 0.3 \) and \( r = 0.05.\)

<table>
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<tr>
<th>Initial Strategy</th>
<th>Average Number of Iteration</th>
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<th>( \epsilon = 0.01 )</th>
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<tr>
<td>Myopic Strategy</td>
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5. The optimal strategy versus the myopic and ignorant strategies

In this section, we compare the optimal strategy with the myopic strategy and a naive strategy which we describe below. We compare these strategies numerically for some test cases, as well as theoretically for extreme parameter values.

Let \( \pi_c(t) \) be a 2-dimensional process whose first coordinate denotes the optimal trading strategy in the complete market (that is, assuming, \( \sigma_y = 0 \)) and whose second coordinate is 0. Thus, \( X^{\pi_c}(t) \) represents the wealth of an agent who tries to optimize utility while ignoring the randomness of the unhedgeable income he/she receives. We will call \( \pi_c(t) \) the ignorant strategy.

Like for Proposition 7, we derive in Appendix B the following closed-form expression for the ignorant strategy.
Proposition 9.

\[
\pi_c(t) = \left(1 + \frac{y(t)}{X^{\pi_c,x}(t)} \left(1 - e^{(t-T)(r-\alpha_y)}\right) \right) \sigma^{-1} \left(\begin{array}{c}
\theta_1 \\
0
\end{array} \right).
\]  \hspace{1cm} (23)

Comparing (22) and (23), we see that the sign of \(\theta_1\) is the same as that of \(\pi_c(t) - \hat{\pi}_0(t)\). For example, if the market price of risk of the risky asset is positive, the agent invests less in this asset under the myopic strategy than under the ignorant one. This difference can be seen as a method to decrease the portfolio’s variance.

**Figure 2.** The above graphs compare the log-utilities of the terminal wealths for the optimal strategy, the myopic strategy and the ignorant strategy for various market prices of risk of the tradable asset. The parameter values are \(x = y = 1\), \(T = 1\), \(\sigma_y = 5\), and \(\alpha_y = r = 0.05\). 1000 simulations were used.

In Figures 2, 3 and 4 we show via simulations that for different values of the market price of risk of the tradable asset, the endowment volatility, and the endowment drift, the myopic strategy outperforms the ignorant strategy. The utilities from the optimal strategy is significantly higher than the corresponding utility from the myopic and ignorant strategies in all the figures.

In Figure 2 the utility is an increasing function of the market price of risk of the tradable asset because a higher market price of risk implies a higher expected return of the traded asset and, therefore, a higher expected utility. In Figure 3 we show that for low endowment volatility \(\sigma_y\), the difference is negligible. This is because with low \(\sigma_y\) we are close to the corresponding complete market situation. For this reason we use high values of \(\sigma_y\) in Figures 2 and 4. As expected, according to Figure 4 the endowment drift increases the expected terminal wealth and the expected utility.
Figure 3. The above graphs compare the log-utilities of the terminal wealths for the optimal strategy, the myopic strategy and the ignorant strategy for various volatility levels of the endowment. The parameter values are $x = y = 1$, $T = 1$, $\theta_1 = 0.5$, $\sigma_1 = 0.3$ and $\alpha_y = r = 0.05$. 1000 simulations were used.

Note that according to the figures, the utilities derived from the ignorant and myopic strategies converge as the market “becomes complete” ($\sigma_y \to 0$) or as the market price of risk of the tradable asset goes to zero ($\theta_1 \to 0$). We prove in Appendix B that this happens in general, and that moreover, both terminal wealths converge to $X_{\pi_0,x}$, the terminal wealth under the optimal portfolio.

**Proposition 10.** If $\theta_1 = 0$, then

$$\hat{\pi}_0(t) = \pi_c(t) = \pi_0(t) = 0.$$  

(24)

That is, all three strategies tell the agent to invest only in the bond.

As $\sigma_y$ approaches zero, both the myopic and the ignorant strategies converge to the optimal strategy; that is

$$\lim_{\sigma_y \to 0} \pi_0(t) = \lim_{\sigma_y \to 0} \pi_c(t) = \lim_{\sigma_y \to 0} \pi_0(t).$$  

(25)

Note that in (24) we have equality while in (25) we only have a statement involving limits. This is because the interval from 0 to $\sigma_y$ approaches the empty set as $\sigma_y \to 0$. Thus, the myopic market price of risk does not exist in the complete market. Alternatively, some algebra shows that (21) cannot have a solution when $\sigma_y = 0$.

