INTERNAL DIFFUSION LIMITED AGGREGATION

NESTOR GUILLEN

Abstract. A random growth model known as Internal Diffusion-Limited Aggregation, is discussed. This model amounts to a Markov chain whose state space is the subsets of $\mathbb{Z}^2$. We review Diaconis and Fulton related notion of "multiplication" for sets of $\mathbb{Z}^2$ and how it relates to this process. We also discuss the long time behavior of the model: after n steps, the state of the chain resembles a circle of a certain radius proportional to $\sqrt{n}$, a fact first proved by Lawler et al.

1. Background

A mathematical model combining some kind of particle dynamics, and a rule by which different particles start coalescing into a growing cluster is known as a *growth model*. Often, such a model amounts to a Markov chain where the state space are subsets of a lattice (or a more general discrete set). In this note, we discuss one such model. It was introduced by Diaconis and Fulton [1] and Lawler et al [2]. Here is an informal description of this process: at time n we have a set A_n, which initialized as

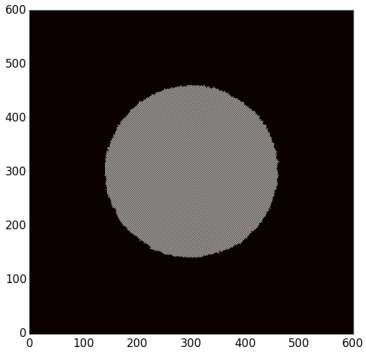
$$A_0 = \{0\}.'$$

The evolution from A_n to A_{n+1} is as follows. We consider a random walk X_t starting at 0 and run it until the first time it hits A_n^c, this is a point $X_{T_{A_n^c}}$, so we set

$$A_{n+1} = A_n \cup \{X_{T_{A_n^c}}\},$$

where for a set B, T_B denotes the first time that X_t reaches B.

This model is a variation on one introduced by Witten and Sander [3] which they dubbed Diffusion-Limited Aggregation. In the orignal model of Witten and Sander (dubbed just DLA) there is again an initial "seed" $A_0 = \{0\}$. Then the way to go from A_n to A_{n+1} is to start a random walk very far from the origin (concretely: one places a the starting point at one point chosen at random uniformly from the sphere of large radius centered at zero), and run it until it reaches for the first time one of the points which have a point of A_n as a neighbor.



In this model any emerging branches will continuously grow further, with unoccupied region near the origin being blocked from being visited. By contrast, in the internal DLA variation,

unoccupied sites closer the origin will be more likely to be visited than far away ones. Therefore just as in DLA one sees the appearance of marked irregularities, for internal DLA one expects a more or less symmetric shape to emerge, in fact, a disc.

This much was suggested by numerical simulations (one example is presented in the previous page). Later Lawler et al [2] used estimates on hitting probabilities of random walks to prove that in fact, as n grows, the cluster A_n begins to resemble with high probability a disc of radius proportional to $n^{1/d}$. More concretely, they showed that given any number $\varepsilon > 0$, with probability 1 there is a (random) number N_ε such that A_n contains a disc of radius $c_d n^{1/d}(1-\varepsilon)$ and is contained in a disc of radius $c_d n^{1/d}(1+\varepsilon)$. This result is stated in precise form later Theorem 2.1. In section 4 we discuss the results of numerical simulations that illustrate the scaling $n^{1/d}$ for the size of the disc.

2. Markov Chain Formalism and asymptotics

We consider a Cartesian grid, denoted by

$$\mathbb{Z}^2 = \Big\{(x,y) \mid x, y \text{ are integers}\Big\}$$

We have a growing sequence of random sets $A_n \subset \mathbb{Z}^2$, and the probability of going from the set A to the set B is given by

$$\mathbb{P}[A_{n+1} = B \mid A_n = A] = \begin{cases} h_0(A^c, \{x\}) & \text{if } B = A \cup \{x\},\ x \notin A, \\ 0 & \text{if } B \text{ otherwise.} \end{cases}$$

Here $h_z(A,B)$ denotes the probability that starting from z, the random walk first hits the set A at a node belonging to the set $B \subset A$.

Lawler et al show in [2] that indeed the asymptotic limit of the A_n is, after a proper rescaling, a ball (this is in fact the main result in [2]).

