Modelling and Hedging Options under Stochastic Pricing Parameters‡

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Modelling and Hedging Options

**Abstract.** We consider the effect of stochastic parameters on the modelling of option prices. Our underlying asset model incorporates negative (down) and positive (up) jumps as well as stochastic volatility. Our empirical analysis with S&P 500 put options reveals that the parameters governing down jumps affect option price processes more significantly than those governing stochastic volatility or up jumps. However, due to the correlation structure between the model parameters and the underlying asset, stochastic volatility, up jumps, and down jumps affect our partial hedging almost in similar magnitude. A test of a portfolio selection strategy and different hedging scenarios shows that hedging parameter uncertainty improves the performance of a simple delta hedge on average by 8 percent and increases the portfolio's Sharpe Ratio by 21 percent.

*Keywords:* Derivative instrument, estimation, transformation analysis, hedging.

1. Introduction

Investors in options face uncertainties both in the underlying asset prices and in the accuracy of their pricing model. This paper considers a specific type of option pricing model, and also allows the model parameters to change over time to analyze the effects of possible parameter misspecifications.

We use the jump-diffusion process of Duffie, Pan and Singleton (2000) to model the underlying asset and the corresponding option prices. Due to this quite general jump-diffusion model we are able to consider three additional characteristics of the underlying asset in addition to the diffusion process: up jumps, down jumps, and stochastic volatility. We consider only simple contingent claims, namely European calls and puts. We empirically analyze an option pricing function that can be derived using transformation analysis.|| We first estimate the model’s implied parameter processes by using time-series models. Using a method of Merton (1976) we then hedge all continuous uncertainties in the option processes that are correlated with the underlying asset price. These uncertainties are due to the changes in the underlying asset and in the implied parameters. We cannot eliminate all of these uncertainties using only the underlying asset. If the unhedgable uncertainties are asset specific, they may be diversifiable and thus have little impact on derivative prices. However, Coval and Shumway (2000) show that options are not redundant securities and that, for instance, volatility risk is priced.¶

Incorporating jumps in our model is important since, for instance Johannes (2002) and Eraker, Johannes and Polson (2002) show that jumps in the underlying asset and volatility improve the fit of stock return and short rate models. However, because of the jump-diffusion process, perfect replication in our setup is possible only by using other derivative instruments. In some cases this kind of strategy can lead to a static


¶ Bakshi and Kapadia (2001) and Brenner, Ou, and Zhang (2001) also show that volatility risk has an important effect on option prices, but Fan, Gupta and Ritchken (2002) show that in the LIBOR swaption market, stochastic volatility does not play an important role.
Our model can be viewed as a parameterization for the implied volatility surface (as a function of maturity and strike price) with ten parameter processes, because there always exists a Black-Scholes price with an implied volatility that corresponds to our model’s price. Many others model implied volatility.∗

Pan (2002) finds that jump risk premia are higher than stochastic volatility premia, and that they become more important in volatile markets. The results of our paper are threefold. First, as in Pan (2002) we find that downward jumps are the most important characteristic in the modelling of option prices. Second, we show that the model parameters are predictable. Third, we use this predictability to hedge the uncertainties in the parameters and show that implied parameter hedging is important. Due to the correlation structure between the model parameters and the underlying asset, down and up jumps as well as stochastic volatility affect the hedging with almost the same magnitude. According to our empirical analysis, hedging the parameter changes improves delta hedging on average by about 8 percent. Our hedging result is similar to those of Jiang and Oomen (2001). They perform a simulation study to evaluate several hedging strategies against various risk factors, including the risk that the model is misspecified. We also measure the performance of different hedging strategies. Hedging parameter uncertainties increases the Sharpe ratio of a delta hedged portfolio by 21 percent.

The paper proceeds as follows. Section 2 reviews the Duffie-Pan-Singleton (2000) model. Section 3 introduces a general parameter estimation model. Section 4 describes the hedging technique, Section 5 considers a numerical analysis with S&P option data, and Section 6 concludes.

2. Duffie-Pan-Singleton model

Duffie, Pan and Singleton (2000) derive a closed form transform for general affine jump-diffusion processes and show that it leads to tractable solutions to many valuation problems. We use one of their specific models and review it in this section.