6. Conclusion

We have proposed a numerical method for portfolio optimization in an incomplete market where the agent maximizes log-utility from terminal wealth and where he/she is unable to hedge his/her endowment uncertainty. We first extended the
initial market to a complete market, where the endowment process can be hedged using a fictitious risky asset. In the extended market there exists such a market price of risk that the agent does not want to use the fictitious asset. Therefore, such a portfolio strategy can be realized in the initial incomplete market.

To derive an explicit solution, we first modeled this market price of risk with a myopic process. The advantage of this model is that we can easily solve for the trading strategy by using a fixed-point theorem. We then showed how to improve the strategy recursively, until a close approximation of the true optimal strategy is achieved.

Appendix A. The optimal strategy

A.1. The proof of Theorem 1. We begin with the following elementary Lemma. Let $f(t)$ and $g(t)$ be stochastic processes, which are (almost surely) continuous in $t$.

Lemma 11. Suppose $\int_0^t |f(s)|ds < \infty$ and for all $s \in [0, t]$, $|g(s)| < \infty$. Then $\int_0^t |f(s)g(s)|ds < \infty$.

We now prove Theorem 1 with three Lemmas.

Lemma 12. If $\pi \in A(x)$ then

$$x + E \left[ \int_0^T H(s)y(s)ds - H(T)\pi^T x(T) \right] \geq 0.$$
Proof. From (3) and Ito’s formula, we get
\[
H(t)X^{\pi,x}(t) - \int_0^t H(s)y(s)ds
\]
\[
= x + \int_0^t H(t)X^{\pi,x}(s)(\sigma^*(s)\pi(s) - \theta(s))^*dW(s).
\]  
(26)

Note that the right hand side is a local martingale.

For the positive integers \(n \in \mathbb{N}\), define a sequence of stopping time \(\tau_n\) by
\[
\tau_n := T \wedge \inf \left\{ t \in [0,T] \mid \int_0^t H(s)y(s)ds \geq n \right\}
\]
For \(t \in [0,\tau_n]\), \(H(t)X^{\pi,x}(t) - \int_0^t H(s)y(s)ds\) is a local martingale bounded below by \(-n\), hence a supermartingale (see Karatzas et al., 1998, p.19).

Thus, for \(t \in [0,\tau_n]\),
\[
E \left[ H(t)X^{\pi,x}(t) - \int_0^t H(s)y(s)ds \right] \leq x.
\]  
(27)

We claim that \(\lim_{n \to \infty} \tau_n = T\) pointwise a.s. Since \(H(t)\exp \left( \int_0^t r(s)ds \right)\) is a non-negative local martingale (Section 2.2), it is also a supermartingale. Hence
\[
E \left[ H(t) \exp \left( \int_0^t r(s)ds \right) \right] \leq E[H(0)1] = 1 \quad \forall t \in [0,T].
\]

Since \(r(t) \geq 0\), we get \(E[H(t)] \leq 1\), and hence \(H(t) < \infty\). Applying this and (2) to Lemma 11, we get that for almost all \(\omega \in \Omega\), there exists \(n(\omega) \in \mathbb{N}\) such that
\[
\int_0^T H(\omega, s)y(\omega, s)ds < n(\omega).
\]
Thus \(\tau_n\) converges to \(T\) pointwise a.s.

Note that \(H(\tau_n)X(\tau_n)\) converges pointwise a.s. to \(H(T)X(T)\) because of the integral definitions of \(H\) and \(X\). (Any \(f(z) := \int_0^z g(w)dw\) is a continuous function in \(z\); similarly for stochastic integrals.) Moreover, (9) and (31) imply \(H(\tau_n)X^{\pi,x}(\tau_n) \geq 0\). By Fatou’s Lemma, then
\[
E[H(T)X(T)] \leq \liminf_{n \to \infty} E[H(\tau_n)X(\tau_n)].
\]  
(28)

Note that \(\int_0^{\tau_n} H(s)y(s)ds\) is a sequence of non-negative functions converging monotonically pointwise a.s. to \(\int_0^T H(s)y(s)ds\). This follows from the non-negativity of \(H(s)y(s)\) and the above-mentioned fact about the continuity of integrals. Thus, by the Monotone Convergence Theorem,
\[
\lim_{n \to \infty} E \left[ \int_0^{\tau_n} H(s)y(s)ds \right] = E \left[ \int_0^T H(s)y(s)ds \right].
\]  
(29)

Combining equations (27-29), we get the Lemma.

\[\square\]

