PERSONAL STATEMENT

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In this personal statement, I give a non-technical description of my research. I only cite my own papers, numbering them as they appear on my CV. This non-technical description mostly discusses my work in symplectic geometry, but also some recent work in string topology. In the separate self-contained research statement, I provide a technical description of the research. In the research statement I use a comprehensive bibliography, citing my papers along with others.

1. Background

Symplectic geometry is sometimes called symplectic topology, depending on the point of view. My research supports the "geometry" point of view.

Perhaps the best way to distinguish the topologist from the geometer is to describe how they would interpret the notion of "equivalence." Both the topologist and the geometer study spaces, often called manifolds. The topologist cares only about large-scale features of the manifold, such as its dimension and possible holes. The geometer studies more precise features such as distance and curvature. Consider the following 2-dimensional manifolds: the surface of a doughnut, not including the interior, known as a torus; and the surface of a coffee mug. The topologist can find a continuous bijective map whose domain is the mug and range the torus, and thus declares the two spaces equivalent. But any such map could not preserve the curvature; thus, the geometer would declare the two spaces different.

To place symplectic geometry in this context, I should first describe its historical precedent, classical mechanics. Mechanics studies the motion of objects that obey the basic laws of physics, such as conservation of energy and momentum, as well as "$F = ma$." We live on the 2-dimensional surface of the planet Earth. But a symplectic geometer would describe this world as a 4-dimensional phase space: two dimensions for an object's latitude and longitude, (q_1, q_2), and two corresponding dimensions for its momentum (p_1, p_2). Recall $p_j = m\frac{dq_j}{dt}$ where m is mass and t is time. To ensure that a moving object conserves its total energy, which is the sum of its kinetic and potential energies, we need to know both its changing position and momentum coordinates.

The Heisenberg Uncertainty Principle links momentum and position: measuring a particle's position affects its momentum, and vice versa. For example, measuring the latitude, q_1, alters p_1 in some semi-random way. A rough way to rephrase the principle is that if you know the object's q_1-coordinate up to an error of size Δq_1 and, similarly with p_1, q_2, p_2, then your experiment for locating an object in phase space can never reduce the quantities $\Delta q_1 \Delta p_1$ and $\Delta q_2 \Delta p_2$ below a certain threshold.

Like topologists and geometers, symplectic geometers study manifolds, focusing solely on even-dimensional manifolds which are themselves phase spaces of smaller manifolds. To prove that the phase spaces of the torus and the coffee mug surface are equivalent, the symplectic geometer must construct a continuous bijective map, known as a symplectic map, from one space to the other which preserves the quantities $\Delta q_1 \Delta p_1$ and $\Delta q_2 \Delta p_2$. This may seem like geometry, where maps must preserve distances like Δq_1. However, there exists a symplectic map from any sufficiently small ball in any phase space of a given dimension to any other small ball in another phase space of the same dimension. This is in sharp contrast to geometry because one small ball might have flat curvature, like the base of the mug, while another has round curvature. For this reason, symplectic maps were first considered more "topological", recording only large-scale features.

Formally, symplectic geometers study symplectic manifolds which generalize phase spaces. A symplectic manifold is $2n$-dimensional and comes equipped with a symplectic form which (locally) is written as $dq_1 dp_1 + \cdots + dq_n dp_n$. Here $dq_j dp_j$ is a "differential form" whose input is a pair of tangent vectors and output is a real number. A symplectic map preserves this symplectic form.

Instead of considering maps from one manifold to another, one can also think of a continuous map which mixes points around within a single symplectic manifold. Think of each point in the manifold as the position and momentum coordinates of a particle. The points are mixed as the particles move about. Some calculus shows essentially that the map preserves the symplectic form if and only if the particles obey the laws of classical mechanics. In different forms, this is called Hamiltonian or Euler-Lagrangian dynamics, or the Least Action Principle. Consider a loop of particles as it moves about under the Least Action Principle. Thinking of time as one coordinate and the parameterization of the loop as the second, one can rewrite the ordinary differential equation behind the Least Action Principle as a partial differential equation (PDE). This PDE is equivalent to the Cauchy-Riemann equations from complex analysis, and hence the cylinder traced out by the moving loop becomes what complex analysts call a "holomorphic" curve. (It looks like a 2-dimensional surface, but mathematicians like to call it a 1-complex-dimensional curve.)

This idea motivated a mathematician, Mikhail Gromov, in 1986 to study holomorphic curves in general symplectic manifolds. He proved that they had a certain property: sets of such curves are compact sets. This result marked the birth of modern symplectic geometry. Symplectic geometers were able to use many of the powerful properties of holomorphic curves which had been developed in the much older field of complex analysis.

One of the first significant applications arose from effective ways to count certain holomorphic curves. I will loosely call these different counts "Gromov-Witten theories" due to the popularity in mathematics and physics of the Gromov-Witten invariants. The counts produce not just numbers, but sometimes groups or algebras – mathematical objects with more information. One particularly useful count, known as Floer homology, led to discoveries in symplectic dynamics which showed that symplectic geometry is quite different than topology. As an example, consider a closed orbit of a dynamical system, which is a path of a particle which closes up on itself. In 1998, Floer homology was critical to proving the Arnold Conjecture, which claims a lower bound on the minimum number of closed orbits of any given Hamiltonian dynamical system. This bound is greater than the bound a topologist could achieve for any given dynamical system. More recently, even topologists have started to use Floer homology. For example, a topologist studying a knot (formed by taking a string, tying a knot, and fusing the ends together) can construct some large-dimensional phase space and compute Floer homology there to recover information about the knot. Examples of this include knot Floer homology and knot contact homology.