Let \( Y_t = \ln(X_t) \) where \( X_t \) is the value of the underlying stock at time \( t \). Assume the stock pays a constant dividend yield \( q \). Suppose \( V_t \) is the stock volatility process. Let \( W_t \) be the \( \mathcal{F}_t \)-standard Brownian motion in \( \mathbb{R}^2 \) where \( \mathcal{F}_t \) is the filtration. Let \( Z_1^t \) and \( Z_2^t \) be two \( \mathcal{F}_t \)-adapted jump processes in \( \mathbb{R} \) with constant mean jump-arrival rates \( \lambda_1 \) and \( \lambda_2 \), whose bivariate jump-size distributions \( \nu_1 \) and \( \nu_2 \) have transforms \( \phi_1 \) and \( \phi_2 \),


respectively. Here, for \( l = 1, 2 \), the transform of \( \nu_l \) is defined as \( \phi_l(c) = \int_{\mathbb{C}} \exp(cz) d\nu_l(z) \) where \( c \in \mathbb{C} \). We consider the special case where \( Z_1^t \) and \( Z_2^t \) represent uncorrelated up-jumps and down-jumps, which for \( l = 1, 2 \), have normally distributed jump size with mean \( \mu_l \) and variance \( \sigma_l^2 \). The first jump is called “up” and the second one “down” because we will set \( \mu_1 > 0 \) and \( \mu_2 < 0 \). One can check that \( \phi_l(c) = \exp \left( \mu_l c + \frac{1}{2} \sigma_l^2 c^2 \right) \).

Consider the process:

\[
\begin{align*}
    d \left( \begin{array}{c} Y_t \\ V_t \end{array} \right) &= \left( \begin{array}{c} r - q - \lambda_1 \left( e^{\mu_1 + \frac{1}{2} \sigma_1^2} - 1 \right) - \lambda_2 \left( e^{\mu_2 + \frac{1}{2} \sigma_2^2} - 1 \right) - \frac{1}{2} V_t \\ \kappa_v (\bar{v} - V_t) \end{array} \right) dt \\
    &+ \sqrt{V_t} \left( \begin{array}{cc} 1 & 0 \\ -\bar{\rho} & \sigma_v \sqrt{1 - \bar{\rho}^2} \end{array} \right) dW_t + \left( \begin{array}{c} d \omega^1_t \\ d \omega^2_t \end{array} \right) \tag{1}
\end{align*}
\]

We assume that the volatility is mean reverting, with mean \( \bar{v} \), \( \kappa_v \) is the mean reversion coefficient, \( \sigma_v \) is the volatility of volatility and \( \bar{\rho} \) is the correlation between the volatility process and the process of the underlying asset.

Note that in this model, the parameter set is

\[
\theta = (\lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_1, \sigma_2, \bar{\rho}, \sigma_v).
\]

Denote by \( E_{\theta}[Y] \) the expectation of \( Y \), under the risk-neutral probability \( Q \), where the distribution of \( Y \) relies on the model’s choice of \( \theta \).

Duffie, Pan and Singleton derive a semi-explicit formula for the option price for this class of models. Since \( Y_t = \ln X_t \), we have for a European call option with payoff \((X_T - c)^+\),

\[
C(t, \theta) = e^{-r(T-t)} E_{\theta} \left[ (e^{Y_T} - c) \mathbf{1}_{Y_T \geq \ln c} \mid \mathcal{F}_t \right] 
= G_t(-\ln c; Y_t, V_t, \theta) - cG_0(-\ln c; Y_t, V_t, \theta).
\]

where

\[
G_d(x; Y_t, V_t, \theta) = e^{-r(T-t)} E_{\theta} \left[ e^{dY_T} \mathbf{1}_{Y_T \leq x} \right]
\]

where \( \mathbf{1}_{Y_T \leq x} \) is the indicator function that takes on the value 1 if \( Y_T \leq x \) and 0 otherwise. Let \( \iota = \sqrt{-1} \). Duffie, Pan and Singleton prove that the Fourier transform (in \( x \)) of \( G_d(\cdot; Y_t, V_t, \theta) \), \( \hat{G}_d(\cdot; Y_t, V_t, \theta) \), is affine and therefore tractable. It is given by