Lemma 13. If \(\pi \in A(x)\) then \(J(\pi, x) \leq E[\log(\xi_0)]\). Moreover, if \(E[\log X^{\pi,x}(T)] = E[\log(\xi_0)]\) then \(X^{\pi,x}(T) = \xi_0\).
To simplify notation, let $\lambda = \frac{1}{x + E[\int_0^T H(s)g(s)ds]}$. First note that by the definition of $\xi_0$ in (12),

$$E \left[ -\int_0^T H(s)y(s)ds + H(T)\xi_0 \right] = x. \quad (30)$$

Note that $\log(\frac{1}{y}) \geq \log x + y(\frac{1}{y} - x)$; thus, for an arbitrary $\pi \in A(x)$

$$\log \left( \frac{1}{\lambda H(T)} \right) \geq \log(X_{\pi,x}^\pi(T)) + \lambda H(T) \left( \frac{1}{\lambda H(T)} - X_{\pi,x}^\pi(T) \right).$$

Since $\xi_0 = \frac{1}{\lambda H(T)}$,

$$E[\log(\xi_0)] \geq J(\pi, x) + \lambda \left( x + E \left[ \int_0^T H(s)y(s, \omega)ds - H(T, \omega)\alpha \right] \right)$$

The inequality now follows from the previous lemma and noting that $\lambda \geq 0$.

To prove uniqueness, fix a sample path $\omega$. It is well known that if $(\mu, \alpha)$ is a solution to the unconstrained problem

$$\inf_{\mu \geq 0} \sup_{\alpha \in \mathbb{R}} \left( \log(\alpha) + \mu \left( x + \int_0^T H(s, \omega)y(s, \omega)ds - H(T, \omega)\alpha \right) \right)$$

then $\alpha$ is a solution to the constrained problem

$$\sup_{\alpha \in \mathbb{R}} \log(\alpha) \text{ such that } x + \int_0^T H(s, \omega)y(s, \omega)ds - H(T, \omega)\alpha \geq 0.$$ 

Moreover, the convexity of $\log$ implies the unconstrained problem has a unique solution, one (and the only one) of which is given, according to the previous discussion, by $\alpha = \xi_0(\omega)$. This is true for all $\omega$, so uniqueness holds.

\[\square\]

**Lemma 14.** $X_{\pi_0,x}(t) = Y(t)$, in particular $X_{\pi_0,x}(T) = \xi_0$. Moreover, the optimal portfolio is unique.

**Proof.** From (13), (14), (16) and (30)

$$H(t)Y(t) - \int_0^t H(s)y(s)ds = x + \int_0^t \psi^*(s)dW(s) = x + \int_0^t (H(s)Y(s)(\sigma^*(s)\pi_0(s) - \theta(s))^*dW(s).$$

From (13), $H(0)Y(0) = 1Y(0) = x$. Since $H(t)X_{\pi_0,x}(t)$ solves the same stochastic integral equation, (26), with the same initial conditions (and $H > 0$), $X_{\pi_0,x} = Y$. From (13) we see $Y(T) = \xi_0$. Thus $X_{\pi_0,x}(T) = Y(T) = \xi_0$.

If $\pi'$ where another optimal portfolio, then by Lemma 13, $X_{\pi_0,x}(T) = \xi_0 = X_{\pi',x}(T)$. The uniqueness of the optimal portfolio now follows from the uniqueness of $\psi$, given by the (local) Martingale Representation Theorem.

\[\square\]

Combining the last two Lemmas proves Theorem 1.
A.2. The proof of Theorem 2. Note that since $x > 0$,
\[ X^{\pi_{x}}(t) > 0. \] (31)
Choose $\theta$ such that $\theta_2$ satisfies (17). Note that although $H(t)$ depends on the stochastic process $\theta_2$, it is an integral formula; thus, $H(t)$ does not depend on the random variable $\theta_2(t)$. Similarly with $X^{\pi_{x}}(t)$. However, we see from (14) that $\psi_2(t)$ depends on $H(s)$ for $s \geq t$ and this in turn depends on $\theta_2(t)$. Thus (17) is a stochastic equation in $\theta_2$. Such an equation is hard to solve. Moreover, it is even difficult to check that the coefficients satisfy appropriate integrability conditions.

However, note that this proof of Theorem 2 demonstrates that (17) is a necessary condition for any solution of the optimization problem. The existence of a solution to this problem is proved in Cvitanic et al. (1999), who consider a more general condition for any solution of the optimization problem. Theorem 1 proves that the solution is unique for our special case. Thus (17) must have a solution.