Theorem 2.1. *Fix $\varepsilon > 0$, then for all n large enough we have*

$$B_{n(1-\varepsilon)}(0) \subset A(\omega_d n^d) \subset B_{n(1+\varepsilon)}(0).$$

Equivalently, for $c_d = \omega_d^{-1/d}$ and all sufficiently large t, we have

$$B_{c_d t^{1/d}(1-\varepsilon)}(0) \subset A(t) \subset B_{c_d t^{1/d}(1+\varepsilon)}(0).$$

An alternative way of expressing this theorem, is if we define the two random quantities

$$R(n) = \max\{|z| \mid z \in \partial A(n)\}$$
$$r(n) = \min\{|z| \mid z \in \partial A(N)\}.$$

where

$$\partial A(n) = \big\{z \in A(n) \mid \text{ one of } A(n) \text{ nearest neighbors lies outside } A(n)\big\}.$$

Then, the above theorem says that, with probability 1, as $n \to \infty$

$$\log(R(n)) = \tfrac{1}{d}\log(n) + (\text{small term}),$$
$$\log(r(n)) = \tfrac{1}{d}\log(n) + (\text{small term}).$$

Where (small term) means a number which is much smaller than $\log(n)$ as $n \to \infty$. In the next subsection, we are going to discuss some of the ideas used in [2] to prove this Theorem. In particular, we only discuss ideas related to "the lower bound", that is, the fact that

$$B_{n(1-\varepsilon)}(0) \subset A(\omega_d n^d), \tag{1}$$

or equivalently,

$$\log(r(n)) \geq \tfrac{1}{d}\log(n) + (\text{small term}).$$

2.1. **A sketch of the lower bound.** Before showing the results of the numerical simulations, we shall briefly review some of the tools used in [2] to prove the inclusion (1).

First, the authors study the properties of an arbitrary random walk in the Cartesian grid. Therefore, they define for a given site (state) z and time t, the random quantity

$$N_t(z) = \#\Big\{ \text{ visits up to time } t \text{ to } z \Big\},$$

and used it to introduce the following function

$$G_n(y,z) = \mathbb{E}_y[N_{T_n-1}(z)].$$

Where T_n denotes the first time a particle exists the ball $B_n(0)$, that is

$$T_n = T_{B_n^c}$$

What $G_n(y,z)$ counts the (expected) number of visits to z before exiting $B_n(0)$, assuming the random walk starts from y.

Lemma 2.1. *Let $d = 2$, then for any z with $\|z\| \geq 1$,*

$$G_n(0,z) = \frac{2}{\pi} \ln\left(\frac{n}{\|z\|}\right) + o\left(\frac{1}{\|z\|}\right) + O\left(\frac{1}{n}\right)$$

On the other hand, if $d \geq 3$, then for any z with $\|z\| \geq 1$,

$$G_n(0,z) = \frac{2}{d-2}\frac{1}{w_d}\left(\frac{1}{\|z\|^{d-2}} - \frac{1}{n^{d-2}}\right) + O\left(\frac{1}{\|z\|^{d-1}}\right)$$

The estimate for $G_n(y,z)$ we really need, is one in average. What we need is that in average, $G_n(y,z)$ is comparable to $G(0,z)$ in the ball $B_n(0)$.

Lemma 2.2.

$$\sum_{y \in B_n} G_n(y,z) \leq w_d n^d G(0,z)$$

These two lemmas, the proof of which we omit, are then used to analyze the behavior of the clusters $A(n)$.

Second, fix $\varepsilon > 0$, and write, for the sake of brevity

$$r(n) = c_d(1-\varepsilon)n^{1/d}$$

Consider the events

$$\begin{aligned} E_z(n) &= \text{ site } z \text{ does not belong to } A_n, \\ F_\varepsilon(n) &= \text{ the disc } B_{r(n)}(0) \text{ is not contained in } A_n. \end{aligned}$$

$$F_\varepsilon(\varepsilon, n) = \bigcup_{z \in B_{r(n)}} E_z(n) \tag{2}$$

Then, a way to restate Theorem 2.1 is that the probability that an infinite number of the events $F_\varepsilon(n)$ occurs simultaneously is zero. Because that means that with probability 1 there is some n_0 such that