\[
\hat{G}_d(\xi; y, v, \theta) = \exp(\alpha(\tau - t, d - i\xi) + (d - i\xi)y + \beta(\tau - t, (d - i\xi)v),
\]

where, letting

\[
\begin{align*}
s &\in \mathbb{R}, \quad u \in \mathbb{C} \\
b(u) &= \sigma_v \bar{\rho} u - \kappa_v, \\
a(u) &= u(1 - u), \\
\gamma(u) &= \sqrt{b^2(u) + a(u)\sigma_v^2} \quad \text{and} \\
\alpha_0(s, u) &= -rs + (r - q)us - \\
\kappa_v \bar{v} \left( \frac{\gamma(u) + b(u)}{\sigma_v^2} s + 2\sigma_v^{-2} \ln \left( 1 - \frac{\gamma(u) + b(u)}{2\gamma(u)} \left( 1 - e^{-\gamma(u)s} \right) \right) \right),
\end{align*}
\]
we have
\[
\bar{\beta}(s, u) = -\frac{a(u)(1 - e^{-\gamma(u)s})}{2\gamma(u) - (\gamma(u) + b(u))(1 - e^{-\gamma(u)s})}
\]
\[
\bar{\alpha}(s, u) = \alpha_0(s, u) + \sum_{l=1}^{2} \lambda_l s(-1 - \phi_l(1)u + u + \phi_l(u)).
\]

The function \(G_d\) in the pricing model can be solved from \(\hat{G}_d\) by taking the inverse Fourier transform,
\[
G_d(x; y, v, \theta) = \frac{1}{\pi} \int_0^\pi \frac{\text{Im} \left\{ \hat{G}_d(\xi; y, v, \theta) e^{-i\xi x} \right\}}{\xi} d\xi,
\]
and the formula for the put price can be obtained with put-call parity.

3. Estimating the model parameters with mean squared errors

This section introduces our parameter estimation method. The objective is to calculate the parameters implied by the market prices of the options. A simple example is the implied volatility of the Black-Scholes model.

Suppose we are given the prices of \(M\) options at \(n\) distinct times \(t_1, \ldots, t_n \in [0, T]\), where \(T\) is bounded. Note that \(t_k\) may occur after the expiration of the \(j\)-th derivative, in which case the sample size is smaller for the dates after the expiration of this instrument. As in the previous section, let \(\theta(t_k)\) denote the vector of real-valued constant parameters which define our model at time \(t_k\). Our goal is to come up with a “best-fit” for \(\theta(t_k)\) at each time \(t_k\).

Let \(\hat{C}_j(t_k)\) denote the historical price of the \(j\)-th option at time \(t_k\). Let \(C_j(t_k, \theta(t_k))\) denote the model price of the option at time \(t_k\) when the parameter set is \(\theta(t_k)\). At each time \(t_k\), we define the best fit parameter set, \(\hat{\theta}(t_k)\), by
\[
\sum_{j=1}^{M} \left( \hat{C}_j(t_k) - C(t_k, \hat{\theta}(t_k)) \right)^2 = \min_{\theta(t_k)} \left\{ \sum_{j=1}^{M} \left( \hat{C}_j(t_k) - C(t_k, \theta(t_k)) \right)^2 \right\}.
\]

Thus, we find the optimal parameters by minimizing the mean squared error. This gives us a sequence of parameter sets \(\hat{\theta}(t_1), \ldots, \hat{\theta}(t_n)\) which we can then model with time series models. Note that since the price history \(\{\hat{C}(t_1), \ldots, \hat{C}(t_n)\}\) is observed under the real-world measure \(P\), the stochastic processes that we derive for the implied parameter \(\{\hat{\theta}(t_1), \ldots, \hat{\theta}(t_n)\}\) will also be given with respect to that measure. We use the time series models in a portfolio selection and hedging.