Having proven that a $\theta_2 = \frac{\mu_2 - r}{\sigma_2}$ exists that satisfies (17), it is easy to specify $\mu_2$ and $\sigma_2$ so that (7) holds.

From (9) and (31) it follows that for all $t \in [0, T]$, \[ \frac{1}{X^{\pi_{x}}(t)H(t)} < \infty. \] Thus, from (15) and Lemma 11,
\[ \int_{0}^{T} \left| \frac{\mu_2(\omega, s) - r}{\sigma_2(\omega, s)} \right|^2 ds = \int_{0}^{T} \left| \frac{\psi_2(\omega, s)}{X^{\pi_{x}}(\omega, s)H(\omega, s)} \right|^2 ds < \infty. \]

This shows that $\theta_2$ satisfies (8). Since (7) and (8) both hold, the completion is admissible.

It remains to prove that $\pi_{02} \equiv 0$. From Theorem 1 we have $X^{\pi_{0}}(t) = Y(t)$. From (16), (17) and the special form of the matrix in (6),
\[ \pi_{02}(t) = \frac{1}{\sigma_2(t)} \left( \frac{\psi_2(t)}{X^{\pi_{0}}(t)H(t)} + \frac{\mu_2(t) - r}{\sigma_2(t)} \right) = 0. \]

Appendix B. The myopic trading strategy

B.1. The proof of Proposition 6. We begin by showing that (20) reduces to a fixed point problem.

From (2), (9) and Ito’s Lemma
\[ d(Hy)(t) = H(t)y(t)(\alpha_y - r - \sigma_y \theta_2(t))dt \]
\[ + H(t)y(t)(-\theta_1 dW_1(t) + (\sigma_y - \theta_2(t))dW_2(t)). \] (32)

Or,
\[ H(s)y(s) = \] (33)
\[ H(t)y(t) \exp \left( \int_{t}^{s} \left( \alpha_y - r - \sigma_y \theta_2(u) - \frac{1}{2} \sigma_y^2 - \frac{1}{2} \theta_1^2 \sigma_y^2 \right) du \right) \]
\[ + \int_{t}^{s} \theta_1 dW_1(u) + \int_{t}^{s} (\sigma_y - \theta_2(u))dW_2(u). \]

We now introduce a special one parameter family of time dependent functions, $\Phi(t, s)$. The first argument $t \in [0, T]$ indexes the family. The second argument $s \in [0, T]$ represents the time. For a given $t$, we set $\Phi(t, s) = \Phi(s, t)$ if $s \leq t$ and $\Phi(s, t) = \Phi(t, t)$ if $s \geq t$. Note that $\Phi(t, t)$ completely determines $\Phi(t, s)$. We will construct $\Phi(t, s)$ such that for a fixed $t$, $\Phi(t, s)$ represents what the myopic
investor (at time $t$) thinks the stochastic process (where time is indicated by $s$) for the market price of risk should be. Note that although the investor sees that the market price of risk depended on time, $s$, in the past ($s < t$), the investor myopically believes the market price of risk will not change in the future ($s > t$).

Denote $\Phi(t, t)$ by $\dot{\theta}_2(t)$. We replace the true market price of risk, $\theta_2(t)$, with this myopic one, $\dot{\theta}_2(t)$.

Since all the other parameters are constant, (33) simplifies to

\[
H(s)y(s) = H(t)y(t)\exp\left(\left(\alpha_y - r - \sigma_y\dot{\theta}_2(t) - \frac{1}{2}(\sigma_y - \dot{\theta}_2(t))^2 - \frac{1}{2}\theta_s^2\right)(s-t)\right) + \theta_1(W_1(s) - W_2(t)) + (\sigma_y - \dot{\theta}_2(t))(W_2(s) - W_2(t)).
\]

Thus,

\[
E\left[H(s)y(s)|F_t\right] = H(t)y(t)e^{(\alpha_y - r - \sigma_y\dot{\theta}_2(t))(s-t)}
\]

Using the Fubini-Tonelli Theorem with (34), we get

\[
d\left(E\left[-\int_0^TH(s)y(s)ds\right]|F_t\right) = -\left[H(t)y(t)\int_t^T e^{(\alpha_y - r - \sigma_y\dot{\theta}_2(t))(s-t)}ds\right] - H(t)y(t)dt
\]

Using (32) to compare the $dW$ terms in (14) and (35) then implies that

\[
\dot{\psi}_2(t) = \frac{e^{(r-\alpha_y+\sigma_y\dot{\theta}_2(t))(t-T)} - 1}{r - \alpha_y + \sigma_y\dot{\theta}_2(t)}(\sigma_y - \dot{\theta}_2(t))H(t)y(t).
\]

Thus, $\dot{\theta}_2(t)$ satisfies the fixed point problem (21).