$$B_{r(n)}(0) \subset A_n \text{ for all } n \geq n_0.$$

In probability, the best known way to show that out of an infinite list of events $F(1), F(2), \ldots$ then with probability 1 at most a finite number of them occur simultaneously, is the celebrated Borel-Cantelli lemma. According to this Lemma, all we need to do is show that

$$\sum_{n=1}^{\infty} \mathbb{P}(F_\varepsilon(n)) < \infty.$$

Therefore, the goal is to find some upper estimate for the probability of $E_z(n)$ (and in turn, $F_\varepsilon(n)$) which goes to zero sufficiently fast in n so as to make the above series converges. With this goal in mind, Lawler et al [2] introduced further auxiliary quantities

$$M(n) = \#\Big\{ \text{ walks that visit } z \text{ before exiting } B_n \Big\}$$
$$L(n) = \#\Big\{ \text{ walks that visit } z \text{ before exiting } B_n \text{ but after leaving the cluster} \Big\}.$$

They show that, for every number $a > 0$, we have the estimate

$$\mathbb{P}(E_z(n)) \leq \mathbb{P}(M(n) \leq a) + \mathbb{P}(L(n) \geq a)$$

Then, [2, Lemma 6] shows that, for the right selection of the constnat a, we have

$$\mathbb{P}(M(n) \leq a) \leq e^{-c_d n},$$
$$\mathbb{P}(L(n) \geq a) \leq e^{-c_d n}.$$

Which in turn implies that

$$\mathbb{P}(E_z(n)) \leq 2e^{-c_d n}.$$

Then, in light of (2), we have

$$\mathbb{P}(F_\varepsilon)(n) \leq \sum_{z \in B_{r(n)}} \mathbb{P}(E_z(n)) \leq 2\omega_d n^d e^{-c_d n}$$

From here, the Borel-Cantelli Lemma says with probability 1, the number of events

$$F_\varepsilon(n)$$

which occur simultaneously must be finite. As explained above, this means that, with probability 1, for all sufficiently large n

$$B_{c_d n^{1/d}(1-\varepsilon)}(0) \subset A(n)$$

3. Diaconis and Fulton's smash sum

Consider two finite sets $A, B \subset \mathbb{Z}^d$. We define a **random set**, denoted $A \oplus B$, defined as follows:

1.If $A \cap B = \emptyset$, then

$$A \oplus B := A \cup B.$$

2.If $A \cap B$ is non-empty, we enumerate the elements of this interesection as $x_1, \ldots, x_n$. Then we define a sequence of sets $C_0, C_1, C_2, \ldots, C_n$ as follows: $C_0 = A \cup B$, given C_{k-1} for some $k \geq 1$ we start a random walk at x_k and let y_k denote the first time this random walk exits the set C_{k-1}. Then we set

$$C_k = C_{k-1} \cup \{y_k\}$$

This determines a random set C_n. We thus set

$$A \oplus B := C_n.$$

The set C_n may depend on the numbering of the points in $A \cap B$, but Diaconis and Fulton discovered that the probability distribution for C_n **does not**. Thus, in reality, the "smash sum" is not a well defined set but a well defined **distribution** over sets.

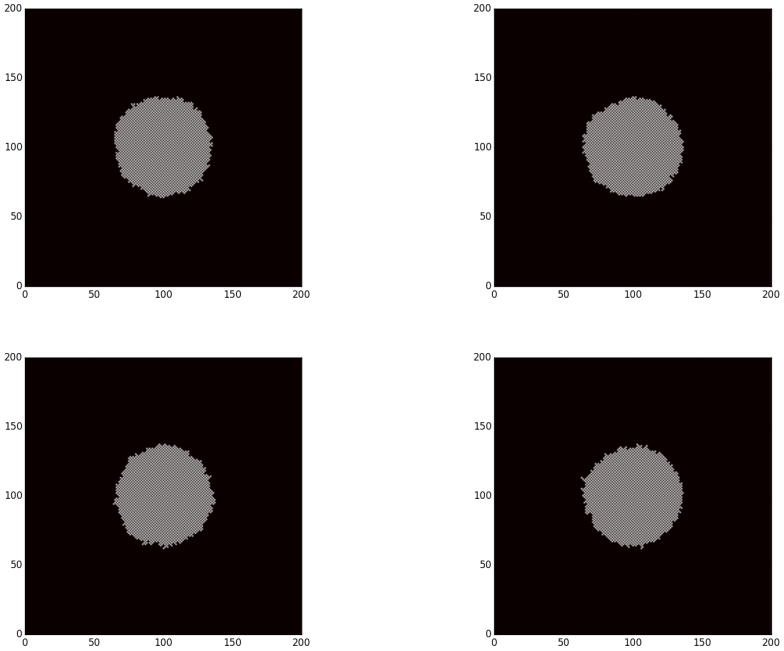
Internal DLA is just the sequence of sets given by