4. Portfolio selection and hedging

One application of the pricing model is in the selection and hedging of option portfolios. In this section, we analyze a simple portfolio strategy with different hedging scenarios.
We assume an investor selects portfolios by calculating $E[\Delta C_j(t, \hat{\theta}(t))]$ at time $t_k$ for all $j \in \{1, \ldots, M\}$, which is based on the estimated parameter set $\hat{\theta}(t)$ and, as we show, also on the estimated processes of the parameters. Given a constant $\beta > 0$, the investor holds a long position of option $j$ if $E[\Delta C_j(t, \hat{\theta}(t))] \geq \beta (\exp(r \Delta t) - 1)$, and a short position if $E[\Delta C_j(t, \hat{\theta}(t))] \leq \beta (\exp(-r \Delta t) - 1)$. At time $t_{k+1}$ the investor clears all positions and repeats the process. Thus, our initial portfolio choice implies that we try to maximize expected portfolio value. Then later in hedging we minimize portfolio risk.

In Section 3 we assumed that there exist 10 parameters whose implied values might differ when they are estimated at different $t_k \in [0, T]$. We assume here that the implied parameters follow the following stochastic differential equation

$$d\hat{\theta}(t) = \alpha(\hat{\theta}, t) dt + \sigma dW_t^\theta,$$

where $\alpha(\hat{\theta}, t) \in \mathbb{R}^{10}$, $\sigma \in \mathbb{R}^{10}$, and $W_t^\theta$ is a 10-dimensional $\mathcal{F}_t$-adapted Brownian motion process. In the next section we describe $\alpha(\hat{\theta}, t)$ by using time series models and then $\alpha(\hat{\theta}, t)$ and $\sigma$ can be used in the portfolio selection to calculate $E[\Delta C_j(t, \hat{\theta}(t))]$. The correlation between the first element of $\hat{W}_t$ (the element that governs the underlying asset process) and the $i$-th element of $W_t^\theta$ is denoted by $\rho_i$. So, for instance, $\rho_1$ is the correlation between $\lambda_1$ and $X$. Further, for simplicity, we assume that the correlation between the parameters is given by correlations with the underlying asset. For instance, the correlation between the first and second elements of $W_t^\theta$ is $\rho_1$.

To calculate $E[\Delta C_j(t, \hat{\theta}(t))]$ we use observation $Y(t_k)$, estimated $\hat{\theta}(t_k)$, equation (4), and Itô’s lemma as follows

$$E[\Delta C_j(t, \hat{\theta}(t))] = \frac{\partial C_j}{\partial Y_i} \left[ r - q - \lambda_1 \left( e^{\mu_1 + \frac{1}{2} \sigma_1^2} - 1 \right) - \lambda_2 \left( e^{\mu_2 + \frac{1}{2} \sigma_2^2} - 1 \right) \right]$$

$$- \frac{1}{2} \frac{\partial^2 C_j}{\partial Y_i^2} \Delta t + \frac{\partial C_j}{\partial V_i} [\kappa_v(\bar{v} - V_i)] \Delta t + \frac{1}{2} \frac{\partial^2 C_j}{\partial Y_i^2} V_i \Delta t$$

$$+ \frac{1}{2} \frac{\partial^2 C_j}{\partial V_i^2} V_i \sigma_v^2 (1 - \rho^2) \Delta t + \frac{\partial^2 C_j}{\partial Y_i \partial V_i} V_i \sigma_v \sqrt{1 - \rho^2} \sigma_v \Delta t$$

$$+ \lambda_1 \Delta t \int_{\mathbb{R}} \left[ C_j \left( t, Y_i + z_1, \hat{\theta}(t) \right) - C_j \left( t, Y_i, \hat{\theta}(t) \right) \right] dv_1(z_1)$$

$$+ \lambda_2 \Delta t \int_{\mathbb{R}} \left[ C_j \left( t, Y_i + z_2, \hat{\theta}(t) \right) - C_j \left( t, Y_i, \hat{\theta}(t) \right) \right] dv_2(z_2)$$

$$+ \sum_{i=1}^{10} \frac{\partial C_j}{\partial \theta_i} \alpha_i(\hat{\theta}, t) \Delta t + \sum_{i=1}^{10} \frac{\partial^2 C_j}{\partial \theta_i^2} \sigma_i \sigma_i \rho_i \rho_i \Delta t$$

$$+ \sum_{i=1}^{10} \frac{\partial^2 C_j}{\partial Y_i \partial \theta_i} \sqrt{V_i} \sigma_i \rho_i \Delta t + \sum_{i=1}^{10} \frac{\partial^2 C_j}{\partial V_i \partial \theta_i} \sqrt{V_i} \rho_i \rho_i \Delta t.$$  (5)

