Math 456 **Math Modelling** **Spring 2019**

HW 5, due Thursday April 5

1. Recall the Ising model (Example 1.33, p.37) with a $(2L+1)$ by $(2L+1)$ square grid of magnetic particles. Show that the distribution $\pi(\xi) = \frac{1}{Z_\beta}\exp\left(\beta \sum_{x,y\equiv x} \xi_x \xi_y\right)$ is indeed a stationary distribution for the Metropolis-Hasting process. (Here $\beta > 0$ is a constant, and $Z_\beta > 0$ is a too-hard-to-compute constant that makes π an actual distribution.) Recall the transition matrix for the process from Metropolis-Hasting is
$$p(\xi, \xi') = q(\xi, \xi')r(\xi, \xi') = q(\xi, \xi') \min\left(\frac{\pi(\xi')q(\xi', \xi)}{\pi(\xi)q(\xi, \xi')}, 1\right)$$
where our particular choice of distribution q is $q(\xi, \xi') = (2L+1)^{-2}$ if ξ and ξ' have only one magnetic particle with a different sign, and $q(\xi, \xi') = 0$ otherwise.

2. Recall the traveling salesperson problem and the stationary distribution $\pi(\xi) = \frac{1}{Z_\beta}\exp(-\beta l(\xi))$ where $\beta > 0$ is a constant and $l(\xi)$ is the distance traveled under itinerary ξ. (The "itinerary" is the order of the fixed set of N cities visited.) Define the initial distribution $q(\xi, \xi')$ as follows. Pick randomly (uniformly) from the list of cities twice. (Note the same city can be picked twice.) If the itinerary ξ' is gotten from the itinerary ξ by switching those two cities' orders and leaving the order of the other cities fixed, then $q(\xi, \xi') = N^{-2}$. Otherwise $q(\xi, \xi') = 0$.

   (a) Explain why the Metropolis-Hastings process is aperiodic and irreducible. Do not worry about showing that it has a stationary distribution as that is similar to the above Ising model problem.

   (b) Recall the animated example of the traveling salesperson from class: the itineraries converged to one where the total distance travelled was (one of) the shortest. Explain why this visual convergence is an application of the Convergence Theorem.

3. Show that the population paradox cannot happen for the Jefferson method of apportionment. (Hint: if a single state's population increases and all other state populations are fixed, you can assume the quote Q does not decrease.)

4. Look at the following 2016 congressional race results for the MA. At the top of the page https://en.wikipedia.org/wiki/2016-United-States-House-of-Representatives-elections-in-Massachusetts you can see that Democratic congressional candidates won 100% of the seats with 79.73% of the vote. Republicans won 0% of the seats and 15.34% of the votes. (This does not add up to 100% due to third party voting. To simplify, assume there are only two parties, so for example, assume Democrats won $\frac{79.73}{79.73+15.34}$ percent of the vote.) Graph these MA results on the efficiency-gap graph.

Currently there are four court cases claiming gerrymandering in NC, MD, WI, PA. (Supreme Court is hearing NC case this week.) Repeat this process to locate these four states on the efficiency-gap graph. Can the efficiency-gap graph justify the gerrymandering claims for (all/some of) these states? Argue why or why not.

5. Do challenge #1 of the Squaretopedia sheet. Here we define compactness via the square version of Reock which works as follows. For each of the ten district, compute the ratio of the area of the district to the area of the smallest square (not circle) enclosing the district. Take the average of the ten ratios, which are each between 0 and 1. The closer this average is to 1, the more compact.