$$X_0 = \{0\}, \quad X_n = X_{n-1} \oplus \{0\}$$

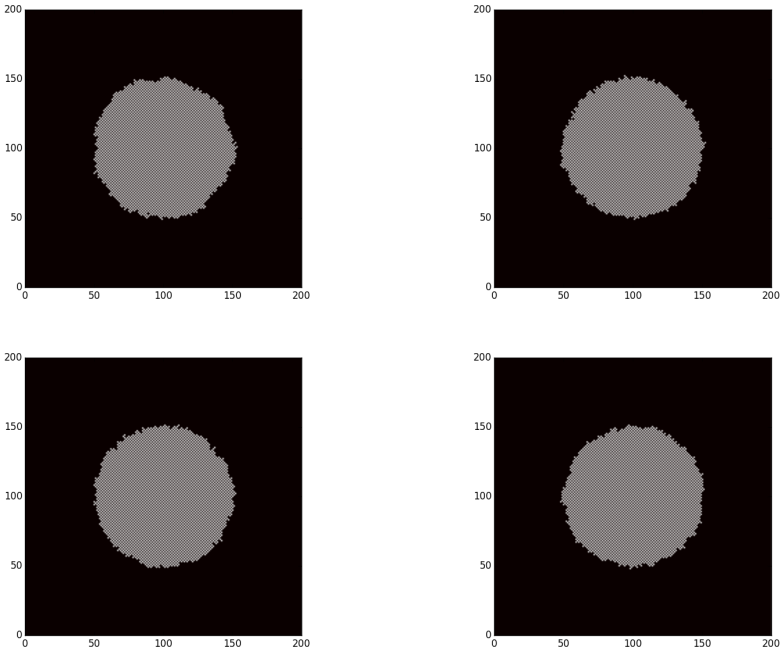
4. Numerical simulations

The graphs generated below are run in a 100x100 grid at first, and later in a 300x300 grid (see the appendix for the python code used).

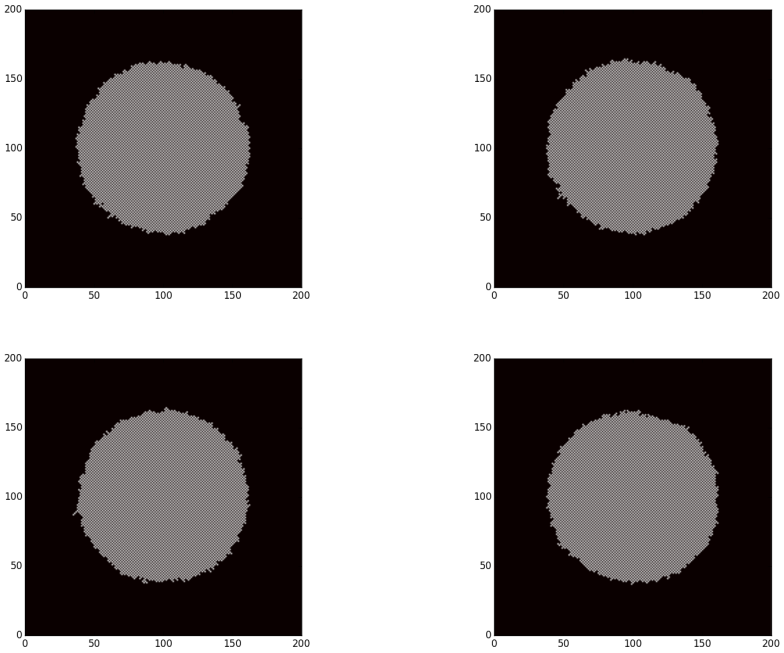
4.1. **Square lattice. Samples with $N = 2000$ points, 100 x 100 grid**