Here $\alpha_i, \sigma_i, \hat{\theta}_i$ are the $1 \leq i \leq 10$-th coordinate of $\alpha, \sigma, \hat{\theta}$, respectively. By using equation (5) and the portfolio rule described above we can select our initial portfolio that we next hedge.
For each fixed set of model parameters, $\theta$, there exist a set of “traditional Greeks”, some of which are useful to hedge in practice. For example, usually one may try to delta-, gamma- and vega-hedge one’s portfolio daily [see e.g. Hull (2002)]. In this section we analyze hedging in the same way as is done in Merton (1976). That is, we try to hedge all the continuous uncertainties of an option price by trading the underlying asset. The continuous uncertainties are not only from the underlying asset but also from the parameter changes. In the presence of jumps, perfect option hedging requires other derivative instruments. Such hedging is considered e.g. in Björk (1998) and Carr, Ellis, and Gupta (1998).

If the correlations are nonzero, we can at least partially hedge the parameter uncertainties by trading the underlying asset. Let $\pi_t$ denote how much of the underlying asset one must hold to hedge the uncertainties that are correlated with the underlying asset. Then from (1), (4), (5), and Itô’s lemma we get

$$\pi_t = \frac{1}{X_t \sqrt{V_t}} \sum_{j=1}^{M} \left[ \frac{\partial C_j}{\partial Y_t} \sqrt{V_t} + \frac{\partial C_j}{\partial V_t} \sqrt{V_t} \rho + \sum_{i=1}^{10} \frac{\partial C_j}{\partial \theta_i} \rho_i \sigma_i \right].$$

Note that if the correlations vanish, then our hedging strategy agrees with the usual delta hedging, i.e., $\pi_t = \sum_{j=1}^{M} \frac{\partial C_j}{\partial X_t}$ since in this case we have $\pi_t = \sum_{j=1}^{M} \frac{\partial C_j}{\partial X_t} \frac{1}{X_t}$ and $\frac{\partial Y_t}{\partial X_t} = \frac{1}{X_t}$ because $Y_t = \ln(X_t)$. Recall that the above hedging strategy is not a perfect hedge because we ignore possible jumps and the correlations are usually less than one. That is, we are unable to hedge all the uncertainties in the option prices by only trading the underlying asset and therefore we consider only partial hedging.

5. Numerical analysis

5.1. Estimation

We use daily closing prices of S&P 500 index put options for September 1999 through June 2001. We use puts because they are more liquid than calls during our sample period. We eliminate all prices below $0.05 and prices that violate no-arbitrage bounds. We use the average 4-week treasury bill rate as the risk-free rate, which was 3% in 1999, 5.7% in 2000 and 3.3% in 2001. We assume the dividend rate on the S&P 500 to be one percent per year, which approximates the 5-year average dividend yield on the S&P 500. We divide the data into two parts: sample data, which ranges from September 1999 through December 2000, and out-of-sample data, which ranges from January 2001 through June 2001. Table 1 summarizes the data.

In our pricing model, we incorporate three additional characteristics of the underlying asset in addition to the diffusion process: up jumps, down jumps, and stochastic volatility. We examine which of these characteristics contribute most to the option price processes in order to avoid modelling phenomena whose effect is negligible. At each trading day, we estimate the implied parameters using least squares. The parameters are rounded to the nearest 0.01. In general, the parameters of the pricing
function differ significantly from zero and they are optimal in the mean square sense. Given this data, our model and parameters do a better job of pricing than the Black-Scholes model with optimal implied volatility and also, as an example, do better than the same model with the parameter values from Duffie, Pan, and Singleton (2000) (this is natural because they use a different data set and their model is calibrated to that data set). For example, on December 22, 2000, the sum of squared pricing error for all of the options was 7,000 for our model, 8,540 for the Black-Scholes model with the best fitting implied volatility, and 152,100 using the parameter values from Duffie, Pan and Singleton (2000).