Contact geometry is the sister field of symplectic geometry. Many if not most symplectic geometers study both even-dimensional symplectic manifolds and odd-dimensional contact manifolds. For an example of a contact manifold, suppose the total energy of a phase space is only kinetic energy, $E(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}$. Conservation of energy implies that a particle must travel on an energy level set, such as $E^{-1}(17)$. This level set is a three-dimensional contact manifold. Contact manifolds have special submanifolds known as Legendrian submanifolds. (Their symplectic analogue is called a Lagrangian submanifold.) An n-dimensional Legendrian submanifold sits inside a $(2n+1)$-dimensional contact manifold. Legendrians, although abstractly defined, appear naturally. Systems which satisfy the first law of thermodynamics form a 2-dimensional Legendrian surface in the contact 5-space (energy, temperature, entropy, pressure and volume). Legendrians of arbitrary dimension represent wavefronts in the field of optics. For a more pictorial example, as a car parallel parks, its front tire traces out the so-called front projection of a one-dimensional Legendrian curve in contact 3-space.

Each complex dimension can be thought of as two real dimensions; thus, complex analysis applies to even-dimension manifolds. For this reason, despite the similarities, contact geometry did not develop as quickly as symplectic geometry after Gromov's seminal work. Only more recently have Gromov-Witten theories been developed to study problems in contact geometry. These include contact homology and its generalization, symplectic field theory. In some papers written before I had tenure, Tobias Ekholm, John Etnyre and I develop Legendrian contact homology (LCH) as a way of counting holomorphic curves to study Legendrian submanifolds.

2. Results as Associate Professor

My more recent work is in contact geometry. I use holomorphic curves to prove results which further demonstrate that the field is more geometry than topology.

Any 1-dimensional knot in a 3-dimensional ambient space can produce a 2-dimensional Legendrian torus in a 5-dimension contact subspace of the (3(position) + 3(momentum) = 6)-dimensional symplectic phase space of the ambient space. Moreover, two knots considered to be equivalent by topologists produce two Legendrian tori that are equivalent to contact geometers; thus, LCH invariants for the tori induce topological invariants for the knots. In this case, the LCH of the torus is called the knot contact homology of the knot. In [12], Tobias Ekholm, John Etnyre, Lenny Ng and I prove that this, usually hard-to-compute LCH invariant, has a combinatorial reformulation. Knot contact homology (with some extra information) has since been shown to be a complete knot invariant: two knots are distinct if and only if their knot contact homologies are. There has also been recent effort by physicists to connect knot contact homology to string theory. In [13], Ekholm, Etnyre, Ng and I show how if the one-dimensional knot is in fact transverse (a complimentary property to being Legendrian) when the ambient 3-space is thought of itself as a contact manifold, the knot contact homology comes equipped with natural extra structure. Our new transverse knot invariant is quite powerful as it can distinguish transverse knots that could not be distinguished by previous such invariants.

As mentioned above, LCH has a nice combinatorial reformulation in the case when the Legendrian is a 2-dimensional torus arising from a 1-dimensional knot. In general, however, LCH, like any holomorphic-curve based Gromov-Witten theory, is *a priori* quite hard to compute. The difficulty lies in the fact that holomorphic-curve based invariants are "analytical": they arise as solutions to PDE as mentioned in Section 1. In a series of papers, I show how LCH is in fact combinatorial when the Legendrian is 2-dimensional. (The combinatorial result for 1-dimensional Legendrian knots was proved by Chekanov in 1997.) In the first paper [14], which grew out of an REU project with Alicia Harper, we show that LCH is local: you only have to compute LCH for pieces of the submanifold. Later, in [17] and [18], Dan Rutherford and I use [14] to enumerate all the holomorphic curves, producing a very "natural" reformulation of the theory. There are many potential applications for this reformulation, including [20], which is work-in-progress.

In another project which grew out of the knot contact homology of [12], Somnath Basu, Jason McGibbon, Dennis Sullivan and I reformulate the knot invariant in terms of string topology [14]. String topology is much easier to define than holomorphic curves. Essentially, we take strings (arcs) which start and end on the knot, and perform operations on them, like cutting and gluing, to form other strings. In the end we produce a theory, which morally at least, is a (topological) field theory. The string topology we develop in [14] is the first such which can detect parts of knot theory. D. Sullivan and I further study this string topology in work-in-progress [22] where we prove that the theory can detect how 3-dimensional spaces are decomposed according to Thurston's famous Geometrization Conjecture.

In another direction, Georgios Dimitroglou Rizell and I in [19] prove a Legendrian version of the Arnold conjecture mentioned in Section 1. For this Legendrian version, we bound from below the number of flows of the Reeb vector field (a special type of flow studied in contact dynamics) which start on a Legendrian and end on the Legendrian pushed off of itself by a contact Hamiltonian map. These are the natural maps to consider when you want to preserve all the information of the contact manifold. Instead of LCH, we use another Floer-type theory defined by holomorphic curves.

Recall the initial dichomotony mentioned between contact (or symplectic) geometry and contact topology. If the field were entirely contact topology, then for two Legendrian submanifolds to be equivalent it would be sufficient if they were equivalent as just submanifolds. This is clearly not the case, as many people have shown. Josh Sabloff and I address this dichotomy in a "higher parameter space" [20]. Consider a Legendrian submanifold and move it about the ambient contact space until it comes back to itself after a minute. This is a loop of Legendrian submanifolds. The loop may or may not contract itself into the "constant loop," which is simply the Legendrian staying in place the whole minute. We provide examples of such Legendrian loops (in any dimension) which can contract to the constant loop when thought of as just submanifolds, but not when thought of as Legendrian submanifolds. Our proof uses generating families, which is an alternative, yet conjecturally equivalent, theory to LCH.