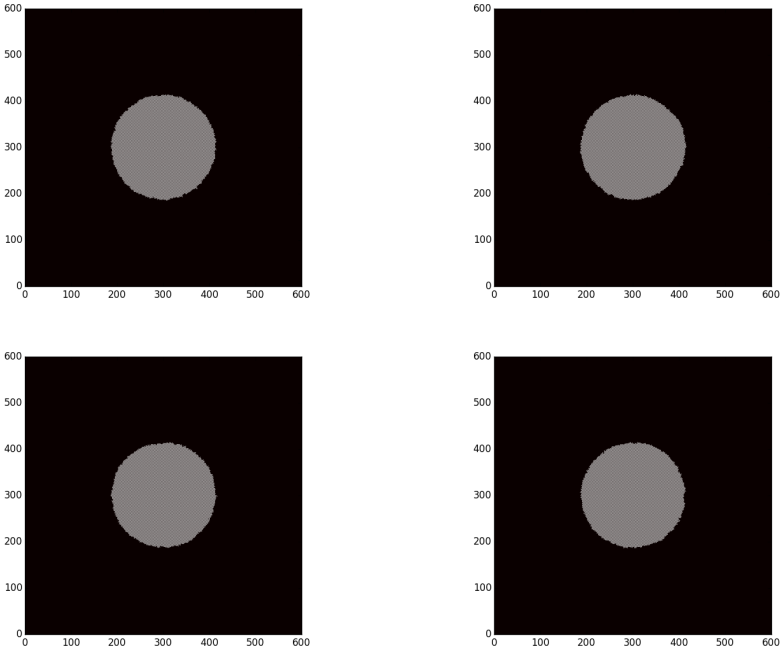
Samples with $N = 4000$ points, 100 x 100 grid



Samples with $N = 6000$ points, 100 x 100 grid



Samples with $N = 20000$ points, 300 x 300 grid



4.2. **Statistics on the size of the cluster.** Below we have a logarithmic plot of the size of the clusters for $d = 2$. We see how as n gets larger, that

$$\log(r(n)) \approx \log(R(n)) \approx \frac{1}{2}.$$

Appendix A. Code

The python code used to generate the images and the data in Section 4 is below.

```
import numpy as np
import random
import matplotlib.pyplot as plt

def random_direction():
    e = np.random.random_integers(0,1,2)
    e[0] = int(2*e[0]-1)
    e[1] = int(2*e[1]-1)
    return (e[0],e[1])

def move(x_location,N_size):
  e_direction = random_direction()
  x = np.zeros(2)
  if x_location[0]+e_direction[0]<=2*N_size and x_location[0]+e_direction[0]>=0:
     x[0] = x_location[0]+e_direction[0]
  if x_location[1]+e_direction[1]<=2*N_size and x_location[1]+e_direction[1]>=0:
     x[1] = x_location[1]+e_direction[1]
  return x

def indicator_complement(x_location,A_set):
    if A_set[x_location[0]][x_location[1]] == 1:
        return False
    else:
        return True

N = 300
n = 40000
x = np.zeros((2))

A = np.zeros((2*N,2*N)) #This is [-N,N]^2 (we gonna use negative indices!)

A[N][N] = 1

#Walker indicator setup
for i in range(n):
  Stop_time = False
  x = (N,N)
  print i
  while Stop_time == False:
    x = move(x,N)
    Stop_time = indicator_complement(x,A)
  A[x[0]][x[1]] = 1
```

```
#It remains to plot A

plt.imshow(A,cmap=plt.cm.hot)
plt.show()
```

References

[1] Diaconis, Persi, and William Fulton. *A growth model, a game, an algebra, Lagrange inversion, and characteristic classes.* Rend. Sem. Mat. Univ. Pol. Torino 49.1 (1991): 95-119.

[2] Lawler, Gregory F and Bramson, Maury and Griffeath, David, *Internal diffusion limited aggregation*, The Annals of Probability (1992), 2117–2140.

[3] Witten Jr, T. A., and Leonard M. Sander. *Diffusion-limited aggregation, a kinetic critical phenomenon.* Physical review letters 47.19 (1981): 1400.