STAT 515

Homework 9

Show work to receive full credit. You may work on these problems with others but your answers must be your own. Please note the names of anyone that you worked with at the end of the homework.

1. Suppose that X is the total time between a customer's arrival in the store and departure from the service window, Y is the time spent in line before reaching the window, and the joint densities of these variables is given by

$$f(x,y) = \begin{cases} e^{-x}, & 0 \leq y \leq x \leq \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find the marginal densities $f_X(x)$ for X and $f_Y(y)$ for Y.

   (b) What is the conditional density of X given that $Y = y$?[1]

   (c) What is the conditional density of Y given that $X = x$?[2]

   (d) Is the conditional density $f(x \mid y)$ that you obtained in part (1b) the same as $f_X(x)$?

   (e) What does your answer in part (1d) imply about the marginal and conditional probabilities that X falls in any interval?

[1] Be sure to specify the values of y for which this density is defined.

[2] Be sure to specify the values of x for which this density is defined

2. Consider the set-up in problem 1. Are X and Y independent? Please support your answer.

3. Again, consider the set-up in problem 1. The random variable $Z = X - Y$ represents the time spent at the service window. Find $\mathbf{E}(Z)$ and $\text{Var}(Z)$. Is it highly likely that a randomly selected customer would spend more than 4 minutes at the service window?

4. Suppose that the probability that a head appears when a coin is tossed is p and the probability that a tail occurs is $q = 1 - p$. Person A tosses the coin until the first head appears and stops. Person B does likewise. The results obtained by persons A and B are assumed to be independent. What is the probability that A and B stop on exactly the same number toss?

5. Consider the set-up in problem 4. Let X and Y denote the number of times that persons A and B toss the coin, respectively. It is reasonable to conclude that X and Y are independent and each has a geometric distribution with parameter p. Consider $Z = X - Y$, the difference in the number of tosses required by the two individuals.

   (b) Find $\mathbf{E}(X)$, $\mathbf{E}(Y)$ and $\mathbf{E}(Z)$.

   (c) Find $\mathbf{E}(X^2)$, $\mathbf{E}(Y^2)$ and $\mathbf{E}(XY)$.

   (d) Find $\mathbf{E}(Z^2)$ and $\text{Var}(Z)$.

(e) Give an interval that will contain Z with probability at least $\frac{8}{9}$.

6. A box contains four balls, numbered 1 through 4. One ball is selected at random from the box. Let

$$X_1 = 1 \text{ if ball 1 or ball 2 is drawn,}$$
$$X_2 = 1 \text{ if ball 1 or ball 3 is drawn,}$$
$$X_3 = 1 \text{ if ball 1 or ball 4 is drawn.}$$

The X_i values are zero otherwise, $i \in \{1, 2, 3\}$. Show that any two of the random variables X_1, X_2 and X_3 are independent but that the three together are not.

7. Suppose that X and Y are uniformly distributed over the triangle shaded in Figure 1.

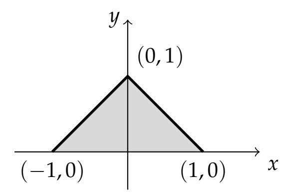

Figure 1: Joint density for X and Y.

(a) Find $\text{Cov}(X, Y)$.

(b) Are X and Y independent?

(c) Find the coefficient of correlation for X and Y.

(d) Does your answer in (7b) lead you to doubt your answer to (7a)? Why or why not?

8. If c is any constant and Y is a random variable such that $\mathbf{E}(Y)$ exists, show that $\text{Cov}(c, Y) = 0$.

9. Let Y_1 and Y_2 have a bivariate normal distribution.

(a) Show that the marginal distribution of Y_1 is normal with mean μ_1 and variance σ_1^2.

(b) What is the marginal distribution of Y_2?

10. Suppose that a company has determined that the number of jobs per week, N, varies from week to week and has a Poisson distribution with mean λ. The number of hourse to complete each job, Y_i, is gamma distributied with parameters α and β. The total time[3] to complete all jobs in a week is $T = \sum_{i=1}^{N} Y_i$.

[3] Note that T is the sum of a random number of random variables.

(a) What is $\mathbf{E}(T \mid N = n)$?

(b) What is $\mathbf{E}(T)$?