Table 2 summarizes the parameters. The variances of the downward and upward jump parameters are quite low but the mean-reverting stochastic volatility parameter, $\kappa_v$, has high variance.Regardless of this high variance, the average option price difference corresponding to the minimum and maximum parameter values is low (the option price effect). The high variation in $\kappa_v$ implies that the parameter is not stable and possibly other models for stochastic volatility should be used. Based on the option price effects downward jump is the most important. For instance, the variation of $\mu_2$ affects over 400% option price changes. According to these results, one would want to concentrate on modelling down jumps at the expense of the other characteristics. However, as is illustrated later in this section, this is not the case if we consider hedging.

<table>
<thead>
<tr>
<th>Type</th>
<th>Up jump</th>
<th>Down Jump</th>
<th>Stochastic Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>$\mu_1$</td>
<td>$\sigma_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>Mean</td>
<td>0.38</td>
<td>0.69</td>
<td>0.09</td>
</tr>
<tr>
<td>Median</td>
<td>0.08</td>
<td>0.07</td>
<td>0</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.53</td>
<td>1.16</td>
<td>0.21</td>
</tr>
<tr>
<td>Min.</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Max.</td>
<td>1.98</td>
<td>4.66</td>
<td>0.76</td>
</tr>
<tr>
<td>Option price effect, %</td>
<td>54.23</td>
<td>69.21</td>
<td>64.29</td>
</tr>
</tbody>
</table>

Table 1. Summary of option data. $P$ is the price of the put option, $K$ is the strike price, $X$ is the level of the S&P 500, and $T$ is the time to maturity in years.
Table 3. Best models for parameter prediction

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Best model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>ARMAX(1,0,1)</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>ARMAX(2,1,1)</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>AR(1)</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>ARMAX(4,0,1)</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>ARMAX(2,1,1)</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>ARMAX(4,2,1)</td>
</tr>
<tr>
<td>$\kappa_v$</td>
<td>ARMAX(2,1,1)</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>ARMAX(4,3,1)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>ARMAX(2,1,1)</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>ARMAX(2,1,1)</td>
</tr>
</tbody>
</table>

We then fit a time series model to the resulting daily changes in the parameter estimates, i.e., we model the drift term in (4) with a time-series model. We consider an ARMAX model, which incorporates the parameters’ lagged values, lagged error terms, and also the lagged values of the underlying asset (see Ljung (1998), for example). We use this model because it is quite general and nests popular time series models such as AR and ARMA. For all the parameters, we choose the ARMAX model order that minimizes prediction error on out-of-sample data. The best model for prediction purposes is generally a full differenced ARMAX model, where we model differenced implied parameters, rather than any of the simpler models. Table 3 shows a summary of the best model for each parameter. Modelling all parameters using the best model decreases prediction error from using the myopic model by 65% (with out-of-sample data). That is, we are able to predict the implied parameters and this can be seen as predictability of the implied volatility surface and thus option prices.

5.2. Portfolio selection and hedging

As an example and test of our hedging strategy of section 4, we select portfolio and hedging positions by using the sample data. We assume an investor has initial wealth $M_0 = 1,000,000$ and the portfolio selection parameter $\beta = 3$ which means that we select options that give at least three times higher expected price change than the risk-free bond.

The investor proceeds as follows. At time $t$, she clears all positions at hand and obtains wealth of $M_t$. Before choosing a new portfolio, she eliminates all out-of-money and almost mature options and calculates $E[\Delta C(t)]$ of the rest of options by using the ARMAX models. Second, the investor decides on a short position. If $E[\Delta C(t)] \leq 3(\exp(-r\Delta t) - 1)$ she sells the option. We assume the total value of her short positions is equal to the wealth at hand, i.e. $M_t$. Third, the investor uses her wealth to buy all options with $E[\Delta C(t_t)] \geq 3(\exp(r\Delta t) - 1)$. If there is no option that
satisfies this condition, she invests the money in the risk-free bond with interest \( r \). The procedure is repeated at time \( t + \Delta t \).