To prove the second statement in the Proposition, we note the following Lemma.

**Lemma 15.** Let $k_1, k_2$ and $k_3$ be positive constants and $k_4$ be any constant. Let

\[
f(x) = -k_1\left(e^{-k_4+k_3x} - 1\right)(k_3-x)\frac{k_4+k_3x}{k_4}.
\]

Then $f(x) = x$ has a solution. Moreover, all solutions lie in the interval $(0, k_3)$.

**Proof.** Note that $f$ can be continuously extended at $x = -\frac{k_4}{k_3}$. Since

\[
\lim_{x \to -\infty} f(x) = \infty > 0, \quad \lim_{x \to \infty} f(x) = \frac{-k_1}{k_2} < 0
\]

$f(x) = x$ must have at least one solution.

Since $\frac{e^{-k_4+k_3x} - 1}{x}$ is always negative, we see that $f$ is negative when $x > k_3$ and positive when $x < k_3$. Thus any solution of $f(x) = x$ must lie between 0 and $k_3$.

Letting

\[
k_1 = \frac{y(T-t)}{X^{\pi_0,x}}, \quad k_2 = \sigma_y(T-t), \quad k_3 = \sigma_y, \quad \text{and} \quad k_4 = (r - \sigma_y)(T-t)
\]
we get the second statement of the Proposition from this Lemma.

**B.2. The proofs of Propositions 5, 7, and 9.** We first prove a Lemma.

**Lemma 16.** If $\theta_2$ is deterministic, then for $\theta_2(t)$ to solve (17) it must satisfy the equation

$$\theta_2(t) = \frac{H(t)y(t)v(t)(-\sigma_y + \theta_2(t))}{H(t)y(t)v(t) - H(T)Y(T)}$$

(37)

where $v(t)$ is a deterministic function of $\theta_2(t)$ and $Y(t)$ is from (13).

**Proof.** Assume $\theta_2$ is deterministic. Then we can calculate like in (34), that for $s \geq t$,

$$E \left[ H(s)y(s) \mid \mathcal{F}_t \right] = H(t)y(t)v(t,s)$$

where

$$v(t,s) = \exp \int_t^s (\alpha_y - r - \sigma_y \theta_2(u)) du.$$ 

Applying the Fubini-Tonelli Theorem gives

$$E \left[ \int_t^T H(s)y(s)ds \mid \mathcal{F}_t \right] = H(t)y(t) \int_t^T v(t,s)ds$$

(38)

where

$$v(t) = \int_t^T v(t,s)ds.$$ 

This time,

$$d \left( E \left[ \int_0^T H(s)y(s)ds \mid \mathcal{F}_t \right] \right) = H(t)y(t)dt + v(t)d(Hy)(t) + H(t)y(t)dv(t).$$

Using (32) to compare the $dW_2$ coefficients of (14) and (39), we get

$$\psi_2(t) = H(t)y(t)v(t)(-\sigma_y + \theta_2(t)).$$

(40)

Note that (13) and (38) imply

$$H(t)Y(t) = -H(t)y(t)v(t) + H(T)Y(T).$$

(41)

Since $Y = X^{\pi_0, x}$, the Lemma now follow from (17), (40) and (41). \hfill \Box

Now for the proof of Proposition 5.

**Proof.** Recall that (12) and (13) imply that $H(T)Y(T) = x + E \left[ \int_0^T H(s)y(s)ds \right]$ is constant.

Assume that $\theta_2$ is deterministic. Rewrite (37) as,

$$\theta_2 = \frac{k(t)Z(t)}{Z(t) + c}$$

(42)

where $k(t) = -\sigma_y + \theta_2(t)$ is deterministic, $Z(t) = H(t)Y(t)v(t)$ is an Ito process and $c = -H(T)Y(T)$ is constant. This can be rewritten as

$$(\theta_2(t) - k(t))Z(t) = -c\theta_2(t).$$
The right-hand side is deterministic while the left-hand side is stochastic. Since \( c \neq 0 \), this implies \( \theta_2(t) - k(t) = 0 \) and \( \theta_2(t) = 0 \), which in turn implies \( \sigma_y = 0 \).

Next assume that the market is complete, that is, \( \sigma_y = 0 \). We claim that \( \theta_2(t) = 0 \) is a solution for (17). To see this, note that we can apply Lemma 16.

Then plugging in 0 for \( \theta_2 \) on both sides of (37) we get \( 0 = 0 \); hence, equality holds.

We end with the proofs of Propositions 7 and 9.

**Proof of Proposition 7.** We first prove (22), the formula for the myopic strategy. From (14), (32) and (35) we get

\[
\hat{\psi}_1(t) = \frac{e^{(t-T)(r-\alpha_y+\sigma_y\hat{\theta}_2(t))} - 1}{r - \alpha_y + \sigma_y\hat{\theta}_2(t)}(-\theta_1)H(t)y(t). \tag{43}
\]

Equation (22) now follows from (19) and (43).

**Proof of Proposition 9.** To prove (23), the formula for the ignorant strategy, we repeat the arguments for equations (32) through (35), replacing \( \sigma_y \) and hence \( \sigma_y\hat{\theta}_2 \) as well as \( \sigma_y\hat{\theta}_2 = \sigma_y\hat{\theta}_2 = 0 \) with 0. Equation (14), and the modified (32) and (35) (setting \( \sigma_y\hat{\theta}_2 = \sigma_y\hat{\theta}_2 = 0 \)) imply that

\[
\psi_1(t) = \frac{e^{(t-T)(r-\alpha_y)} - 1}{r - \alpha_y}(-\theta_1)H(t)y(t). \tag{44}
\]

Equation (23) now follows from (44) and the 1-dimensional version of (16).

**B.3. Proof of Proposition 8.** In the optimization method \( \theta_2^{k+1}(i,j) \) is calculated as follows

\[
\theta_2^{k+1}(i,j) = -\frac{\psi_2^{k+1}(i,j)}{X^k(i,j)H^{k+1}(i,j)},
\]

where \( \psi_2^{k+1}(i,j) \), \( X^k(i,j) \) and \( H^{k+1}(i,j) \) are functions of \( \theta_2^k(m,n) \) for \( 1 \leq m \leq M \) and \( 1 \leq n \leq N \). We can therefore think of the above equation as the \( (i,j) \)-entry of a matrix equation of the form

\[
\theta_2^{k+1} = F(\theta_2^k)
\]

for some continuous function \( F \). This is a fixed-point problem.

Let \( z_k = (\theta_2^k, \theta_2^{k+1}) \in \mathbb{R}^{2MN} \). Let \( D = \{(x,x) \mid x \in \mathbb{R}^{MN}\} \subseteq \mathbb{R}^{2MN} \) denote the diagonal. Let \( \Gamma_F = \{(x,F(x)) \mid x \in \mathbb{R}^{MN}\} \subseteq \mathbb{R}^{2MN} \) denote the graph of \( F : \mathbb{R}^{MN} \to \mathbb{R}^{MN} \). Note that \( z_k \in \Gamma_F \), the sequence \( \{z_k\} \) is bounded and inside the box \([-K_1,K_1]^{2MN} \subseteq \mathbb{R}^{2MN} \), and if we denote the usual Euclidean distance by \( dist \) than \( dist(z_k,D) \leq dist(\theta_2^k,\theta_2^{k+1}) \to 0 \).

Thus there is a subsequence \( w_l = z_{k_l} \) which converges to \( w_\infty \) which (by the continuity of \( F \)) is in the intersection of \( D \) and \( \Gamma_F \), that is, \( w_\infty \) represents a fixed point of \( F \), \( w_\infty = (x_\infty, x_\infty) = (x_\infty, F(x_\infty)) \), and \( x_\infty = \lim_{l \to \infty} \theta_2^{k_l} \in \mathbb{R}^{MN} \).

It is clear that the corresponding trading strategy \( \lim_{l \to \infty} \pi^{k_l}(i,j) \) is optimal.
B.4. Proof of Proposition 10. Assume $\theta_1 = 0$. From (9), we see that the state price deflator, $H$, is driven only by $W_2$. Since $y$ also only has randomness from $W_2$, (14) implies that $\psi_1(t) = 0$. (24) now follows from (16).

The second equality in (25) is clear. To prove the first equality, recall that $\hat{\theta}_2(t) \in (0, \sigma_y)$. Thus
\[
\lim_{\sigma_y \to 0} \sigma_y \hat{\theta}_2(t) = 0.
\]
The claim now follows by comparing (23) and (22).

References