Together with the above initial portfolio selection we consider different hedging strategies. We assume the investor can partly hedge the risk by buying and selling future contracts on S&P500 Index. We choose futures contracts because they have the same delta as the underlying asset and because they have zero price.

In calculating hedging strategies we assume that the correlations \( \rho_i \) and standard deviation of the parameters \( \sigma_i \) (estimated using the sample data) are constant over the hedging period (out-of-sample time period). Thus, this example is a test of the stationarity of these parameters as well. We consider three strategies. The first strategy holds the initial portfolio and none of the future contracts. The second strategy is a regular delta hedge. The third strategy uses Section 4 to determine its position in the future contracts, i.e. hedging all parameters as well as possible. The hedging error is defined as follows

\[
\text{stdev} [\ln(S(t+1)) - \ln(S(t))],
\]

where \( S \) is the value of the initial portfolio plus the value of the futures position.

The correlations between the parameters and the underlying asset are presented in Table 4. The correlations of the stochastic volatility parameters are more significant than the correlations of the other characteristics. The first derivatives of the put prices with respect to the parameters are presented in Table 5. These are the average values for our put options and we expect them to be different for other options with different strikes. This is because under the risk-neutral measure the jump size and intensity are compensated by an equal and opposite change in drift. Therefore, only the higher moments of the distribution change, and this will affect the probability that the option is in-the-money differently for different strike prices. Table 5 implies that the most important one is the derivative with respect to the average volatility, \( \bar{v} \). Table 5 also illustrates the hedging due to the parameter changes. According to (6) the parameter hedging is given by the parameter’s standard deviation, Table 2, correlation with the underlying asset, Table 4, and the first derivative, Table 5. Table 5 implies that all underlying characteristics are important in the hedging and, therefore, the modelling of all these characteristics is essential.

### Table 4. Correlation between the parameters and S & P Index

<table>
<thead>
<tr>
<th>Type</th>
<th>Up jump</th>
<th>Down Jump</th>
<th>Stochastic Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>( \mu_1 )</td>
<td>( \sigma_1 )</td>
<td>( \lambda_1 )</td>
</tr>
<tr>
<td>Correlation</td>
<td>0.07</td>
<td>0</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Table 6 presents the results of the initial portfolio with different hedging strategies and the out-of-sample data. The measure of investment performance is the Sharpe
Table 5. First derivatives and hedging effects

<table>
<thead>
<tr>
<th>Type</th>
<th>Up jump</th>
<th>Down Jump</th>
<th>Stochastic Vol.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>( \mu_1 )</td>
<td>( \sigma_1 )</td>
<td>( \lambda_1 )</td>
</tr>
<tr>
<td>Derivative</td>
<td>16.93</td>
<td>23.74</td>
<td>86.02</td>
</tr>
<tr>
<td>Hedging, % of the delta hedge</td>
<td>-2.21</td>
<td>-0.29</td>
<td>-17.69</td>
</tr>
</tbody>
</table>

Figure 1 shows the wealth process with different hedging strategies from April 03, 2001 to June 03, 2001. The three different curves show the wealth processes of the initial portfolio without hedging, with delta hedging and with the hedging of all parameters, like in Table 6. This illustrates that the maximally hedged portfolio process is close to that of the delta-hedged process, but somewhat less volatile. The unhedged process diverges from the other two and follows a less predictable path.

6. Conclusion

This paper models the implied parameters of a relatively general option pricing setup using market data, and finds that downward jumps have a larger effect on option processes than do stochastic volatility or upward jumps. We find that a time-series model explains 65% of the variation in the option pricing parameters. We hedge the parameter uncertainties with futures contracts on the underlying asset and, in our example, this improves the hedging performance of a simple delta hedge on average by 8 percent and increased the portfolio’s Sharpe Ratio by 21 percent. In this hedging, we found that...
Figure 1. **Wealth process with different hedging strategies.** The three different curves show the wealth processes without hedging, with delta hedging and with hedging of all the parameters.

Modelling and Hedging Options

all underlying characteristics (stochastic volatility, up and down jumps) are all almost equally significant. Since our model can be viewed as a parameterization for implied volatility, this paper gives a method to predict and hedge implied volatility movements.


